Name: _____ Name-code: _____

University of Michigan Physics Department
Graduate Qualifying Examination

Part II – Modern Physics
Saturday, January 13, 2007 9:00am-1:00pm

Please write your name-code on each page of the exam.

This is a closed book exam – but you may use the "Constants, Conversions, and Formulas" sheet we provide for this exam.

Show your work in an organized manner to receive partial for it. If you need to make an assumption or an estimate, indicate it clearly.

You must answer the first 8 obligatory questions and two of the optional four questions. Indicate which of the latter you wish us to grade (e.g., circle the question number). We will only grade the indicated optional questions.

Please do not write on this page. The hatched column to the right and the total score will be filled in by the graders.

Good Luck!

Total Score:

1	
2	
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Part A: Obligatory Problems

1. Consider a particle with mass m in a two-dimensional square potential of width L and height V. There is also a weak potential in the box given by

$$\Delta V(x,y) = V_0 L^2 \delta(x - x_0)\delta(y - y_0)$$

a) Under what conditions for V and/or L would you expect the following to be good approximations to the eigenfunctions?

$$\psi^0(x,y) = \frac{2}{L}\sin\left(\frac{\pi m}{L}x\right)\sin\left(\frac{\pi n}{L}y\right)$$

where m and n are (small) integers.

b) Find the energy eigenvalues of the unperturbed system with $\Delta V(x,y) = 0$, assuming the eigenfunctions of part (a) above.

c) Evaluate the first-order correction to the energy of the ground state due to $\Delta V(x,y)$. (Use table of integrals available if needed.)

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2. A particle with mass m moves in one spatial dimension subject to the potential

$$V(x) = \frac{1}{8} g (x^2 - x_0^2)^2$$

Here g and x_0 are constants. Assume that the de Broglie wave length $\lambda \ll x_0$. Estimate the splitting ΔE between the two lowest lying states.

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3. At time $t = 0$, an electron and a positron are formed in a state with total spin equal to zero, perhaps from the decay of a spinless particle. The particles interact with a uniform magnetic field B_0 in the z-direction through a Hamiltonian of the form

$$\hat{H} = \omega_0(\hat{S}_{1z} - \hat{S}_{2z})$$

What is the probability that the total spin of the system is $S = 1$ at time t?

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4. Consider a simple model for the thermal properties of a solid, the Einstein model. Here the atoms (which are bound in place by their mutual interactions) are supposed to vibrate around their equilibrium positions as if they were N independent simple harmonic oscillators with natural frequency ω_E. This Einstein frequency is of order 10^{14}/sec.

   a) First suppose that the atoms can be treated classically, and are distinguishable because they are fixed in position. Find the partition function, Z, the free energy, F, and the heat capacity.

   b) Now treat the oscillators quantum mechanically. Again find Z, F, and the heat capacity. Show for what T limit the quantum heat capacity becomes classical.

   [Recall that $\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}$.]

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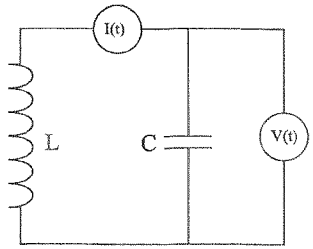
5. Consider an LC circuit as a quantum system at a fixed temperature T.

   a) Find the RMS values of the current and of the voltage in the circuit.

   b) Find the high-temperature and low-temperature limits of your results in (a).



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6. A tritium atom (^{3}H) in the electronic ground state decays into a Helium ion ($^3He^+$). What is the probability that the ion emerges in the electronic ground state?

   Note, $\int_0^\infty x^n \exp(-px)dx = n!/p^{n+1}$.

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7. Consider an anisotropic Fermi gas for which the relation between energy and momentum is given by

$$E = \frac{\hbar^2}{2}\left(\frac{k_x^2}{m_1} + \frac{k_y^2}{m_1} + \frac{k_z^2}{m_2}\right)$$

Find the Fermi energy as a function of density by figuring out the volume in k-space of the Fermi surface, and dividing by the volume per k-vector $(2\pi)^3/(2V)$. (Where does the 2 in the denominator come from?)

Hint: It is well but not widely known that the volume of an ellipsoid defined by

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

is $4\pi abc/3$.

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8. Deep inelastic electron proton scattering is an important process in the study of strong interactions in particle physics. The process is $e^- + p \rightarrow e^- + X$, where X denotes final state hadrons.

(a) This process has recently been studied at the HERA collider where electrons with 30 GeV are collided head on with 820 GeV protons. Calculate the center of mass energy of this reaction. What energy does an electron beam which hits a stationary proton target have to have to reproduce this center of mass energy?

(b) For the case of inelastic scattering, there are two independant parameters, the invariant momentum transfer Q^2 and the Bjorken scaling variable, $x = \frac{Q^2}{2p \cdot q}$. Here $q_\mu = l'_\mu - l_\mu, Q^2 = -q^2$, l_μ, l'_μ are respectively the four momenta of the incoming and outgoing electron and p_μ is the four momenta of the proton. From the requirement that the invariant mass of the X state satisfies $M_X^2 \geq M^2$ (M is the proton mass), what is the allowed range of values for the Bjorken scaling variable x. What is its value in the elastic limit?

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Part B: Optional Problems

9. Due to the fine-structure splitting, the $5S$ and $5P$ transition of rubidium has two components at wavelengths $\lambda_1 = 780\,\text{nm}$ and $\lambda_2 = 795\,\text{nm}$. Estimate the inner-atomic magnetic field that gives rise to this splitting.

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10. Suppose the E vs. k relation for a band of a simple cubic solid is given by

$$E(\vec{k}) = D(3 - \cos(ak_x) - \cos(ak_y) - \cos(ak_z))$$

where a is the lattice constant.
a) What is the shape of the first Brillouin zone?
b) Plot $E(\vec{k})$ for $\vec{k}$ along the 110 direction in the first zone.
c) At what values of k in the zone do the top (maximum E) and bottom (minimum E) lie?
d) Suppose this band is the valence band of a material with 2 electrons/unit cell. Suppose the energy gap is constant and equal to D everywhere on the surface of the zone. Is the material a metal or a insulator? Why?

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11. The muon decays by the reaction, $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$. The mass of the muon is 105.6 MeV and its lifetime is $\tau_\mu = 2.2 \times 10^{-6}$ sec.

(a) In the context of the Fermi theory of weak interaction, and neglecting the electron mass argue essentially from dimensional considerations that the decay width of the muon to leading order is of the form,

$$\Gamma_\mu = \frac{\hbar}{\tau_\mu} = A G_F^2 m_\mu^5, \tag{1}$$

where, A is a constant, G_F is the Fermi coupling and m_μ is the muon mass.

(b) In the standard model the weak interactions couple universally to all the quarks and leptons so eq. (1) holds for all possible charged decays of fundamental fermions to lighter leptons and quarks. Thus all channels contribute equally to the decay width except for phase space corrections arising due to different masses. In particular for the τ lepton, assume that there are three channels open, i.e., (i) $\tau^- \to \nu_\tau + \bar{\nu}_e + e^-$ (ii) $\tau^- \to \nu_\tau + \bar{\nu}_\mu + \mu^-$ (iii) $\tau^- \to \nu_\tau + \bar{u} + d$. Neglecting the masses of the electron, muon and the up and down quarks compared to the τ mass, estimate the lifetime of the τ lepton.

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12. Naturally occuring Uranium is a mixture of 99.28% ^{238}U and 0.72% ^{235}U. It is believed that all heavy elements were produced in supernova explosions and the ejected material used to build new stars and their solar systems (if any). Thus, in the following you may assume that at the time of the creation of the our solar system both of the Uranium isotopes were present in equal amounts.

    (a) Given that the lifetime of ^{238}U is 6.5×10^9 yrs and that of ^{235}U is 1.015×10^9 yrs, how old is the material of the solar system?

    (b) What fraction of the ^{238}U has decayed since the formation of the Earth's crust 2.5×10^9 yrs ago?