Contents

1 Problem Statement 2
    1.1 Why Variance Swap? ........................................ 2
    1.2 Why signature? ........................................... 2

2 Signatures: Definitions and Models 3
    2.1 Notions ...................................................... 3
    2.2 Definitions ................................................... 3

3 Variance Swap Pricing Model 5

4 Calculation 8
    4.1 Explicit form for $E[\int_{t_k}^{t_{k+1}} \sigma^2_s ds]$ ................................. 8
    4.2 Explicit form for $E[(\int_{t_k}^{t_{k+1}} \sigma^2_s ds)^2]$ ........................................ 9
    4.3 Explicit form for $E[\int_{t_k}^{t_{k+1}} \sigma^2_s ds \int_{t_k}^{t_{k+1}} \sigma_s dB_s]$ ................... 11
        4.3.1 Explicit part B ........................................ 12
        4.3.2 Explicit part C ....................................... 13
        4.3.3 Explicit part D ....................................... 14

5 Future Work 15
1 Problem Statement

1.1 Why Variance Swap?

Variance swap is a type of financial derivative that allows investors to trade or hedge against the volatility of an underlying asset. It is a contract in which two parties agree to exchange the realized variance of the underlying asset for a predetermined fixed payment. The underlying asset of a variance swap is typically an equity index, such as the S&P 500, or a specific stock. The swap’s payoff depends on the difference between the realized variance of the asset’s returns over a specified period and a pre-agreed variance strike price. The realized variance represents the actual volatility observed in the market during the contract’s duration. It is calculated based on the squared returns of the underlying asset over a specific time frame. The variance strike price $K$ is a fixed level set at the inception of the swap, representing the expected or implied volatility of the underlying asset.

The payoff of variance swap during time period $0=t_0<t_1<...<t_N=T$ we use in this report

$$\sum_{k=0}^{N} \left[ \ln \left( \frac{S_{t_k}+1}{S_{t_k}} \right) \right]^2 - K$$

At the expiration of the swap, if the realized variance is higher than the variance strike price, the party receiving the fixed payment (typically the party who sold the swap) pays the difference to the counterparty. Conversely, if the realized variance is lower than the variance strike price, the party receiving the fixed payment receives the difference from the counterparty. Variance swaps are commonly used by investors and traders to hedge against or speculate on changes in volatility. By entering into a variance swap, market participants can effectively isolate and trade the volatility component of an asset’s price movement, separate from the asset’s direction. This allows investors to manage their exposure to market volatility independently from their exposure to the asset’s returns. Variance swaps provide a flexible and efficient means of gaining exposure to volatility, and they are often utilized by institutional investors, hedge funds, and other sophisticated market participants. For more on Variance Swap, we refer to (P. Allen, S. Einchcomb, N. Granger, 2006)\(^3\).

1.2 Why signature?

The signature method is a mathematical technique used in asset pricing to capture and analyze the dynamics of financial price processes. Here are several properties of signature-based models in asset pricing:

- **Linearity**: Financial markets often exhibit nonlinear relationships and dependencies between asset prices and other variables. The signature method allows for the capture and analysis of these nonlinear dependencies with linear signature functions, enabling a more comprehensive and accurate representation of the underlying dynamics.

- **Good extraction on higher-order information**: The signature method provides a systematic way to extract higher-order information from price processes. By considering not only the prices themselves but also the path of the price movements, the signature captures patterns, trends, and dependencies that may not be apparent at lower orders.

- **Good Model Calibration and Parameter Estimation Results**: In asset pricing, signature method is helpful to calibrate and estimate models that accurately capture the behavior of financial markets. This involves finding appropriate parameter values for the model that match the observed market data as closely as possible.
2 Signatures: Definitions and Models

2.1 Notions

This section is referenced heavily on C. Cuchiero, G. Gazzani, S. Svaluto-Ferro(2022)[1].

- (Tensor Algebra) For \( n,d \in \mathbb{N} \), the \( n \)-fold tensor product of \( \mathbb{R}^d \) is given by
  \[
  (\mathbb{R}^d)^{\otimes n} = (\mathbb{R}^d \otimes \cdots \otimes \mathbb{R}^d)
  \]
  where we construct an \( nd \)-dimensional space out of \( n \) vectors from \( d \)-dimensional space.
  
  For \( d \in \mathbb{N} \), the extended tensor algebra on \( \mathbb{R}^d \) is given by
  \[
  T(\mathbb{R}^d) = \{ a=(a_0,a_1,\ldots,a_n) : a_i \in (\mathbb{R}^d)^{\otimes i}, \ i=0,1,2,\ldots,n \}
  \]
  which could provide a structural understanding of term \textit{signature}, which should be represented as a collection of iterated integrals of a path.

- (Operations) Suppose \( a=(a_i)_{i=0}^{\infty}, \ b=(b_i)_{i=0}^{\infty} \in T(\mathbb{R}^d) \), define the sum + and product \( \otimes \) by
  \[
  a + b := (a_i+b_i)_{i=0}^{\infty}, \quad a \otimes b := (\sum_{k=0}^{\infty} a_k \otimes b_{i-k})_{i=0}^{\infty}
  \]

- (Multi-index) Suppose a multi-index \( I := (i_1,i_2,\ldots,i_n) \), then we set \(|I| := n\). Remark that we also set \( I' := (i_1,\ldots,i_{n-1}) \) and \( I'' := (i_1,\ldots,i_{n-2}) \) whenever needed.
  
  We also have the notation \( \{ I : |I| = n \} := \{ 1,2,\ldots,d \}^n \).
  
  Now we combine the multi-index with the tensor basis:
  \[
  e_I = e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n}
  \]
  where \( e_{i_1},e_{i_2},\ldots,e_{i_n} \) denotes the canonical basis vectors of \( \mathbb{R}^d \). Denoting that \( e_0 \) is the basis element corresponding to \((\mathbb{R}^d)^{\otimes 0}\).

- Suppose we have \( e_I : |I| = N \) to be an orthonormal basis of \((\mathbb{R}^d)^{\otimes N}\). Then for any \( a \in T((\mathbb{R}^d)) \), it can be written as
  \[
  a = \sum_{|I| \geq 0} a_I e_I
  \]
  where \( a_I \in \mathbb{R} \) is a coefficient for a vector basis on multi-index \( I \). Equivalently,
  \[
  a_I = \langle e_I, a \rangle
  \]

2.2 Definitions

**Definition 2.1** (C. Cuchiero, G. Gazzani, S. Svaluto-Ferro(2022)[1]). Let \((X_t)_{t \in [0,T]}\) be a continuous \( \mathbb{R}^d \)-valued semimartingale. The \textit{signature} of \( X \) is the \( T((\mathbb{R}^d)) \)-valued process \((s,t) \mapsto X_{s,t})\) whose components are recursively defined as
  \[
  \langle e_0, X_{s,t} \rangle = 1
  \]
  \[
  \langle e_I, X_{s,t} \rangle = \int_s^t \langle e_{I'}, X_{s,r} \rangle \circ dX_r^{i_n}
  \]
  for each \( I = (i_1,i_2,\ldots,i_n) \), \( I' = (i_1,i_2,\ldots,I_{n-1}) \) and \( 0 \leq s \leq t \leq T \), where \( \circ \) denotes the Stratonovich integral. Its projection \( X^N \) on \( T^N((\mathbb{R}^d)) \) is given by
  \[
  X^N_{s,t} = \sum_{|I| \leq N} \langle e_I, X_{s,t} \rangle e_I
  \]
  and is called \textit{signature of \( X \) truncated at level \( N \)}.

The equivalent notation:
  \[
  X_t = (1, \int_0^t 1 \circ dX^1_s, \ldots, \int_0^t 1 \circ dX^d_s, \int_0^t (\int_0^t 1 \circ dX^1_s) \circ dX^1_s, \ldots, \int_0^t (\int_0^t 1 \circ dX^d_s) \circ dX^d_s, \ldots )
  \]

**Definition 2.2** (C. Cuchiero, G. Gazzani, S. Svaluto-Ferro(2022)[1]). (Shuffle Product) Given two multi-
indices $I=(i_1,i_2,\ldots,i_n)$ and $J=(j_1,j_2,\ldots,j_m)$, the **shuffle product** is defined recursively as
\[
    e_I \shuffle e_J = (e_I \shuffle e_J) \otimes e_i + (e_I \shuffle e_J') \otimes e_j,
\]
where $e_I \shuffle e_J = e_I = e_J$.

**Proposition 2.1** (Shuffle Property). [C. Cuchiero, G. Gazzani, S. Svaluto-Ferro(2022) [1]] Let $((X_t))_{t \in [0,T]}$ be a continuous $\mathbb{R}^d$-valued semimartingale and $I$, $J$ be two multi-indices. Then
\[
    \langle e_I, X_{s,t} \rangle \langle e_J, X_{s,t} \rangle = \langle e_I \shuffle e_J, X_{s,t} \rangle
\]

**Proof.** We prove this proposition by induction with Stratonovich integrals.

**Induction Basis.** $\langle e_\emptyset, X_{s,t} \rangle \langle e_J, X_{s,t} \rangle = 1 \langle e_J, X_s \rangle = \langle e_\emptyset \shuffle e_J, X_{s,t} \rangle$, which agrees with the proposition.

**Induction Steps.** Assumed that for arbitrary subsets $S_1$ and $S_2$ of multi indices $I$ and $J$, we have
\[
    \langle e_{S_1}, X_{s,t} \rangle \langle e_{S_2}, X_{s,t} \rangle = \langle e_{S_1 \shuffle e_{S_2}}, X_{s,t} \rangle
\]
After that,
\[
    \langle e_I, X_{s,t} \rangle \langle e_J, X_{s,t} \rangle = \int_s^t \langle e_I, X_{s,u} \rangle \odot d\langle e_J, X_{s,u} \rangle + \int_s^t \langle e_I, X_{s,r} \rangle \odot d\langle e_J, X_{s,r} \rangle
\]
\[
    = \int_s^t \langle e_I, X_{s,r} \rangle \langle e_J, X_{s,r} \rangle \odot dX_{r}^I + \int_s^t \langle e_I, X_{s,r} \rangle \langle e_J, X_{s,r} \rangle \odot dX_{r}^J
\]
\[
    = \langle e_I \shuffle e_J, X_{s,t} \rangle + \langle e_I \shuffle e_J', X_{s,t} \rangle
\]
\[
    = \langle e_I \shuffle e_J, X_{s,t} \rangle
\]

**Proposition 2.2** (Chen’s identity). [C. Cuchiero, G. Gazzani, S. Svaluto-Ferro(2022) [1]] Let $((X_t))_{t \in [0,T]}$ be an $\mathbb{R}^d$-valued semimartingale. Then
\[
    X_{s,t} = X_{s,u} \otimes X_{u,t}
\]
for each $s \leq u \leq t \leq T$. This can equivalently be written as
\[
    \langle e_I, X_{s,t} \rangle = \sum_{e_{I_1} \otimes e_{I_2} = e_I} \langle e_{I_1}, X_{s,u} \rangle \langle e_{I_2}, X_{u,t} \rangle
\]
where $I$ represents an arbitrary multi-index.

**Proof.** We prove this by induction on the length of the multi-index $I$.

**Induction Basis.** When $|I| = 0$, obviously the statement holds.

**Induction Steps.** Assumed that the statement holds for all multi-indices $J \subseteq I$, where $|J| < n$ and $I=(i_1,i_2,\ldots,i_n)$. Then,
\[
    \langle e_I, X_{s,t} \rangle = \int_s^t \langle e_I', X_{s,r} \rangle \odot dX_{r}^I
\]
\[
    = \sum_{u} \langle e_I', X_{s,r} \rangle \odot dX_{r}^I + \int_u^t \langle e_I', X_{s,r} \rangle \odot dX_{r}^I
\]
\[
    = \langle e_I, X_{s,u} \rangle + \sum_{u} \langle e_I', X_{s,u} \rangle \langle e_{I_2}, X_{u,t} \rangle \odot dX_{r}^I
\]
\[
    = \langle e_I, X_{s,u} \rangle + \sum_{e_{I_1} \otimes e_{I_2} = e_I} \langle e_{I_1}, X_{s,u} \rangle \langle e_{I_2}, X_{u,t} \rangle
\]
Note: paying attention to the difference between shuffle $\shuffle$ and tensor product $\otimes$.

**Definition 2.3** (Signature Model). [C. Cuchiero, G. Gazzani, S. Svaluto-Ferro(2022) [1]] A **signature model**
is a stochastic process of the form

$$S_n(l)_t = \ell_0 + \sum_{0 \leq |I| \leq n} I_I \langle e_I, \hat{X}_t \rangle$$

where \(n \in \mathbb{N}\) and \(l = \{\ell_0, l_I : 0 < |I| \leq n\} \).

### 3 Variance Swap Pricing Model

We consider asset price models whose dynamics are described by linear functions of the (time extended) signature of a primary underlying process, which is Brownian motion in our case. Throughout the paper we fix a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})\) on which we define the stochastic process

$$dS_t = \sigma_t S_t dB_t, \quad S_0 \in \mathbb{R}_+$$

where \(S_t\) is the pricing process, \(\sigma_t\) is the volatility process that will be represented by signature model. \(B_t\) here is a Brownian motion which is correlated with \(\sigma_t\). Now we parameterize the volatility process \(\sigma_t\) as a signature model (Sig-SDE) of a primary process \(X\):

$$\sigma_t(l) = \ell_0 + \sum_{|I| \leq n} \ell_I \langle e_I, \hat{X}_t \rangle$$

Note that \(\hat{X}\) is the signature of time extended Brownian motion \(\hat{X} = (t, X_t)\).

Since every polynomial function in the signature has a linear representation, suppose the \(d\)-dimensional real-valued process \(X\) is a weak solution of

$$dX_t = b(\hat{X}_t)dt + \sqrt{a(\hat{X}_t)}dW_t$$

where \(W\) is a \(d\)-dimensional Brownian motion and \(a\) and \(b\) are linear maps. Thus we can define \(X\) as a polynomial process:

$$dX_t = b(X_t)dt + \sqrt{a(X_t)}dW_t, \quad X_0 = x_0$$

where maps \(a : \mathbb{R}^d \rightarrow \mathbb{R}^d_+\) and \(b : \mathbb{R}^d \rightarrow \mathbb{R}^d\) such that each \(a_{ij}\) is a polynomial of degree at most 2 and \(b_j\) is a polynomial of degree at most 1 for each \(i, j = 1, \ldots, d + 1\). Furthermore, to define the correlation between processes \(X\) and \(B\), we introduce process \(Z = (X_t, B_t)\) and denote its time extension by \(\hat{Z} = (t, X, B)\) with its signature \((\hat{Z}_t)_{t \geq 0}\). Thus, the correlation matrix process between \(X\) and \(B\) can be defined as

$$\rho_{ij} = \frac{[Z^i, Z^j]}{\sqrt{[Z^i, Z^i]} \sqrt{[Z^j, Z^j]}} \in [-1, 1]$$

for all \(i, j = 1, \ldots, d + 1\), where \([ , \) denotes the quadratic variation. In this case, the volatility process \(\sigma_t\) can be defined equivalently as a signature model (Sig-SDE) of the primary process \(Z\):

$$\sigma_t(l) = \ell_0 + \sum_{|I| \leq n, c_i \neq \phi, d + 1} \ell_I \langle e_I, \hat{Z}_t \rangle$$

**Lemma 3.1** \([\text{2}^+\text{]}\). Let \((X_t)_{t \geq 0}\) be the polynomial process given by (4) and \(b, a\) be the corresponding drift and diffusion coefficients. Then

$$b_j(x) = b^c_j + \sum_{k=1}^{d} b^k_j x_k$$

$$a_{ij}(x) = a^c_{ij} + \sum_{k=1}^{d} a^k_{ij} x_k + \sum_{k,h=1}^{d} a^{kh}_{ij} x_k x_h$$

for \(b^c_j, b^k_j, a^c_{ij}, a^k_{ij}, a^{kh}_{ij} \in \mathbb{R}\)

Moreover we have,

$$b_j(Y_t) = \langle b_j, Y^1_t \rangle$$

and

$$a_{ij}(Y_t) = \langle a_{ij}, Y^2_t \rangle$$

where

$$b_j = b^c_j e_0 + \sum_{k=1}^{d} (b^k_j X^k_0 e_0 + b^k_j e_k) = (b^c_j + \sum_{k=1}^{d} b^k_j X^k_0) e_0 + \sum_{k=1}^{d} b^k_j e_k$$
where

Then we fix a labelling injective function

such that

the

d

process given by (4).

Theorem 3.1 (C. Cuchiero, G. Gazzani, J.Moller, S. S. Ferro (2022) [2].) Let \((X_t)_{t \geq 0}\) be the polynomial process given by (4) and take \(b_0\) from lemma 1. The truncated signature \((X^n_t)_{t \geq 0}\) is a polynomial process and for each \(|I| \leq n\) it holds that

\[
\langle e_I, X^n_t \rangle = \int_0^t \langle L e_I, X^n_s \rangle ds + \int_0^t \langle e_I, X^n_s \rangle \sigma_{|I|}(X_s) dW_s
\]

where the operator \(L: T(\mathbb{R}^d) \rightarrow T(\mathbb{R}^d)\) satisfies \(L(T^{(n)}(\mathbb{R}^d)) \subseteq T^{(n)}(\mathbb{R}^d)\) and is given by

\[
L e_I = e_I \| b_{|I|} + \frac{1}{2} \sigma_{|I|} \| a_{|I|} e_h \|
\]

Definition 3.1. [C. Cuchiero, G. Gazzani, J.Moller, S. S. Ferro (2022) [2].] For each \(|I| \leq n\) set then \(\eta_{IJ} \in \mathbb{R}\) such that

\[
L e_I = \sum_{|J| \leq n} \eta_{IJ} e_J
\]

Then we fix a labelling injective function \(L: \{I: |I| \leq n\} \rightarrow \{1,2,...,d_n\}\). We then call the matrix \(G \in \mathbb{R}^{d_n \times d_n}\)

the \(d_n\) dimensional matrix representative of \(L\).

Theorem 3.1 (C. Cuchiero, G. Gazzani, J.Moller, S. S. Ferro (2022) [2].) Let \((X_t)_{t \geq 0}\) be the polynomial process given by (4). \(((F_t))_{t \geq 0}\) be the filtration generated by \((Y_t)_{t \geq 0}\) and let \(G\) be the \(d_n\)-dimensional matrix representative of the dual operator corresponding to \(X\). Then for each \(T, t \geq 0, |I| \leq n\) we have

\[
\mathbb{E}[\text{vec}(X^n_{T+1}) | \mathcal{F}_T] = e^{G^T} \text{vec}(X^n_T)
\]

To price discretely sampled variance swaps, we first define pricing formula for vanilla variance swaps. Let the sampling dates be defined as \(0 = t_0 < t_1 < ... < t_N = T\), where \(T\) is the maturity date of the option. The payoff of vanilla variance swap at \(T\) is defined as

\[
\sum_{k=0}^N [\ln(S_{t_{k+1}}/S_{t_k})]^2 - K
\]

where \(K\) is the strike price of the variance swap. To make the market free of arbitrage, the fair strike price \(K\) under risk-free measure \(Q\) is given by

\[
K = \mathbb{E}_Q^Q[\sum_{k=1}^N [\ln(S_{t_{k+1}}/S_{t_k})]^2] | \mathcal{F}_0
\]

From our definition of \(S_t\) above, we get

\[
S_{t_{k+1}} = S_{t_k} e^{-\int_{t_k}^{t_{k+1}} \sigma^2_s^2 ds + \int_{t_k}^{t_{k+1}} \sigma_s dB_s}
\]

Then

\[
K = \mathbb{E} \sum_{k=0}^N \left(- \int_{t_k}^{t_{k+1}} \frac{\sigma^2_s^2}{2} ds + \int_{t_k}^{t_{k+1}} \sigma_s dB_s \right)^2 | \mathcal{F}_0
\]

\[
= \sum_{k=0}^N \mathbb{E} \left[ \frac{1}{2} \int_{t_k}^{t_{k+1}} \sigma^2_s ds \right]^2 + \sum_{k=0}^N \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \sigma_s dB_s \right]^2 - \sum_{k=0}^N \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \sigma^2_s ds \int_{t_k}^{t_{k+1}} \sigma_s dB_s \right]
\]

\[
= \sum_{k=0}^N \left\{ \frac{1}{4} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \sigma^2_s ds \right]^2 - \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \sigma^2_s ds \int_{t_k}^{t_{k+1}} \sigma_s dB_s \right] \right\} + \mathbb{E} \left[ \int_0^T \sigma^2_s ds \right]
\]

\[
= \sum_{k=0}^N \left\{ \frac{1}{4} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \sigma^2_s ds \right]^2 - \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \sigma^2_s ds \int_{t_k}^{t_{k+1}} \sigma_s dB_s \right] + \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \sigma^2_s ds \right] \right\}
\]
Our goal is to reformulate this explicitly with signature models, which is, applying signature model to describe the dynamics of $S_t$. With definition 2.3, we write $\sigma_t$ as
\[
\sigma_t(\ell) = \ell_0 + \sum_{|I| \leq n} \ell_I \langle e_I, \tilde{\mathbb{W}}_t \rangle
\]
where $W$ is the underlying $d$-dimensional Brownian motion. With the same motivation,
\[
(\sigma_t(\ell))^2 = \ell_0 + \sum_{|I|, |J| \leq n} \ell_I \ell_J \langle e_I \sqcup e_J, \tilde{\mathbb{W}}_t \rangle
\]
4 Calculation

In this section we would layout the explicit formula for each of the three terms separately. Recall our target formula:

$$K = \sum_{k=0}^{N-1} \frac{1}{4} \mathbb{E} [\int_{t_k}^{t_{k+1}} \sigma_s^2 ds] - \mathbb{E} [\int_{t_k}^{t_{k+1}} \sigma_s^2 ds \int_{t_k}^{t_{k+1}} \sigma_s dB_s] + \mathbb{E} [\int_{t_k}^{t_{k+1}} \sigma_s^2 ds]$$

4.1 Explicit form for $\mathbb{E} [\int_{t_k}^{t_{k+1}} \sigma_s^2 ds]$

$$\mathbb{E} [\int_{t_k}^{t_{k+1}} \sigma_s^2 ds] = \mathbb{E} \left[ \sum_{|I|,|J| \leq n} \ell_I \ell_J < e_I \otimes e_J, \widehat{\mathbb{W}}_s > ds \right]$$

$$= \sum_{|I|,|J| \leq n} \ell_I \ell_J \mathbb{E} \left[ \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} < e_I \otimes e_J, \widehat{\mathbb{W}}_s > ds | \mathcal{F}_{t_k} \right] | \mathcal{F}_0 \right]$$

$$= \ell^T Q^{(1)}_{\mathcal{L}(t_0, t_{k+1}-t_k) \mathcal{L}(J)} = \mathbb{E} \left[ \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} < e_I \otimes e_J, \widehat{\mathbb{W}}_s > ds | \mathcal{F}_{t_k} \right] | \mathcal{F}_0 \right]$$

where we apply definition 3.1 and theorem 8, then

$$Q^{(1)}_{\mathcal{L}(t_0, t_{k+1}-t_k)} = \mathbb{E} \left[ \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} < e_I \otimes e_J, \widehat{\mathbb{W}}_s > ds | \mathcal{F}_{t_k} \right] | \mathcal{F}_0 \right]$$

$$= \mathbb{E} \left[ \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} < e_I \otimes e_J, \widehat{\mathbb{W}}_s > ds - \int_{0}^{t_k} < e_I \otimes e_J, \widehat{\mathbb{W}}_s > ds | \mathcal{F}_{t_k} \right] | \mathcal{F}_0 \right]$$

$$= \mathbb{E} \left[ \mathbb{E} \left[ \left( \int_{0}^{t_{k+1}} < e_I \otimes e_J, \widehat{\mathbb{W}}_s > ds - \int_{0}^{t_k} < e_I \otimes e_J, \widehat{\mathbb{W}}_s > ds | \mathcal{F}_{t_k} \right) | \mathcal{F}_0 \right] \right]$$

$$= \mathbb{E} \left[ \sum_{e_K = (e_I \otimes e_J) \otimes e_0} \sum_{|H| \leq 2n+1} \left( e^{\ell_{k+1}}_{(k+1)-t_k} G^T \right) \mathcal{L}(K) \mathcal{L}(H) \langle e_H, \widehat{\mathbb{W}}_t \rangle | \mathcal{F}_0 \right]$$

$$- \sum_{e_K = (e_I \otimes e_J) \otimes e_0} \sum_{|H| \leq 2n+1} \left( e^{\ell_k} G^T \right) \mathcal{L}(K) \mathcal{L}(H) \langle e_H, \widehat{\mathbb{W}}_0 \rangle$$

$$= \mathbb{E} \left[ \sum_{e_K = (e_I \otimes e_J) \otimes e_0} \sum_{|H| \leq 2n+1} \left( e^{\ell_{k+1}}_{(k+1)-t_k} G^T \right) \mathcal{L}(K) \mathcal{L}(H) \langle e_H, \widehat{\mathbb{W}}_t \rangle | \mathcal{F}_0 \right]$$

$$- \sum_{e_K = (e_I \otimes e_J) \otimes e_0} \sum_{|H| \leq 2n+1} \left( e^{\ell_k} G^T \right) \mathcal{L}(K) \mathcal{L}(H) \langle e_H, \widehat{\mathbb{W}}_0 \rangle$$

$$= \mathbb{E} \left[ \sum_{e_K = (e_I \otimes e_J) \otimes e_0} \sum_{|H| \leq 2n+1} \left( e^{\ell_{k+1}}_{(k+1)-t_k} G^T \right) \mathcal{L}(K) \mathcal{L}(H) \langle e_H, \widehat{\mathbb{W}}_t \rangle | \mathcal{F}_0 \right]$$

$$- \sum_{e_K = (e_I \otimes e_J) \otimes e_0} \sum_{|H| \leq 2n+1} \left( e^{\ell_k} G^T \right) \mathcal{L}(K) \mathcal{L}(H) \langle e_H, \widehat{\mathbb{W}}_0 \rangle$$
4.2 Explicit form for $E[(\int_{t_k}^{t_{k+1}} \sigma^2_s ds)^2]$ 

$$
E[(\int_{t_k}^{t_{k+1}} \sigma^2_s ds)^2] = E[(\int_0^{t_{k+1}} \sigma^2_s ds - \int_0^{t_k} \sigma^2_s ds)^2 | \mathcal{F}_0] 
$$

$$
= E[(\int_0^{t_{k+1}} \sigma^2_s ds)^2 + (\int_0^{t_k} \sigma^2_s ds)^2 - 2(\int_0^{t_{k+1}} \sigma^2_s ds \int_0^{t_k} \sigma^2_s ds)| \mathcal{F}_0] 
$$

$$
= E[\left( \sum_{|I|,|J| \leq n} \ell_I \ell_J < e_I \otimes e_J \otimes e_0, \widehat{\mathcal{W}}_{t_{k+1}} > \right)^2 + \left( \sum_{|I|,|J| \leq n} \ell_I \ell_J < e_I \otimes e_J \otimes e_0, \widehat{\mathcal{W}}_{t_k} > \right)^2 
- 2(\int_0^{t_{k+1}} \sigma^2_s ds \int_0^{t_k} \sigma^2_s ds)| \mathcal{F}_0] 
$$

$$
= E[\left( \sum_{|I|,|J|,|M|,|N| \leq n} \ell_I \ell_J \ell_M \ell_N < e_I \otimes e_J \otimes e_0 \otimes e_0, \widehat{\mathcal{W}}_{t_{k+1}} > \right)| \mathcal{F}_0] 
+ \sum_{|I|,|J|,|M|,|N| \leq n} \ell_I \ell_J \ell_M \ell_N < e_I \otimes e_J \otimes e_0 \otimes e_0, \widehat{\mathcal{W}}_{t_k} > 
- 2(\sum_{|I|,|J|,|M|,|N| \leq n} \ell_I \ell_J \ell_M \ell_N < e_I \otimes e_J \otimes e_0 \otimes e_0, \widehat{\mathcal{W}}_{t_k} > 
+ \int_0^{t_{k+1}} \sigma^2_s ds \int_0^{t_k} \sigma^2_s ds)| \mathcal{F}_0] 
$$

$$
= \sum_{|I|,|J|,|M|,|N| \leq n} \ell_I \ell_J \ell_M \ell_N Q_{(L)(N)}^{(3)} (0,t_{k+1}) - \sum_{|I|,|J|,|M|,|N| \leq n} \ell_I \ell_J \ell_M \ell_N Q_{(L)(N)}^{(3)} (0,t_k) - 2A 
$$

where

$$
Q_{(L)(N)}^{(3)} (0,t_{k+1}) = E[\left( \sum_{e_K = (e_I \otimes e_J \otimes e_0) \otimes (e_M \otimes e_N \otimes e_0)} < e_K, \widehat{\mathcal{W}}_{t_{k+1}} > \right)| \mathcal{F}_0] 
$$

$$
= \sum_{e_K = (e_I \otimes e_J \otimes e_0) \otimes (e_M \otimes e_N \otimes e_0)} \sum_{|H| \leq 4n+2} (e_{k+1}^{G^{+}})_{L(H)} e_K |H| \leq 4n+2 
$$

and

$$
Q_{(L)(N)}^{(3)} (0,t_k) = E[\left( \sum_{e_K = (e_I \otimes e_J \otimes e_0) \otimes (e_M \otimes e_N \otimes e_0)} < e_K, \widehat{\mathcal{W}}_{t_k} > \right)| \mathcal{F}_0] 
$$

$$
= \sum_{e_K = (e_I \otimes e_J \otimes e_0) \otimes (e_M \otimes e_N \otimes e_0)} \sum_{|H| \leq 4n+2} (e_{k}^{G^{+}})_{L(H)} e_K |H| \leq 4n+2 
$$
Now we formulate part $A$:

$$A = E[\int_{t_k}^{t_{k+1}} \sigma_s^2 ds \int_0^{t_k} \sigma_s^2 ds | \mathcal{F}_0]$$

$$= E[\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s^2 ds \int_0^{t_k} \sigma_s^2 ds | \mathcal{F}_{t_k}] | \mathcal{F}_0]$$

$$= E[\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s^2 ds | \mathcal{F}_{t_k}] \mathbb{E}[\int_0^{t_k} \sigma_s^2 ds | \mathcal{F}_{t_k}] | \mathcal{F}_0]$$

$$= E[\mathbb{E}[(\int_0^{t_{k+1}} \sigma_s^2 ds - \int_0^{t_k} \sigma_s^2 ds) | \mathcal{F}_{t_k}] | \mathcal{F}_0]$$

$$= E[\mathbb{E}[\int_0^{t_{k+1}} \sigma_s^2 ds - \int_0^{t_k} \sigma_s^2 ds | \mathcal{F}_{t_k}] | \mathcal{F}_0]$$

$$= E[\int_0^{t_{k+1}} \sum_{|I|, |J| \leq n} \ell_{I,J} < e_I \cup e_J, \widetilde{\mathcal{W}}_s > ds$$

$$- \int_0^{t_k} \sum_{|I|, |J| \leq n} \ell_{I,J} < e_I \cup e_J, \widetilde{\mathcal{W}}_s > ds | \mathcal{F}_{t_k}] | \mathcal{F}_0]$$

$$= E[\sum_{|I|, |J| \leq n} \ell_{I,J} \mathbb{E}[(< (e_I \cup e_J) \otimes e_0, \widetilde{\mathcal{W}}_{t_{k+1}} > - < (e_I \cup e_J) \otimes e_0, \widetilde{\mathcal{W}}_{t_k} > ) | \mathcal{F}_{t_k}] | \mathcal{F}_0]$$

$$= E[\sum_{|I|, |J| \leq n} \ell_{I,J} \mathbb{E}[< (e_I \cup e_J) \otimes e_0, \widetilde{\mathcal{W}}_{t_{k+1}} > - < (e_I \cup e_J) \otimes e_0, \widetilde{\mathcal{W}}_{t_k} > ] | \mathcal{F}_0]$$

$$= E[\sum_{|I|, |J| \leq n} \ell_{I,J} \mathbb{E}[< (e_I \cup e_J) \otimes e_0, \widetilde{\mathcal{W}}_{t_{k+1}} > | \mathcal{F}_{t_k}] | \mathcal{F}_0]$$

$$- E[\sum_{|I|, |J| \leq n} \ell_{I,J} < e_I \cup e_J, \widetilde{\mathcal{W}}_s > | \mathcal{F}_{t_k}]$$

$$= \sum_{|I|, |J|, |M|, |N| \leq n} \ell_{I,J,M,N} \mathbb{E}[(< (e_I \cup e_J) \otimes e_0, \widetilde{\mathcal{W}}_{t_{k+1}} > | \mathcal{F}_{t_k}) < (e_M \cup \varepsilon e_N \otimes e_0) \varepsilon \tilde{\mathcal{W}}_{t_k} > | \mathcal{F}_0]$$

$$- \sum_{|I|, |J|, |M|, |N| \leq n} \ell_{I,J,M,N} \mathbb{E}[< (e_I \cup e_J) \otimes e_0, \varepsilon (e_M \cup \varepsilon e_N \otimes e_0), \tilde{\mathcal{W}}_{t_k} > | \mathcal{F}_0]$$

$$= \sum_{|I|, |J|, |M|, |N| \leq n} \ell_{I,J,M,N} Q_{\mathcal{L}(I)\mathcal{L}(J)\mathcal{L}(M)\mathcal{L}(N)}^{(2)}(0, t_{k+1} - t_k)$$

$$+ \sum_{|I|, |J|, |M|, |N| \leq n} \ell_{I,J,M,N} Q_{\mathcal{L}(I)\mathcal{L}(J)\mathcal{L}(M)\mathcal{L}(N)}^{(3)}(0, t_{k+1} - t_k)$$

where

$$Q_{\mathcal{L}(I)\mathcal{L}(J)\mathcal{L}(M)\mathcal{L}(N)}^{(2)}(t_0, t_{k+1} - t_k) = E[\mathbb{E}[(e_I \cup e_J) \otimes e_0, \varepsilon \widetilde{\mathcal{W}}_{t_{k+1}}) | \mathcal{F}_{t_k}] | ((e_M \cup \varepsilon e_N) \otimes e_0, \varepsilon \widetilde{\mathcal{W}}_{t_k}) | \mathcal{F}_0]$$

$$= E[\sum_{e_K = (e_I \cup e_J) \otimes e_0} \sum_{|H| \leq 2n+1} (e_i^k \varepsilon e_K) \mathcal{L}(K) \mathcal{L}(H) \mathcal{L}(e_H, \varepsilon \widetilde{\mathcal{W}}_{t_k}) \mathcal{L}(e_M \cup \varepsilon e_N) \otimes e_0, \varepsilon \widetilde{\mathcal{W}}_{t_k}) | \mathcal{F}_0]$$

$$= E[\sum_{e_K = (e_I \cup e_J) \otimes e_0} \sum_{|H| \leq 2n+1} (e_i^k \varepsilon e_K) \mathcal{L}(K) \mathcal{L}(H) \mathcal{E}[e_H \varepsilon ((e_M \cup \varepsilon e_N) \otimes e_0), \varepsilon \widetilde{\mathcal{W}}_{t_k}) | \mathcal{F}_0]$$

$$= \sum_{e_K = (e_I \cup e_J) \otimes e_0} \sum_{|H| \leq 2n+1} (e_i^k \varepsilon e_K) \mathcal{L}(K) \mathcal{L}(H) \sum_{e_A = e_H \varepsilon ((e_M \cup \varepsilon e_N) \otimes e_0)} \sum_{|B| \leq 3n+1} (e_i^k \varepsilon e_K) \mathcal{L}(A) \mathcal{L}(B) \mathcal{L}(e_B, \varepsilon \widetilde{\mathcal{W}}_{t_k})$$

and

$$Q_{\mathcal{L}(I)\mathcal{L}(J)\mathcal{L}(M)\mathcal{L}(N)}^{(3)}(t_0, t_k) = E[< (e_I \cup e_J) \otimes e_0, \varepsilon (e_M \cup \varepsilon e_N \otimes e_0), \widetilde{\mathcal{W}}_{t_k} > | \mathcal{F}_0]$$

$$= \sum_{e_K = (e_I \cup e_J) \otimes e_0} \sum_{|H| \leq 4n+2} (e_i^k \varepsilon e_K) \mathcal{L}(K) \mathcal{L}(H) \mathcal{L}(e_H, \varepsilon \widetilde{\mathcal{W}}_{t_k})$$
4.3 Explicit form for $\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s^2 ds \int_{t_k}^{t_{k+1}} \sigma_s dB_s]$

Before we dive into the actual computation of the cross term, we introduce the special case when we take the extra correlated Brownian process $B$ into the picture.

If we take $W$ as our multidimensional brownian process and the underlying primary process under the signature model of $\sigma = \sum_{|l| \leq n} \ell_I(e_I, \mathcal{W})$, here we need to define the process $Z = (W, B)$ and the time-extended version $\hat{Z} = (t, W, B)$ as well as its signature $\hat{\mathcal{Z}}$.

**Assumption 4.1.** C. Cuchiero, G. Gazzani, S. Svaluto-Ferro (2022) For all $i \in 1, 2, \ldots, d$, we have

$$d[W^i, Z^{d+1}]_t = \sum_{|J| \leq m} a^J_{i(d+1)} (e_J, \hat{Z}_t) dt$$

for some $m \in \mathbb{N}$ where $\hat{Z} = (t, W, B)$.

Now we compute the special case correlated with $B$.

Now we compute the explicit formula of the cross term.

$$\int_0^T \sigma_s dZ^{d+1}_s = \int_0^T \sigma_s \circ dZ^{d+1}_s - \frac{1}{2} \mathbb{E} [\sigma_0, Z^{d+1}_T]$$

$$= \int_0^T \sum_{|J| \leq n} \ell_I(e_I, \hat{Z}_s) \circ dZ^{d+1}_s - \frac{1}{2} \mathbb{E} [\ell_I(e_I, \hat{Z}), Z^{d+1}_T]$$

$$= \sum_{|J| \leq n} \ell_I(e_I \otimes e_{d+1}, \hat{Z}_T) - \frac{1}{2} \mathbb{E} [\ell_I(e_I \otimes e_{d+1}, \hat{Z}_s) d[Z^{d+1}, Z^{d+1}]_s]$$

$$= \sum_{|J| \leq n} \ell_I(e_I \otimes e_{d+1}, \hat{Z}_T) - \frac{1}{2} \mathbb{E} [\sum_{|J| \leq m} a^J_{i(d+1)} e_{J \sqcup i} \otimes e_0, \hat{Z}_T]$$

$$= \sum_{|J| \leq n} \ell_I e_{d+1}, \hat{Z}_T$$

where $\hat{e}^{d+1} = e_I \otimes e_{d+1} - \frac{1}{2} \sum_{|J| \leq n} a^J_{i(d+1)} e_{J \sqcup i} \otimes e_0$.

(Note that the highest order of $\hat{e}^{d+1}$ is $n+m$)

From (1) to (2) with assumption (1):

$$\int_0^T \langle e_I', \hat{Z}_s \rangle d[Z^{d+1}, Z^{d+1}]_s = \int_0^T \langle e_I', \hat{Z}_s \rangle \sum_{|J| \leq m} a^J_{i(d+1)} (e_J, \hat{Z}_s) ds$$

$$= \int_0^T \sum_{|J| \leq m} a^J_{i(d+1)} (e_I', \hat{Z}_s) (e_J, \hat{Z}_s) ds$$

$$= \sum_{|J| \leq m} a^J_{i(d+1)} (e_I' \sqcup e_J \otimes e_0, \hat{Z}_T)$$

Now we compute the explicit formula of the cross term.

$$\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s^2 ds \int_{t_k}^{t_{k+1}} \sigma_s dB_s]$$
\[ = \mathbb{E}\left[ \int_{t_k}^{t_{k+1}} \sigma_s^2 ds \int_{t_k}^{t_{k+1}} \sigma_s dZ_s^{d+1} | \mathcal{F}_0 \right] \]

\[ = \mathbb{E}\left[ \left( \int_{0}^{t_{k+1}} \sigma_s^2 ds - \int_{0}^{t_k} \sigma_s^2 ds \right) \left( \int_{0}^{t_{k+1}} \sigma_s dZ_s^{d+1} s - \int_{0}^{t_k} \sigma_s dZ_s^{d+1} s \right) | \mathcal{F}_0 \right] \]

\[ = \mathbb{E}\left[ \int_{0}^{t_{k+1}} \sigma_s^2 ds \int_{0}^{t_{k+1}} \sigma_s dZ_s^{d+1} s + \int_{0}^{t_k} \sigma_s^2 ds \int_{0}^{t_k} \sigma_s dZ_s^{d+1} s - \int_{0}^{t_{k+1}} \sigma_s^2 ds \int_{0}^{t_{k+1}} \sigma_s dZ_s^{d+1} s - \int_{0}^{t_k} \sigma_s^2 ds \int_{0}^{t_k} \sigma_s dZ_s^{d+1} s \right] \]

\[ = \mathbb{E}\left[ \int_{0}^{t_{k+1}} \sigma_s^2 ds \int_{0}^{t_{k+1}} \sigma_s dZ_s^{d+1} s - \int_{0}^{t_k} \sigma_s^2 ds \int_{0}^{t_k} \sigma_s dZ_s^{d+1} s \right] \]

\[ = \mathbb{E}\left[ \int_{0}^{t_{k+1}} \sigma_s^2 ds \int_{0}^{t_{k+1}} \sigma_s dZ_s^{d+1} s | \mathcal{F}_0 \right] - \mathbb{E}\left[ \int_{0}^{t_k} \sigma_s^2 ds \int_{0}^{t_k} \sigma_s dZ_s^{d+1} s | \mathcal{F}_0 \right] \]

4.3.1 Explicit part B

\[ B = \mathbb{E}\left[ \int_{0}^{t_{k+1}} \sigma_s^2 ds \int_{0}^{t_{k+1}} \sigma_s dZ_s^{d+1} s | \mathcal{F}_0 \right] - \mathbb{E}\left[ \int_{0}^{t_k} \sigma_s^2 ds \int_{0}^{t_k} \sigma_s dZ_s^{d+1} s | \mathcal{F}_0 \right] \]

\[ = \mathbb{E}\left[ \sum_{|I|, |J| \leq n} \ell_I \ell_J \langle e_I \cup e_J \otimes e_0, \widehat{Z}_{t_{k+1}} \rangle \sum_{|M| \leq n} \ell_M \langle e_M^{d+1}, \widehat{Z}_{t_{k+1}} \rangle | \mathcal{F}_0 \right] - \mathbb{E}\left[ \sum_{|I|, |J| \leq n} \ell_I \ell_J \langle e_I \cup e_J \otimes e_0, \widehat{Z}_{t_{k+1}} \rangle | \mathcal{F}_0 \right] \]

\[ = \mathbb{E}\left[ \sum_{|I|, |J| \leq n} \ell_I \ell_J \langle e_I \cup e_J \otimes e_0, \widehat{Z}_{t_{k+1}} \rangle | \mathcal{F}_0 \right] - \mathbb{E}\left[ \sum_{|I|, |J| \leq n} \ell_I \ell_J \langle e_I \cup e_J \otimes e_0, \widehat{Z}_{t_{k+1}} \rangle | \mathcal{F}_0 \right] \]

\[ = \sum_{|I|, |J| \leq n} \ell_I \ell_J Q^{(4)}(0, t_{k+1})_{L(I), L(J), L(M)} - \sum_{|I|, |J| \leq n} \ell_I \ell_J Q^{(4)}(0, t_k)_{L(I), L(J), L(M)} \]

where

\[ Q^{(4)}(0, t_{k+1})_{L(I), L(J), L(M)} = \mathbb{E}\left[ \langle (e_I \cup e_J \otimes e_0) \cup e_M^{d+1}, \widehat{Z}_{t_{k+1}} \rangle | \mathcal{F}_0 \right] \]

\[ = \sum_{|H| \leq 3n+1+m} \sum_{\epsilon_n = (e_I \cup e_J \otimes e_0) \cup e_M^{d+1}} \langle e_{n+1}^{d+1} G_{L(K) L(H)} e_H, \widehat{Z}_0 \rangle \]

\[ Q^{(4)}(0, t_k)_{L(I), L(J), L(M)} = \mathbb{E}\left[ \langle (e_I \cup e_J \otimes e_0) \cup e_M^{d+1}, \widehat{Z}_{t_k} \rangle | \mathcal{F}_0 \right] \]

\[ = \sum_{|H| \leq 3n+1+m} \sum_{\epsilon_n = (e_I \cup e_J \otimes e_0) \cup e_M^{d+1}} \langle e_{n+1}^{d+1} G_{L(K) L(H)} e_H, \widehat{Z}_0 \rangle \]
4.3.2 Explicit part C

\[ C = \mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s^2 ds \int_0^{t_k} \sigma_s dZ^{d+1,s} | \mathcal{F}_0] = \mathbb{E}[\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s^2 ds \int_0^{t_k} \sigma_s dZ^{d+1,s} | \mathcal{F}_{t_k}] | \mathcal{F}_0] \\
= \mathbb{E}[\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s^2 ds | \mathcal{F}_{t_k}] \int_0^{t_k} \sigma_s dZ^{d+1,s} | \mathcal{F}_0] \\
= \mathbb{E}[\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s^2 ds \mathcal{F}_{t_k}] \int_0^{t_k} \sigma_s dZ^{d+1,s} | \mathcal{F}_0] \\
= \mathbb{E}[\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s^2 ds | \mathcal{F}_{t_k}] \int_0^{t_k} \sigma_s dZ^{d+1,s} - \int_0^{t_k} \sigma_s^2 ds \int_0^{t_k} \sigma_s dZ^{d+1,s} | \mathcal{F}_0] \\
= \mathbb{E}[\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s^2 ds | \mathcal{F}_{t_k}] \int_0^{t_k} \sigma_s dZ^{d+1,s} - \int_0^{t_k} \sigma_s^2 ds \int_0^{t_k} \sigma_s dZ^{d+1,s} | \mathcal{F}_0] \\
= \mathbb{E}[\sum_{|I|,|J| \leq n} \ell_I \ell_J \mathbb{E}[\langle \mathbb{1}_I \mathbb{1}_J \otimes e_0, \hat{\mathbb{Z}}_{t_k+1} \rangle | \mathcal{F}_{t_k}] \sum_{|M| \leq n} \ell_M \langle \hat{\mathbb{e}}_{d+1}^f, \hat{\mathbb{Z}}_{t_k} \rangle | \mathcal{F}_0] \\
- \sum_{|I|,|J| \leq n} \ell_I \ell_J \mathbb{E}[\mathbb{E}[\langle \mathbb{1}_I \mathbb{1}_J \otimes e_0, \hat{\mathbb{Z}}_{t_k+1} \rangle | \mathcal{F}_{t_k}] \langle \hat{\mathbb{e}}_{d+1}^f, \hat{\mathbb{Z}}_{t_k} \rangle | \mathcal{F}_0] \\
= \mathbb{E}[\sum_{|I|,|J|,|M| \leq n} \ell_I \ell_J \mathbb{E}[\mathbb{E}[\langle \mathbb{1}_I \mathbb{1}_J \otimes e_0, \hat{\mathbb{Z}}_{t_k+1} \rangle | \mathcal{F}_{t_k}] \langle \hat{\mathbb{e}}_{d+1}^f, \hat{\mathbb{Z}}_{t_k} \rangle | \mathcal{F}_0] \\
- \sum_{|I|,|J|,|M| \leq n} \ell_I \ell_J \mathbb{E}[\mathbb{E}[\langle \mathbb{1}_I \mathbb{1}_J \otimes e_0, \hat{\mathbb{Z}}_{t_k+1} \rangle | \mathcal{F}_{t_k}] \langle \hat{\mathbb{e}}_{d+1}^f, \hat{\mathbb{Z}}_{t_k} \rangle | \mathcal{F}_0] \\
= \sum_{|I|,|J|,|M| \leq n} \ell_I \ell_J \mathbb{E}[\mathbb{E}[\langle \mathbb{1}_I \mathbb{1}_J \otimes e_0, \hat{\mathbb{Z}}_{t_k+1} \rangle | \mathcal{F}_{t_k}] \langle \hat{\mathbb{e}}_{d+1}^f, \hat{\mathbb{Z}}_{t_k} \rangle | \mathcal{F}_0] \\
- \sum_{|I|,|J|,|M| \leq n} \ell_I \ell_J \mathbb{E}[\mathbb{E}[\langle \mathbb{1}_I \mathbb{1}_J \otimes e_0, \hat{\mathbb{Z}}_{t_k+1} \rangle | \mathcal{F}_{t_k}] \langle \hat{\mathbb{e}}_{d+1}^f, \hat{\mathbb{Z}}_{t_k} \rangle | \mathcal{F}_0]
\]

Where

\[ Q^{(5)}_{\mathcal{L}(I),\mathcal{L}(J),\mathcal{L}(M)}(0, t_{k+1} - t_k) = \mathbb{E}[\mathbb{E}[\langle \mathbb{1}_I \mathbb{1}_J \otimes e_0, \hat{\mathbb{Z}}_{t_k+1} \rangle | \mathcal{F}_{t_k}] \langle \hat{\mathbb{e}}_{d+1}^f, \hat{\mathbb{Z}}_{t_k} \rangle | \mathcal{F}_0] \\
= \mathbb{E}[\sum_{e_K = (\mathbb{1}_I \mathbb{1}_J \otimes e_0)} \sum_{|H| \leq 2n+1} (e_{t_k+1-t_k})^{GT}_{\mathcal{L}(K)\mathcal{L}(H)} \langle e_H, \hat{\mathbb{Z}}_{t_k} \rangle \langle \hat{\mathbb{e}}_{d+1}^f, \hat{\mathbb{Z}}_{t_k} \rangle | \mathcal{F}_0] \\
= \sum_{e_K = (\mathbb{1}_I \mathbb{1}_J \otimes e_0)} \sum_{|H| \leq 2n+1} (e_{t_k+1-t_k})^{GT}_{\mathcal{L}(K)\mathcal{L}(H)} \mathbb{E}[\langle e_H, \hat{\mathbb{Z}}_{t_k} \rangle | \mathcal{F}_0] \\
= \sum_{e_K = (\mathbb{1}_I \mathbb{1}_J \otimes e_0)} \sum_{|H| \leq 2n+1} (e_{t_k+1-t_k})^{GT}_{\mathcal{L}(K)\mathcal{L}(H)} \sum_{e_A = e_H, |e_{d+1}^f| = 1} \sum_{|B| \leq 3n+1+m} (e_{t_k}^{GT})_{\mathcal{L}(A)\mathcal{L}(B)} \langle e_B, \hat{\mathbb{Z}}_0 \rangle \\
\]

and

\[ Q^{(4)}_{\mathcal{L}(I),\mathcal{L}(J),\mathcal{L}(M)}(0, t_k) = \mathbb{E}[\langle \mathbb{1}_I \mathbb{1}_J \otimes e_0, \hat{\mathbb{Z}}_{t_k+1} \rangle | \mathcal{F}_0] \\
= \sum_{e_K = (\mathbb{1}_I \mathbb{1}_J \otimes e_0)} \sum_{|H| \leq 3n+1+m} (e_{t_k}^{GT})_{\mathcal{L}(K)\mathcal{L}(H)} \langle e_H, \hat{\mathbb{Z}}_0 \rangle \\
\]
4.3.3 Explicit part D

\[
\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s dZ^{d+1} s \int_0^{t_k} \sigma_s^2 ds | \mathcal{F}_0] = \mathbb{E}[\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s dZ^{d+1} s \int_0^{t_k} \sigma_s^2 ds | \mathcal{F}_t] | \mathcal{F}_0]
\]

\[
= \mathbb{E}[\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s dZ^{d+1} s \int_0^{t_k} \sigma_s^2 ds | \mathcal{F}_t] - \int_0^{t_k} \sigma_s dZ^{d+1} s \int_0^{t_k} \sigma_s^2 ds | \mathcal{F}_t] | \mathcal{F}_0]
\]

\[
= \mathbb{E}[\mathbb{E}[\int_{t_k}^{t_{k+1}} \sigma_s dZ^{d+1} s \mid \mathcal{F}_t] \int_0^{t_k} \sigma_s^2 ds | \mathcal{F}_0] - \int_0^{t_k} \sigma_s dZ^{d+1} s \int_0^{t_k} \sigma_s^2 ds | \mathcal{F}_0]
\]

\[
= \mathbb{E}[\sum_{|I|,|J| \leq n} \ell_I \ell_J \mathbb{E}[\epsilon^{(k+1)}_{d+1} \hat{Z}_{t_{k+1}}] | \mathcal{F}_t] \sum_{|I|,|J| \leq n} \ell_I \ell_J (e_I \wedge e_J \otimes e_0, \hat{Z}_{t_k})
\]

where

\[
Q^{(6)}(0, t_{k+1} - t_k)_{\mathcal{L}(I), \mathcal{L}(J), \mathcal{L}(M)}(0, t_{k+1} - t_k) = \mathbb{E}[\mathbb{E}[\epsilon^{(k+1)}_{d+1} \hat{Z}_{t_{k+1}}] | \mathcal{F}_t] \langle e_I \wedge e_J \otimes e_0, \hat{Z}_{t_k} | \mathcal{F}_0]
\]

\[
= \mathbb{E} \bigg[ \sum_{e_K = \epsilon^{(k+1)}_{d+1}} \sum_{|H| \leq n+m} \epsilon^{(k+1)-t_k}_H^{G} \mathcal{L}(K) \mathcal{L}(H) \langle e_H, \hat{Z}_{t_k} \rangle \langle e_I \wedge e_J \otimes e_0, \hat{Z}_{t_k} | \mathcal{F}_0 \bigg]
\]

and

\[
Q^{(4)}_{\mathcal{L}(I), \mathcal{L}(J), \mathcal{L}(M)}(0, t_k) = \mathbb{E}[\langle (e_I \wedge e_J \otimes e_0) \wedge \epsilon^{(k+1)}_{d+1} \hat{Z}_{t_k} | \mathcal{F}_0]
\]

\[
= \mathbb{E} \bigg[ \sum_{e_K = (e_I \wedge e_J \otimes e_0) \wedge \epsilon^{(k+1)}_{d+1}} \sum_{|H| \leq 3n+1+m} \epsilon^{(k+1)}_H^{G} \mathcal{L}(K) \mathcal{L}(H) \langle e_H, \hat{Z}_{t_k} \rangle \bigg]
\]
5 Future Work

The signature method provides a concise representation of price processes, capturing important characteristics and patterns. Given a set of observed price paths, their signatures can be calculated. Thus we will implement algorithm to fit daily price data of variance swaps based on different underlying indexes and derive calibration results. [1]

References

