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Contents

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1. Introduction

Let k be a finite field. This article is about representation theory of the group $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$. Representation theory as a subject studies groups G via their linear actions on vector spaces; we refer to [Pia83] for the relevant background on the representation theory discussed in this article. For finite-dimensional complex representations of a finite group G, such representations can always be decomposed into irreducible ones, so the goal of representation theory is to understand these irreducible representations as well as possible.

For example, [Pia83] fully classifies the irreducible representations of the group $\mathrm{GL}_2(k)$, and these ideas can approximately be extended to classify representations of $\mathrm{GL}_n(k)$ for $n \geq 1$. However, as the group becomes more complicated, explicit enumeration becomes unreasonable. Instead, one can hope to attach invariants to these representations and then hope to understand desirable properties of these invariants and perhaps show that the invariants are enough to classify the irreducible representations.

For this paper, we interest ourselves in the "γ-factor" attached to a representation. To explain the motivation, we note that the groups of interest to us, such as $\mathrm{GL}_n(k)$ or $\mathrm{GL}_n(k) \times \mathrm{GL}_m(k)$, have number-theoretic significance. Notably, understanding representations of these groups when k is replaced by a local field is the main content of the local Langlands correspondences, and it is in this context that we first find the γ-factor as a largely analytic normalizing factor attached to some representations.

In number theory, one frequently expects finite fields to have the most controlled structure, so with such strong conjectures on the local field situation, we might hope to gain some traction by finding finite-field analogues for these results. And indeed, in recent years, there has been work both to establish what the analogues are [Nie14; SZ23; YZ20] as well as to relate the two situations together [Ye19; YZ20].

This paper is a continuation of the work described in the previous paragraph. In short, our goal is to tell the relevant story for the group $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$. Notably, a nontrivial part of our exposition closely follows the corresponding work over local fields as worked out by [Ike89; PR87].

1.1. **Layout.** We briefly explain the sections of this paper. In section 2, we review the relevant background on Whittaker models and Bessel functions needed in this article; we refer to, for example, [Pia83] for any other background on representation theory needed. In section 3, we review the theory for the group $\mathrm{GL}_n(k) \times \mathrm{GL}_n(k)$ with the goal of providing a direct proof of the γ-factor at $n = 2$, which is achieved in Theorem 20. In section 4, the main content of the article begins, where we define and prove the basic properties of the γ-factor for

$\mathrm{GL}_2(k)\times\mathrm{GL}_2(k)\times\mathrm{GL}_2(k)$. This discussion is then continued in section 5 where we compare the built theory with the story for local fields. Lastly, section 6 defines and computes these same γ-factors on the "Galois" side of the Langlands program; notably, the rest of the article lives on the "automorphic" side. The appendices contain some miscellaneous computations to supplement the content of section 4.

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1.3. **Notation.** In this article, all representations are complex and finite-dimensional. Let q be a prime-power, and let k be the finite field with q elements. We fix now once and for all an additive character ψ of k.

For now, fix a positive integer n, and we will name some subgroups and elements of $\mathrm{GL}_n := \mathrm{GL}_n(k)$ of interest. For any partition $(n_1, n_2, \ldots, n_r)$ of n into positive integers, we define the diagonal subgroup

$$D_{n_1,n_2,\ldots,n_r} := \left\{ \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_r \end{bmatrix} : d_i \in \mathrm{GL}_{n_i}(k) \text{ for each } i \right\},$$

and the unipotent subgroup

$$U_{n_1,n_2,\ldots,n_r} := \left\{ \begin{bmatrix} I_{n_1} & * & \cdots & * \\ & I_{n_2} & \cdots & * \\ & & \ddots & \vdots \\ & & & I_{n_r} \end{bmatrix} : I_{n_i} \in \mathrm{GL}_{n_i}(k) \text{ for each } i \right\},$$

each sitting inside the parabolic subgroups $P_{n_1,n_2,\ldots,n_r} := U_{n_1,n_2,\ldots,n_r} \rtimes D_{n_1,n_2,\ldots,n_r}$. (Here, $*$ signifies an arbitrary submatrix.) Most notably, we set $D_n := D_{1,1,\ldots,1}$ and $U_n := U_{1,1,\ldots,1}$, and we let $P_n := P_{n-1,1}$ be the mirabolic subgroup. Observe that P_n is the stabilizer of $e_n := (0, 0, \ldots, 0, 1)$.

Continuing, we define the Weyl elements W_n to be the permutation matrices in GL_n. Particularly important are the elements

$$w_{n_1,\ldots,n_r} := \begin{bmatrix} & & & I_{n_r} \\ & & \iddots & \\ & I_{n_2} & & \\ I_{n_1} & & & \end{bmatrix}.$$

Most notable is the long Weyl element $w_n := w_{1,1,\ldots,1}$.

Note that ψ extends naturally to a character ψ_n on U_n defined by

$$\psi_n\left(\begin{bmatrix}1 & u_1 & * & \cdots & * \\ & 1 & u_2 & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & & & 1 & u_{n-1} \\ & & & & 1\end{bmatrix}\right) := \psi(u_1+u_2+\cdots+u_{n-1}).$$

Later on, we will want to denote the symmetric $n\times n$ matrices by Sym_n and the invertible symmetric $n\times n$ matrices by $\operatorname{Sym}_n^\times$.

2. Whittaker Models

In this section, we review properties of Whittaker models.

2.1. Existence of Whittaker Models.

Here is our definition.

Definition 1 (Whittaker type)**.** A representation π of GL_n is *of Whittaker type* if and only if $\operatorname{Res}_{U_n}\pi$ has exactly one eigenvector (up to scalar) with eigenvalue ψ_n.

Remark 2. One can show that the definition above is independent of the chosen character ψ.

For example, it is known that any cuspidal irreducible representation is of Whittaker type. By definition, a representation π of Whittaker type has $\dim\operatorname{Hom}_{U_n}(\psi_n,\pi)=1$, which by reciprocity is equivalent to

$$\dim\operatorname{Hom}_{\mathrm{GL}_n}\left(\pi,\operatorname{Ind}_{U_n}^{\mathrm{GL}_n}\psi_n\right)=1.$$

Thus, we see that any representation π of Whittaker type has unique image $\mathcal{W}(\pi,\psi)$ in $\operatorname{Ind}_{U_n}^{\mathrm{GL}_n}\psi_n$, which is called a Whittaker model. Throughout, we may choose to write a specific Whittaker model by $W_v\in\operatorname{Ind}_{U_n}^{\mathrm{GL}_n}\psi_n$ for each $v\in V_\pi$. Note that this image $\{W_v:v\in V_\pi\}$ of π is only unique up to scalar.

While we are here, we provide a relatively explicit Whittaker model for a representation π of GL_n of Whittaker type.

Lemma 3. *Fix an irreducible representation π of GL_n of Whittaker type. Further, let $\langle\cdot,\cdot\rangle$ be a G-form on V_π, and let $v_\psi\in V_\pi$ be an eigenvector with eigenvalue ψ_n with $\langle v_\psi,v_\psi\rangle$ (which exists and is unique by scaling). Now, for each $v\in V_\pi$ we define*

$$W_v(g):=\langle gv,v_\psi\rangle.$$

Then $\mathcal{W}(\pi,\psi):=\{W_v:v\in V_\pi\}$ is a Whittaker model for π.

Proof. We run the checks directly. The map $W_\bullet\colon\pi\to\operatorname{Ind}_{U_n}^{\mathrm{GL}_n}\psi_n$ is well-defined because

$$W_v(ug)=\langle ugv,v_\psi\rangle=\left\langle gv,u^{-1}v_\psi\right\rangle=\langle gv,\overline{\psi_n(u)}v_\psi\rangle=\psi_n(u)\langle gv,v_\psi\rangle.$$

Continuing, the map is of course linear in v, and it is G-equivariant because

$$W_{\pi(h)v}(g)=\langle ghv,v_\psi\rangle=W_v(gh).$$

Lastly, the map is injective because $v\neq0$ implies that $\{gv:g\in\mathrm{GL}_n\}$ spans V_π because π is irreducible. Thus, $\langle gv,v_\psi\rangle\neq0$ for some $g\in\mathrm{GL}_n$. $\square$

2.2. **Bessel Functions.** Fix an irreducible representation π of GL_n of Whittaker type. Here is our definition.

Definition 4 (Bessel function)**.** Fix a Whittaker model $\mathcal{W}(\pi,\psi)$ for π. Then the *Bessel function* $\mathcal{J}_{\pi,\psi}$ is the unique eigenvector in $\mathcal{W}(\pi,\psi)$ with eigenvalue ψ_n and with $\mathcal{J}_{\pi,\psi}(I_n) = 1$.

Certainly if $\mathcal{J}_{\pi,\psi}$ exists, then it is unique because the eigenvectors with eigenvalue ψ_n are unique up to scalar. To see that $\mathcal{J}_{\pi,\psi}$ exists, we have the following lemma.

Lemma 5. *Fix an irreducible representation π of GL_n of Whittaker type. Then the Bessel function exists.*

Proof. Use the notation of Lemma 3 to set

$$\mathcal{J}_{\pi,\psi}(g) \coloneqq W_{v_\psi}(g) = \langle g v_\psi, v_\psi \rangle.$$

Because v_ψ is an eigenvector with eigenvalue ψ_n, the same is true for $\mathcal{J}_{\pi,\psi}$, and further, we see $\mathcal{J}_{\pi,\psi}(I_n) = \langle v_\psi, v_\psi \rangle = 1$, as required. $\square$

We will want the following property of Bessel functions, whose proof we omit. Roughly speaking, the following result is useful when combined with the Bruhat decomposition $\mathrm{GL}_n = U_n D_n W_n U_n$.

Proposition 6 ([Gel70, Proposition 4.9])**.** *Fix an irreducible representation π of GL_n of Whittaker type. For $d \in D_n$ and $w \in W_n$, if $\mathcal{J}_{\pi,\psi}(dw) \neq 0$, then*

$$dw = \begin{bmatrix} & & & \lambda_{n_r} I_{n_r} \\ & & \iddots & \\ & \lambda_{n_2} I_{n_2} & & \\ \lambda_{n_1} I_{n_1} & & & \end{bmatrix}$$

for some partition $(n_1, \ldots, n_r)$ of n and elements $\lambda_{n_1}, \ldots, \lambda_{n_r} \in k$.

2.3. **A Symmetry on Whittaker Models.** Whittaker models of irreducible representations have a special symmetry which will be important later. Given $g \in \mathrm{GL}_n$, we set $g^\iota \coloneqq g^{-\intercal}$. Then note $(\cdot)^\iota \colon \mathrm{GL}_n \to \mathrm{GL}_n$ is an automorphism of GL_n. This symmetry on GL_n can be upgraded to a symmetry on representations by taking a representation π of GL_n to π^ι defined by $\pi^\iota(g) \coloneqq \pi(g^\iota)$. This π^ι is useful because of the following lemma.

Lemma 7. *Let π be an irreducible representation of GL_n of Whittaker type. Then $\pi^\iota \cong \pi^\vee$.*

Proof. It suffices to show that $\pi^\vee$ and π^ι have the same character. Well, for any $g \in \mathrm{GL}_n$, we note g is conjugate to $g^\intercal$, so

$$\operatorname{tr} \pi^\iota(g) = \operatorname{tr} \pi \left(\left(g^{-1}\right)^\intercal \right) = \operatorname{tr} \pi \left(g^{-1} \right) = \operatorname{tr} \pi^\vee(g),$$

so we are done. $\square$

Namely, π^ι provides a way to talk about $\pi^\vee$ without dual spaces. Observe, however, that $(\cdot)^\vee$ is a contravariant functor while $(\cdot)^\iota$ is covariant, so we should not think about these as the same functor.

Continuing, we may upgrade our symmetry on representations to a symmetry on Whittaker models.

Lemma 8. *Let π be a representation of GL_n of Whittaker type, and let $W_\bullet \colon \pi \to \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n$ be a Whittaker model. Then the map $\widetilde{\cdot} \colon \mathcal{W}(\pi,\psi) \to \mathcal{W}(\pi^\iota, \psi^{-1})$ defined by*

$$\widetilde{W}_v(g) \coloneqq W_v(w_n g^\iota)$$

provides a Whittaker model $\mathcal{W}(\pi,\psi^{-1})$ of π^ι.

Proof. It suffices to show that $\widetilde{W}_\bullet \colon \pi^\iota \to \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n$ is a well-defined injective map of GL_n-representations.

- Well-defined: for $v \in V_\pi$, we show that $\widetilde{W}_v \in \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1}$. The main computation here is that
$$w_n \begin{bmatrix} 1 & u_1 & * & \cdots & * \\ & 1 & u_2 & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & & & 1 & u_{n-1} \\ & & & & 1 \end{bmatrix}^\iota w_n^{-1} = \begin{bmatrix} 1 & -u_1 & * & \cdots & * \\ & 1 & -u_2 & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & & & 1 & -u_{n-1} \\ & & & & 1 \end{bmatrix}.$$
Thus, for any $u \in U_n$, we see that $w_n u^\iota w_n^{-1} \in U_n$, and $\psi_n(w_n u^\iota w_n^{-1})^{-1} = \psi_n(u)$. As such, we see
$$\widetilde{W}_v(ug) = W_v\left(w_n u^\iota g^\iota\right) = \psi_n\left(w_n u^\iota w_n^{-1}\right) W_v(w_n g^\iota) = \psi_n^{-1}(u) \widetilde{W}_v(g).$$
- Homomorphism: of course the map $v \mapsto \widetilde{W}_v$ is linear in v. This is G-equivariant because
$$\widetilde{W}_{\pi^\iota(h)v}(g) = W_{\pi(h^\iota)v}(w_n g^\iota) = W_v(w_n (gh)^\iota) = \widetilde{W}_v(gh).$$
- Injective: note that if $v \in V_\pi$ has $\widetilde{W}_v = 0$, then $W_v(w_n g^\iota) = 0$ for any $g \in \mathrm{GL}_n$, so $W_v = 0$, so $v = 0$ because $\mathcal{W}(\pi,\psi)$ is already a Whittaker model. $\square$

To properly view the map $W \mapsto \widetilde{W}$ as a symmetry, we note that we have the following lemma.

Lemma 9. *For any $g \in \mathrm{GL}_n$, the following diagram commutes.*

$$\begin{array}{ccc} \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n & \xrightarrow{\ \widetilde{\cdot}\ } & \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g^\iota} \\ \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n & \xrightarrow{\ \widetilde{\cdot}\ } & \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \end{array}$$

Proof. This is a direct computation. Fix some $g \in \mathrm{GL}_n$. For any $W \in \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n$, we want to show that $\widetilde{gW} = g^\iota \widetilde{W}$. Well, for any $g_0 \in \mathrm{GL}_n$, we compute

$$\widetilde{gW}(g_0) = (gW)\left(w_n g_0^\iota\right) = W\left(w_n g_0^\iota g\right) = W\left(w_n (g_0 g^\iota)^\iota\right) = \widetilde{W}\left(w_n g_0 g^\iota\right).$$

This equals $(g^\iota \widehat{W})(w_n g_0)$, as desired. $\square$

3. Gamma Factors for $\mathrm{GL}_n \times \mathrm{GL}_n$

In this section, we review the construction of the γ-factor attached to two cuspidal irreducible representations σ and τ of GL_n. We use the Rankin–Selberg method.

3.1. **A Multiplicity One Result.** The backbone of the approach is a multiplicity one result which we will prove in the present subsection. The correct statement requires test functions, which we introduce now.

Definition 10. Let $S(k^n)$ denote the space of functions $k^n \to \mathbb{C}$, and we make $S(k^n)$ into a GL_n-representation by defining

$$(g\varphi)(v) \coloneqq \varphi(vg)$$

for any $g \in \mathrm{GL}_n$ and $\varphi \in S(k^n)$ and $v \in k^n$. Lastly, we let $S_0(k^n)$ denote the G-subrepresentation of functions $\varphi\colon k^n \to \mathbb{C}$ vanishing at 0.

And here is our result.

Proposition 11. *Fix cuspidal representations σ and τ of GL_n. Further, let $S_0(k^n)$ denote the functions $k^n \to \mathbb{C}$ vanishing at 0. Then*

$$\dim \operatorname{Hom}_{\mathrm{GL}_n}(\sigma \otimes \tau \otimes S_0(k^n), \mathbb{C}) = 1.$$

Proof. Because σ and $\tau^\vee$ are cuspidal, we see $\operatorname{Res}^{\mathrm{GL}_n}_{P_n} \sigma \cong \operatorname{Res}^{\mathrm{GL}_n}_{P_n} \tau^\vee \cong \operatorname{Ind}^{P_n}_{U_n} \psi_n$ are isomorphic irreducible representations, so

$$\dim \operatorname{Hom}_{P_n}(\mathbb{C}, \sigma^\vee \otimes \tau^\vee) = 1.$$

The main result now follows from Frobenius reciprocity. Indeed, we claim that $S_0(k^n) \cong \operatorname{Ind}^{\mathrm{GL}_n}_{P_n} \mathbb{C}$. In one direction, send $\varphi \in S_0(k^n)$ to the function $f_\varphi(g) \coloneqq \varphi(e_1 g)$; in the other direction, send $f \in \operatorname{Ind}^{\mathrm{GL}_n}_{P_n} \mathbb{C}$ to the function $\varphi(e_1 g) \coloneqq f(g)$. One can check that these maps are G-equivariant and mutually inverse, which provides our isomorphism. Anyway, the point is that

$$\dim \operatorname{Hom}_{\mathrm{GL}_n}(\sigma \otimes \tau \otimes S_0(k^n), \mathbb{C}) = \dim \operatorname{Hom}_{\mathrm{GL}_n}(S_0(k^n), \sigma^\vee \otimes \tau^\vee) = \dim \operatorname{Hom}_{P_n}(\mathbb{C}, \sigma^\vee \otimes \tau^\vee),$$

which is 1. $\square$

A multiplicity one result is not very useful without actually have elements in the needed vector space, so we go ahead and exhibit an element. Because σ and τ are cuspidal and hence of Whittaker type, we see that we may embed $\sigma \to \operatorname{Ind}^{\mathrm{GL}_n}_{U_n} \psi_n$ and $\tau \to \operatorname{Ind}^{\mathrm{GL}_n}_{U_n} \psi_n^{-1}$, so it suffices to exhibit a map

$$\operatorname{Ind}^{\mathrm{GL}_n}_{U_n} \psi_n \otimes \operatorname{Ind}^{\mathrm{GL}_n}_{U_n} \psi_n^{-1} \otimes S_0(k^n) \to \mathbb{C}.$$

Here is our definition.

Definition 12 (Z-function)**.** For any $W \in \operatorname{Ind}^{\mathrm{GL}_n}_{U_n} \psi_n$ and $W' \in \operatorname{Ind}^{\mathrm{GL}_n}_{U_n} \psi_n^{-1}$ and $\varphi \in S(k^n)$, we define

$$Z(W, W', \varphi; \psi) \coloneqq \sum_{g \in U_n \backslash \mathrm{GL}_n} W(g) W'(g) \varphi(e_n g),$$

where $e_n = (0, 0, \ldots, 0, 1)$.

Let's run our checks.

Lemma 13. *As defined above, Z provides a well-defined G-equivariant map $\operatorname{Ind}^{\mathrm{GL}_n}_{U_n} \psi_n \otimes \operatorname{Ind}^{\mathrm{GL}_n}_{U_n} \psi_n^{-1} \otimes S(k^n) \to \mathbb{C}$.*

Proof. For now, fix $W \in \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n$ and $W' \in \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1}$ and $\varphi \in S(k^n)$. Quickly, we check that each summand $W(g)W'(g)\varphi(e_ng)$ is independent of the coset U_ng. Indeed, for any $u \in U_n$, we see that

$$W(ug)W'(ug) = \psi_n(u)W(g)\psi_n^{-1}(u)W'(g) = W(g)W'(g)$$

by definition of W and W', and $\varphi(e_nug) = \varphi(e_ng)$ because $u \in U_n \subseteq P_n$.

Additionally, we see that Z is linear in W and W' and φ, so we have defined a linear map

$$Z\colon \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n \otimes \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \otimes S(k^n) \to \mathbb{C}.$$

It remains to check that Z is G-equivariant. Well, for any $h \in \mathrm{GL}_n$, we compute

$$Z(hW \otimes hW' \otimes h\varphi) = \sum_{g \in U_n \backslash \mathrm{GL}_n} W(gh)W'(gh)\varphi(e_ngh) = Z(W \otimes W' \otimes \varphi),$$

where the last equality has reindexed the sum. $\square$

With some effort, we can even show that $Z \neq 0$ in our cases of interest.

Lemma 14. *Let σ and τ be cuspidal representations of GL_n. Let $\varphi_n\colon k^n \to \mathbb{C}$ denote the indicator function for $e_n := (0, 0, \ldots, 0, 1)$. Then $Z(\mathcal{J}_{\sigma,\psi}, \mathcal{J}_{\tau,\psi^{-1}}, \varphi_n; \psi) = 1$. In particular, $Z \neq 0$ as an element of $\operatorname{Hom}_{\mathrm{GL}_n}(\sigma \otimes \tau \otimes S_0(k^n), \mathbb{C})$.*

Proof. By definition,

$$Z(\mathcal{J}_{\sigma,\psi}, \mathcal{J}_{\tau,\psi^{-1}}, \varphi_n; \psi) = \sum_{g \in U_n \backslash \mathrm{GL}_n} \mathcal{J}_{\sigma,\psi}(g)\mathcal{J}_{\tau,\psi^{-1}}(g)\varphi_n(e_ng).$$

The main point is to use Proposition 6. Fix some $g \in \mathrm{GL}_n$, and suppose that the summand $\mathcal{J}_{\sigma,\psi}(g)\mathcal{J}_{\tau,\psi^{-1}}(g)\varphi_n(e_ng)$ is nonzero. We claim that $g \in U_n$, which will complete the proof.

Indeed, $\varphi_n(e_ng) \neq 0$ requires $e_ng = e_n$, so $g \in P_n$. Now, using the Bruhat decomposition, we may write $g = udwu'$ where $u, u' \in U_n$ and $d \in D_n$ and $w' \in W_n$. Now, $\mathcal{J}_{\sigma,\psi}(g) \neq 0$, so

$$0 \neq \mathcal{J}_{\sigma,\psi}(udwu') = \psi_n(uu')\mathcal{J}_{\sigma,\psi}(dw),$$

so Proposition 6 forces dw to have the form

$$dw = \begin{bmatrix} & & & \lambda_{n_r} I_{n_r} \\ & & \iddots & \\ & \lambda_{n_2} I_{n_2} & & \\ \lambda_{n_1} I_{n_1} & & & \end{bmatrix}.$$

However, $e_ng = e_n$ requires $e_ndw = e_n$, so $dw \in P_n$ as well, and the only matrix of the above form which lives in P_n is $dw = I_n$, so $g = uu' \in U_n$, as promised. $\square$

3.2. **The Functional Equation.** Thus far, we have provided an element Z which spans $\operatorname{Hom}_{\mathrm{GL}_n}(\sigma \otimes \tau \otimes S_0(k^n), \mathbb{C})$. To produce our functional equation, we want to find another element in that space. For this, we will exhibit a map

$$\sigma \otimes \tau \otimes S(k^n) \to \sigma^\iota \otimes \tau^\iota \otimes S(k^n)$$

with good duality properties, and then we will pass Z through. In fact, this map will be found by restricting a map

$$\mathcal{F}\colon \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n \otimes \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \otimes S(k^n) \to \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \otimes \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n \otimes S(k^n)$$

with good duality properties, and then we will restrict it. Well, this last map will be found component-wise. The map $W \mapsto \widetilde{W}$ of section 2.3 provides maps $\operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n \to \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1}$ and $\operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \to \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n$. Lastly, we desire a map $S(k^n) \to S(k^n)$, for which we use the Fourier transform: for $\varphi \in S(k^n)$, set

$$\widehat{\varphi}(x) \coloneqq \sum_{y \in k^n} \varphi(y)\psi(\langle x, y\rangle),$$

where $\langle\cdot,\cdot\rangle$ is the standard symmetric bilinear form on k^n. Each of the component maps we defined are linear, so they will glue into a linear map

$$\mathcal{F}\colon \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n \otimes \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \otimes S\left(k^n\right) \to \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \otimes \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n \otimes S\left(k^n\right)$$

defined by $\mathcal{F}(W \otimes W' \otimes \varphi) \coloneqq \widetilde{W} \otimes \widetilde{W'} \otimes \widehat{\varphi}$. The aforementioned "good duality properties" are recorded in the following lemma.

Lemma 15. *For any $g \in \mathrm{GL}_n$, the following diagram commutes.*

$$\begin{array}{ccc}
\operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n \otimes \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \otimes S\left(k^n\right) & \xrightarrow{\mathcal{F}} & \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \otimes \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n \otimes S\left(k^n\right) \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g^\iota} \\
\operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n \otimes \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \otimes S\left(k^n\right) & \xrightarrow{\mathcal{F}} & \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n^{-1} \otimes \operatorname{Ind}_{U_n}^{\mathrm{GL}_n} \psi_n \otimes S\left(k^n\right)
\end{array}$$

Proof. We can check this on each component. The diagram commutes on the left two components by Lemma 9. Lastly, for any $\varphi \in S(k^n)$ and $x \in k^n$, we see

$$\widehat{g\varphi}(x) = \sum_{y \in k^n} \varphi(yg)\psi(\langle x, y\rangle) = \sum_{y \in k^n} \varphi(y)\psi\left(\langle x, yg^{-1}\rangle\right) = \sum_{y \in k^n} \varphi(y)\psi\left(\langle xg^\iota, y\rangle\right) = \left(g^\iota\widehat{\varphi}\right)(x).$$

The claim follows. □

We now restrict $\mathcal{F}$ and extract our functional equation.

Lemma 16. *The function $\mathcal{F}$ restricts to a function $\sigma \otimes \tau \otimes S\left(k^n\right) \to \sigma^\iota \otimes \tau^\iota \otimes S\left(k^n\right)$.*

Proof. We check this componentwise. We already know that the map $W \mapsto \widetilde{W}$ restricts to a map $\mathcal{W}(\sigma, \psi) \to \mathcal{W}\left(\sigma^\iota, \psi^{-1}\right)$ by Lemma 8, and similar holds for τ. □

Theorem 17. *Fix cuspidal representations σ and τ of GL_n. There is a unique complex number $\gamma(\sigma \times \tau, \psi)$ such that*

$$Z\left(\widetilde{W}, \widetilde{W'}, \widehat{\varphi}; \psi\right) = \gamma(\sigma \times \tau, \psi) Z(W, W', \varphi; \psi)$$

for any $W \in \mathcal{W}(\sigma, \psi)$ and $W' \in \mathcal{W}(\tau, \psi^{-1})$ and $\varphi \in S_0(k^n)$.

Proof. Define $\mathcal{F}Z\colon \sigma \otimes \tau \otimes S_0(k^n) \to \mathbb{C}$ by $\mathcal{F}Z(W \otimes W' \otimes \varphi) \coloneqq Z(\widetilde{W}, \widetilde{W'}, \widehat{\varphi}; \psi)$. The square in Lemma 15 implies that $\mathcal{F}Z$ is GL_n-invariant because Z is, so the result follows from Proposition 11. Technically, we must know that $Z \neq 0$ to carry this argument out, which is established in Lemma 14. □

Corollary 18. *Fix cuspidal representations σ and τ of GL_n. Then*

$$\gamma(\sigma \times \tau, \psi) = \sum_{g \in U_n \backslash \mathrm{GL}_n} \mathcal{J}_{\sigma,\psi}(g)\mathcal{J}_{\tau,\psi^{-1}}(g)\psi\left(\langle e_n g^{-1}, e_1\rangle\right),$$

where $e_n \coloneqq (0, \dots, 0, 1)$ *and* $e_1 \coloneqq (1, 0, \dots, 0)$.

Proof. Combining Theorem 17 with the computation of Lemma 14, we see upon plugging everything in that

$$\gamma(\sigma \times \tau, \psi) = \sum_{g \in U_n \backslash \mathrm{GL}_n} \mathcal{J}_{\sigma,\psi}(w_n g^\iota) \mathcal{J}_{\tau,\psi^{-1}}(w_n g^\iota) \widehat{\varphi_n}(e_n g).$$

Note that $g \mapsto w_n g^\iota$ is a well-defined involution $U_n \backslash \mathrm{GL}_n \to U_n \backslash \mathrm{GL}_n$ (this is included in the computation of Lemma 8), so we may reindex the sum as

$$\gamma(\sigma \times \tau, \psi) = \sum_{g \in U_n \backslash \mathrm{GL}_n} \mathcal{J}_{\sigma,\psi}(g) \mathcal{J}_{\tau,\psi^{-1}}(g) \widehat{\varphi_n}(e_1 g^\iota).$$

We must now compute the Fourier transform as

$$\widehat{\varphi_n}(e_1 g^\iota) = \sum_{y \in k^n} \varphi_n(y) \psi(\langle e_1 g^\iota, y \rangle) = \psi(\langle e_1 g^\iota, e_n \rangle) = \psi\left(\langle e_n g^{-1}, e_1 \rangle\right).$$

Plugging this in completes the proof. $\square$

3.3. **Computation for** $n = 2$**.** In this subsection, we compute $\gamma(\sigma \times \tau, \psi)$ as the product of two Gauss sums when σ and τ are cuspidal representations of GL_2. For brevity, let ω_σ and ω_τ denote the central characters of σ and τ, respectively. Because σ and τ are cuspidal, the characters ω_σ and ω_τ arise from non-decomposable characters on $\ell^\times$, which we will continue to denote by ω_σ and ω_τ respectively.

Lemma 19. *A set of representatives for* $U_2 \backslash \mathrm{GL}_2$ *is given by* $D_2 \sqcup D_2 w_2 U_2$.

Proof. By the Bruhat decomposition, we may write $\mathrm{GL}_2 = B_2 \sqcup B_2 w_2 U_2$, but $U_2 \backslash B_2$ is represented by D_2 because any element of B_2 takes the form

$$\begin{bmatrix} a & b \\ & d \end{bmatrix} = \begin{bmatrix} 1 & b/d \\ & 1 \end{bmatrix} \begin{bmatrix} a & \\ & d \end{bmatrix}.$$

Thus, we see that $D_2 \sqcup D_2 w_2 U_2$ succeeds in representing $U_2 \backslash \mathrm{GL}_2$. To see that each element of $D_2 \sqcup D_2 w_2 U_2$ belongs to a unique equivalence class, note that there are $(q-1)^2 + (q-1)^2 q = (q-1)^2(q+1)$ elements in $D_2 \sqcup D_2 w_2 U_2$ and $\left(q^2 - 1\right)\left(q^2 - q\right)/q = (q-1)^2(q+1)$ elements in $U_2 \backslash \mathrm{GL}_2$. $\square$

Thus, Corollary 18 gives

$$\gamma(\sigma \times \tau, \psi) = \underbrace{\sum_{d \in D_2} \mathcal{J}_{\sigma,\psi}(d) \mathcal{J}_{\tau,\psi^{-1}}(d)}_{S_D \coloneqq} + \underbrace{\sum_{\substack{d \in D_2 \\ u \in U_2}} \mathcal{J}_{\sigma,\psi}(d w_2 u) \mathcal{J}_{\sigma,\psi^{-1}}(d w u) \psi\left(\langle e_2 (d w u)^{-1}, e_1 \rangle\right)}_{S_{DwU} \coloneqq}.$$

The sum over D_2 can be evaluated to

$$\begin{aligned} S_D &= \sum_{a,d\in k^\times} \mathcal{J}_{\sigma,\psi}\left(\begin{bmatrix} a & \\ & d\end{bmatrix}\right)\mathcal{J}_{\tau,\psi^{-1}}\left(\begin{bmatrix} a & \\ & d\end{bmatrix}\right) \\ &\overset{*}{=} \sum_{a\in k^\times} \mathcal{J}_{\sigma,\psi}\left(\begin{bmatrix} a & \\ & a\end{bmatrix}\right)\mathcal{J}_{\tau,\psi^{-1}}\left(\begin{bmatrix} a & \\ & a\end{bmatrix}\right) \\ &= \sum_{a\in k^\times} \omega_\sigma(a)\omega_\tau(a) \\ &= \begin{cases} q-1 & \text{if } \omega_\sigma|_k = \omega_\tau^{-1}|_k, \\ 0 & \text{else.}\end{cases} \end{aligned}$$

Importantly, $\overset{*}{=}$ has used Proposition 6. The sum over $D_2w_2U_2$ is harder to simplify. The $u \in U_2$ does not alter any summand, so we can begin by writing out

$$\begin{aligned} S_{DwU} &= q\sum_{a,d\in k^\times} \mathcal{J}_{\sigma,\psi}\left(\begin{bmatrix} & a \\ d & \end{bmatrix}\right)\mathcal{J}_{\tau,\psi^{-1}}\left(\begin{bmatrix} & a \\ d & \end{bmatrix}\right)\psi(1/a) \\ &= q\sum_{a,d\in k^\times} \omega_\sigma(a)\omega_\tau(a)\mathcal{J}_{\sigma,\psi}\left(\begin{bmatrix} & 1 \\ a^{-1}d & \end{bmatrix}\right)\mathcal{J}_{\tau,\psi^{-1}}\left(\begin{bmatrix} & 1 \\ a^{-1}d & \end{bmatrix}\right)\psi(1/a) \\ &= q\sum_{a,d\in k^\times} \omega_\sigma(a)^{-1}\omega_\tau(a)^{-1}\mathcal{J}_{\sigma,\psi}\left(\begin{bmatrix} & 1 \\ -d^{-1} & \end{bmatrix}\right)\mathcal{J}_{\tau,\psi^{-1}}\left(\begin{bmatrix} & 1 \\ -d^{-1} & \end{bmatrix}\right)\psi(a). \end{aligned}$$

Now, [Pia83, p. 63] computes

$$\mathcal{J}_{\sigma,\psi}\left(\begin{bmatrix} & 1 \\ -d^{-1} & \end{bmatrix}\right) = -\frac{\omega_\sigma(d)^{-1}}{q}\sum_{\substack{t_\sigma\in\ell^\times \\ \mathrm{N}\,t_\sigma=d}} \psi(\operatorname{tr} t_\sigma)\omega_\sigma(t_\sigma) = -\frac{1}{q}\sum_{\substack{t_\sigma\in\ell^\times \\ \mathrm{N}\,t_\sigma=d}} \psi(\operatorname{tr} t_\sigma)\omega_\sigma(t_\sigma)^{-1},$$

where ℓ/k is the quadratic extension, and $\operatorname{tr}\colon \ell \to k$ and $\mathrm{N}\colon \ell \to k$ denote the trace and norm maps, respectively. A similar formula holds for τ, so we see

$$\begin{aligned} S_{DwU} &= \frac{1}{q}\sum_{a\in k^\times} \omega_\sigma(a)^{-1}\omega_\tau(a)^{-1}\psi(a)\sum_{\substack{t_\sigma,t_\tau\in\ell^\times \\ \mathrm{N}\,t_\sigma=\mathrm{N}\,t_\tau}} \psi(\operatorname{tr} t_\sigma - \operatorname{tr} t_\tau)\omega_\sigma(t_\sigma)^{-1}\omega_\tau(t_\tau)^{-1} \\ &= \frac{\omega_\tau(-1)}{q}\sum_{a\in k^\times} \omega_\sigma(a)^{-1}\omega_\tau(a)^{-1}\psi(a)\sum_{\substack{t_\sigma,t_\tau\in\ell^\times \\ \mathrm{N}\,t_\sigma=\mathrm{N}\,t_\tau}} \psi(\operatorname{tr} t_\sigma + \operatorname{tr} t_\tau)\omega_\sigma(t_\sigma)^{-1}\omega_\tau(t_\tau)^{-1}. \end{aligned}$$

To continue, we note that there is a group homomorphism $\ell^\times \times \ell^\times \to \{(t_\sigma, t_\tau) : \mathrm{N}\,t_\sigma = \mathrm{N}\,t_\tau\}$ given by $(x, y) \mapsto (xy, xy^q)$. Observe that this homomorphism is surjective: for any (t_σ, t_τ) with $\mathrm{N}\,t_\sigma = \mathrm{N}\,t_\tau$, we have $\mathrm{N}(t_\sigma/t_\tau) = 1$, so Hilbert's theorem 90 promises some $z \in \ell^\times$ such that $t_\sigma/t_\tau = y/y^q$, so $x := t_\sigma/y$ yields $(t_\sigma, t_\tau) = (xy, xy^q)$. Now, the kernel of this homomorphism requires $xy = xy^q = 1$, or $x = y^{-1} = y^{-q}$, meaning $x = y^{-1} \in k^\times$. Thus, the kernel has $q-1$ elements, implying

$$\begin{aligned} S_{DwU} &= \frac{\omega_\tau(-1)}{q(q-1)} \sum_{\substack{a\in k^\times \\ x,y\in\ell^\times}} \psi(\operatorname{tr}(xy)+\operatorname{tr}(xy^q)+a)\omega_\sigma(axy)^{-1}\omega_\tau(axy^q)^{-1} \\ &= \frac{\omega_\tau(-1)}{q(q-1)} \sum_{\substack{a\in k^\times \\ x,y\in\ell^\times}} \psi(\operatorname{tr}(x)\operatorname{tr}(y)+a)\omega_\sigma(axy)^{-1}\omega_\tau(axy^q)^{-1} \\ &= \frac{\omega_\tau(-1)}{q(q-1)} \sum_{\substack{a\in k^\times \\ x,y\in\ell^\times}} \psi\left(\frac{\operatorname{tr}(x)\operatorname{tr}(y)}{a}+a\right)\omega_\sigma(xy)^{-1}\omega_\tau(xy^q)^{-1}. \end{aligned}$$

At this point, we would like to send $a\mapsto \operatorname{tr}(y)/a$, but this is only legal when $\operatorname{tr}(y)\neq 0$. Thus, we go ahead and isolate the $\operatorname{tr}(y)\neq 0$ terms now: over these terms, the summation is

$$\frac{\omega_\tau(-1)}{q(q-1)} \sum_{a\in k^\times}\psi(a)\sum_{x\in\ell^\times}\omega_\sigma(x)^{-1}\omega_\tau(x)^{-1}\sum_{\substack{y\in\ell^\times \\ \operatorname{tr}(y)=0}}\omega_\sigma(y)^{-1}\omega_\tau(y^q)^{-1}.$$

If $\omega_\sigma\neq\omega_\tau^{-1}$, then the second sum vanishes. Otherwise, we can collapse the sum down to

$$-\frac{q^2-1}{q(q-1)}\sum_{\substack{y\in\ell^\times \\ \operatorname{tr}(y)=0}}\underbrace{\omega_\sigma(y)^{-1}\omega_\tau(-y^q)^{-1}}_{1} = -\frac{q^2-1}{q}.$$

Thus,

$$\begin{aligned} S_{DwU} &= \frac{\omega_\tau(-1)}{q(q-1)} \sum_{\substack{a\in k^\times \\ x,y\in\ell^\times \\ \operatorname{tr}(y)\neq 0}} \psi\left(\operatorname{tr}(ax)+\operatorname{tr}(y/a)\right)\omega_\sigma(xy)^{-1}\omega_\tau(xy^q)^{-1} - \frac{q^2-1}{q}1_{\omega_\sigma=\omega_\tau^{-1}} \\ &= \frac{\omega_\tau(-1)}{q} \sum_{\substack{x,y\in\ell^\times \\ \operatorname{tr}(y)\neq 0}} \psi\left(\operatorname{tr}(x)+\operatorname{tr}(y)\right)\omega_\sigma(xy)^{-1}\omega_\tau(xy^q)^{-1} - \frac{q^2-1}{q}1_{\omega_\sigma=\omega_\tau^{-1}}. \end{aligned}$$

We would now like to re-add the $y\in\ell^\times$ with $\operatorname{tr}(y)=0$, where the summation looks like

$$S_0 := \frac{\omega_\tau(-1)}{q}\sum_{x\in\ell^\times}\psi(\operatorname{tr}(x))\omega_\sigma(x)^{-1}\omega_\tau(x)^{-1}\sum_{\substack{y\in\ell^\times \\ \operatorname{tr}(y)=0}}\omega_\sigma(y)^{-1}\omega_\tau(y^q)^{-1}.$$

If $\omega_\sigma|_k\neq\omega_\tau^{-1}|_k$, then the right sum will vanish because we can send $y\mapsto cy$ where $c\in k^\times$ to pick up a factor of $\omega_\sigma(c)^{-1}\omega_\tau(c)^{-1}\neq 1$; thus, $S_0=0$ in this case. If $\omega_\sigma\cong\omega_\tau^{-1}$, then the summation collapses to

$$S_0 = -\frac{1}{q}\sum_{\substack{y\in\ell^\times \\ \operatorname{tr}(y)=0}}\underbrace{\omega_\sigma(y)^{-1}\omega_\tau(-y^q)^{-1}}_{1} = -\frac{q-1}{q} = (q-1)-\frac{q^2-1}{q}.$$

Lastly, suppose $\omega_\sigma|_k=\omega_\tau^{-1}|_k$ but $\omega_\sigma\neq\omega_\tau^{-1}$. This case is harder because we must evaluate the Gauss sum. For brevity, set $\chi:=\omega_\sigma^{-1}\omega_\tau^{-1}$, which we know is nontrivial but trivial on $k^\times$.

The right-hand sum is

$$\omega_\tau(-1) \sum_{\substack{y\in\ell^\times \\ \mathrm{tr}(y)=0}} \omega_\sigma(y)^{-1}\omega_\tau(y^q)^{-1} = \sum_{\substack{y\in\ell^\times \\ \mathrm{tr}(y)=0}} \chi(y).$$

There are $q-1$ elements $y \in \ell^\times$ such that $\mathrm{tr}(y) = 0$, and multiplying by an element of $k^\times$ preserves this property. Thus, fixing some $y_0 \in \ell^\times$ such that $\mathrm{tr}(y_0) = 0$, we see that the above summation is $(q-1)\chi(y_0)$.

It remains to evaluate the Gauss sum. To begin, we use the fact that χ is trivial on $k^\times$ to write

$$\sum_{x\in\ell^\times} \psi(\mathrm{tr}(x))\chi(x) = \sum_{x\in\ell^\times/k^\times} \chi(x) \sum_{c\in k^\times} \psi(c\,\mathrm{tr}(x)).$$

If $\mathrm{tr}(x) \neq 0$ (of which the above class shows is true for all but $x = y_0 \in \ell^\times/k^\times$), then the inner sum is a sum of ψ on $k^\times$ and so evaluates to -1. However, $\sum_{x\in\ell^\times/k^\times} \chi(x) = 0$ because χ is nontrivial, so we have

$$\sum_{x\in\ell^\times/k^\times} \chi(x) \sum_{c\in k^\times} (c\,\mathrm{tr}(x)) = (q-1)\chi(y_0) + \sum_{\substack{x\in\ell^\times/k^\times \\ x\neq y_0}} -\chi(x) = q\chi(y_0).$$

Bringing everything together, we see that $S_0 = (q-1)\chi(y_0)^2$, but $y_0^2 = -\mathrm{N}\,y_0 \in k^\times$, so actually $S_0 = (q-1)$.

Combining all cases of S_0, we see

$$S_{DwU} = \frac{\omega_\tau(-1)}{q} \sum_{x,y\in\ell^\times} \psi(\mathrm{tr}(x) + \mathrm{tr}(y))\omega_\sigma(xy)^{-1}\omega(xy^q)^{-1} - (q-1)1_{\omega_\sigma|_k = \omega_\tau|_k}.$$

Adding back in S_D, we have proven the following result.

Theorem 20. *Let σ and τ be cuspidal representations of GL_2 with central characters ω_σ and ω_τ, respectively. Then*

$$\gamma(\sigma\times\tau,\psi) = \frac{\omega_\tau(-1)}{q} \sum_{x\in\ell^\times} \psi(\mathrm{tr}\,x)\omega_\sigma(x)^{-1}\omega_\tau(x)^{-1} \sum_{y\in\ell^\times} \psi(\mathrm{tr}\,y)\omega_\sigma(y)^{-1}\omega_\tau(y^q)^{-1}.$$

Remark 21. The above work has also shown the following: let ℓ/k be a quadratic extension of finite fields, where k has order q. Let ψ be a nontrivial character on k, and let χ be a nontrivial character on $\ell^\times$ with is trivial on $k^\times$. Then

$$\sum_{x\in\ell^\times} \psi(\mathrm{tr}\,x)\chi(x) = \chi(x_0)q,$$

where $x_0 \in \ell^\times \setminus k^\times$ satisfies $x_0^2 \in k^\times$.[1]

4. Gamma Factors for $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$

In this section, we define and prove some basic properties of γ-factors of $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$. Throughout this section, k is a finite field with q elements, where q is odd.

[1] This appears in Elad's work on the Bessel function, Proposition A.2

4.1. **Review of Symplectic Spaces and Notation.** We review basic properties of symplectic spaces and define some subgroups. In this subsection, k is a field of characteristic not equal to 2.

Definition 22 (symplectic)**.** Fix a k-vector space V. A form $\langle\cdot,\cdot\rangle\colon V\times V\to k$ on V is *symplectic* if and only if $\langle\cdot,\cdot\rangle$ is bilinear, non-degenerate, and skew-symmetric. Once equipped with the symplectic form, V is called a *symplectic space*.

Note that any $v\in V$ has $\langle v,v\rangle=-\langle v,v\rangle$ and hence $\langle v,v\rangle=0$ because $\operatorname{char}k\neq 2$. Just to make the point that we can, we define the group $\mathrm{GSp}(V)$ now.

Definition 23 (symplectic group of similitudes)**.** Fix a symplectic k-vector space V. Then the *symplectic group of similitudes* $\mathrm{GSp}(V)$ is given by

$$\mathrm{GSp}(V)\coloneqq\{g\in\mathrm{GL}(V): \text{there is } \lambda(g)\in k^\times \text{ with } \langle gv,gv'\rangle=\lambda(g)\langle v,v'\rangle \text{ for } v,v'\in V\}.$$

Here, $\lambda(g)$ is called the *multiplier* of g.

This definition is perfectly adequate, but it will be helpful to have access to explicit models of symplectic spaces in the sequel. The following lemma explains how to explicitly think about symplectic spaces.

Lemma 24. *Let V be a symplectic space of finite dimension d. Then d is even, and there is a basis $\{x_1,\ldots,x_{d/2},y_1,\ldots,y_{d/2}\}$ of V such that*

$$\langle x_i,x_j\rangle=\langle y_i,y_j\rangle=0\qquad\textit{and}\qquad\langle x_i,y_j\rangle=1_{i=j}$$

for any indices i and j.

Proof. For this, we use a modified Gram–Schmidt process. Pick up any basis $\{v_1,\ldots,v_n\}$ of V, and begin with $x_1\coloneqq v_1$. Because $\langle\cdot,\cdot\rangle$ is non-degenerate, we can find some basis vector v_i such that $\langle x_1,v_i\rangle\neq 0$. Note $v_i\neq x_1$, so without loss of generality, we say $\langle x_1,v_2\rangle\neq 0$, and by scaling v_2, we may assume $\langle x_1,v_2\rangle=1$, so we set $y_1\coloneqq v_2$. Now, for each v_i with $i\geq 3$, we replace v_i with

$$v_i'\coloneqq v_i-\langle v_i,y_1\rangle x_1+\langle v_i,x_1\rangle y_1.$$

A direct computation shows that $\langle v_i',x_1\rangle=\langle v_i',y_1\rangle=0$ for each v_i', so we can repeat the above process (namely, set $x_2\coloneqq v_3$ and extract y_2 so that $\langle x_2,y_2\rangle\neq 0$ and scale) inductively. $\square$

Because the dimension of a finite-dimensional symplectic space is always even, we set the convention to say that V has dimension $2n$ for a positive integer n. Lemma 24 allows us to express each symplectic space in some standard way. In particular, writing vectors $v,v'\in V$ in terms of our basis as $v\coloneqq a_1x_1+\cdots+a_nx_n+b_1y_1+\cdots+b_ny_n$ and similarly for v', we see

$$\langle v,v'\rangle=\begin{bmatrix}a^\intercal & b^\intercal\end{bmatrix}\widehat{w}_{2n}\begin{bmatrix}a'\\ b'\end{bmatrix},$$

where

$$\widehat{w}_{2n}\coloneqq\begin{bmatrix} & -I_n\\ I_n & \end{bmatrix}.$$

Using the above as an explicit basis for k^{2n}, we can write the condition $\langle gv,gv'\rangle=\lambda(g)\langle v,v'\rangle$ for all $v,v'\in V$ as $v^\intercal g^\intercal\widehat{w}_{2n}gv'=\lambda(g)v^\intercal\widehat{w}_{2n}v'$ for all $v\in v'\in V$. Equivalently, we are asking for $g^\intercal\widehat{w}_{2n}g=\lambda(g)\widehat{w}_{2n}$, so we may explicitly define

$$\mathrm{GSp}_{2n}(k)\coloneqq\mathrm{GSp}\left(k^{2n}\right)=\left\{g\in\mathrm{GL}_n(k):\widehat{w}_ng\widehat{w}_n^{-1}=\lambda(g)g^\iota\right\}.$$

Our next point of discussion is of isotropic subspaces.

Definition 25 (isotropic)**.** Fix a symplectic k-vector space V. Then a subspace $W \subseteq V$ is *isotropic* if and only if $\langle w, w' \rangle = 0$ for any $w, w' \in W$.

Example 26. Given the symplectic space V a basis as in Lemma 24. Then we see that $X \coloneqq \operatorname{span}\{x_1, x_2, x_3\}$ is an isotropic subspace.

Lemma 27. *Fix a symplectic k-vector space V of finite dimension. Then an isotropic subspace $X \subseteq V$ is a maximal isotropic subpsace if and only if $\dim X = \frac{1}{2}\dim V$.*

Proof. Given any isotropic subspace $X \subseteq V$, we can extract any basis of X, extend it to a basis of V, and then use a modified version of the Gram–Schmidt process akin to the argument of Lemma 24 to show that X is contained in an isotropic subspace of dimension $\frac{1}{2}\dim V$, so $\dim W \leq \frac{1}{2}\dim V$. On the other hand, if X is maximal, we see equality must hold, so we conclude. $\square$

Remark 28. The proof of Lemma 27 shows that any maximal isotropic subspace X of V has a "dual" maximal isotropic subspace Y such that $V = X \oplus Y$. Indeed, this follows from letting the "rest" of the n basis vectors extracted via Lemma 24 be a basis for Y. Note that this choice of Y is not unique because extending the basis was not unique.

Thus, we will want to let $P^{\mathrm{sp}}(V) \subseteq \mathrm{GSp}(V)$ denote the subgroup fixing some given maximal isotropic subspace. In our concrete situation, we define $P^{\mathrm{sp}}_{2n}(k) \subseteq \mathrm{GSp}_{2n}(k)$ as fixing the subspace $\{x_1, \ldots, x_n\}$, so

$$P^{\mathrm{sp}}_{2n}(k) = \left\{ \begin{bmatrix} A & B \\ & D \end{bmatrix} \in \mathrm{GSp}_{2n}(k) \right\}.$$

Now, to be in $\mathrm{GSp}_{2n}(k)$, we are asking for

$$\begin{bmatrix} A^\intercal & \\ B^\intercal & D^\intercal \end{bmatrix} \widehat{w}_{2n} \begin{bmatrix} A & B \\ & D \end{bmatrix} = \begin{bmatrix} & -D^\intercal A \\ A^\intercal D & -B^\intercal D + D^\intercal B \end{bmatrix}$$

to be a multiple of $\widehat{w}_{2n}$. As such, we see that we require $D = \lambda A^\iota$ for some $\lambda \in k^\times$ and $A^{-1}B$ to be a symmetric matrix. Thus, any matrix in $P^{\mathrm{sp}}_{2n}(k)$ can be uniquely written as

$$\begin{bmatrix} \lambda A & \\ & A^\iota \end{bmatrix} \begin{bmatrix} I_n & Z \\ & I_n \end{bmatrix} = \begin{bmatrix} \lambda A & \lambda A Z \\ & A^\iota \end{bmatrix}$$

where $A \in \mathrm{GL}_n(k)$ and $\lambda \in k^\times$ and $Z \in M_n(k)$ is symmetric. This motivates us to define the subgroups

$$\begin{aligned} D^{\mathrm{sp}}_{2n}(k) &\coloneqq D_{n,n} \cap \mathrm{GSp}_{2n}(k) \\ &= \left\{ \begin{bmatrix} \lambda A & \\ & A^\iota \end{bmatrix} : A \in \mathrm{GL}_n(k) \right\}, \\ U^{\mathrm{sp}}_{2n}(k) &\coloneqq U_{n,n} \cap \mathrm{GSp}_{2n}(k) \\ &= \left\{ \begin{bmatrix} I_n & Z \\ & I_n \end{bmatrix} : Z \in k^{n \times n} \text{ is symmetric} \right\} \end{aligned}$$

so that $P_{2n}^{\mathrm{sp}}(k) = D_{2n}^{\mathrm{sp}}(k)U_{2n}^{\mathrm{sp}}(k)$. For completeness, we note that the corresponding Borel subgroup contained in $P_{2n}^{\mathrm{sp}}(k)$ is

$$B_{2n}^{\mathrm{sp}}(k) \coloneqq \left\{ \begin{bmatrix} \lambda A & \\ & A^\iota \end{bmatrix} : A \in \mathrm{GL}_n(k) \text{ is upper-triangular} \right\} \cdot U_{2n}^{\mathrm{sp}}(k).$$

For example, the diagonal matrices of $\mathrm{GSp}_{2n}(k)$ make up a maximal torus T_{2n}^{sp} of $B_{2n}^{\mathrm{sp}}(k)$, and the unipotent radical in $B_{2n}^{\mathrm{sp}}(k)$ is

$$U_{2n}^{+} \coloneqq \left\{ \begin{bmatrix} A & \\ & A^\iota \end{bmatrix} : A \in \mathrm{GL}_n(k) \text{ is upper-triangular and unipotent} \right\} \cdot U_{2n}^{\mathrm{sp}}.$$

We let U_{2n}^{-} denote the analogous family of lower-triangular matrices, namely

$$U_{2n}^{-} \coloneqq \left\{ \begin{bmatrix} A & \\ & A^\iota \end{bmatrix} : A \in \mathrm{GL}_n(k) \text{ is lower-triangular and unipotent} \right\} \cdot \{u : u^\intercal \in U_{2n}^{\mathrm{sp}}\}.$$

Continuing, we for brevity let $W(\mathrm{GSp}_{2n}(k))$, and for each $w \in W$, we define the subgroups

$$U_w^- \coloneqq U_{2n}^+ \cap wU_{2n}^- w^{-1} \qquad \text{and} \qquad U_w^+ \coloneqq U_{2n}^- \cap wU_{2n}^+ w^{-1}.$$

This allows us to define the standard intertwining operator attached to a Weyl group element $w \in W(\mathrm{GSp}_{2n}(k))$: fix a character χ of T_{2n}^{sp}, and extend it to B_{2n}^{sp} using $B_{2n}^{\mathrm{sp}} = T_{2n}^{\mathrm{sp}} \ltimes U_{2n}^{\mathrm{sp}}$. Then we define the operator $M_w \colon \operatorname{Ind}_{B_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi \to \operatorname{Ind}_{B_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} {}^w\chi$ by

$$(M_w f)(g) \coloneqq \sum_{u \in U_w^-} f\left(w^{-1}ug\right).$$

Most notable is

$$(M_{w_{2n}} f)(g) = \sum_{u \in U_{2n}^{\mathrm{sp}}} f(w_{2n}ug).$$

One can check that M_w is well-defined and G-invariant. Furthermore, it is a general fact about root systems and the Weyl groups attached to them that the multiplication map $U_w^+ \times U_w^- \to U_{2n}^+$ is a bijection. Thus, if $\ell \colon W(\mathrm{GSp}_{2n}) \to \mathbb{Z}$ denotes the length function, then $\ell(w_1) + \ell(w_2) = \ell(w_1w_2)$ implies that $M_{w_1} \circ M_{w_2} = M_{w_1w_2}$.

The whole point of investigating $\mathrm{GSp}_{2n}(k)$ is that we will be able to approximately embed $\mathrm{GL}_2(k)^n$ into $\mathrm{GSp}_{2n}(k)$. Indeed, using the basis provided by Lemma 24, we embed $\mathrm{GL}_2(k)^n \hookrightarrow \mathrm{GL}_{2n}(k)$ by having the g_i in the tuple $(g_1, \ldots, g_n) \in \mathrm{GL}_2(k)^n$ permute $\operatorname{span}\{x_i, y_i\}$. Concretely, this looks like

$$\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \ldots, \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \right) \mapsto \begin{bmatrix} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_n & & & b_n \\ c_1 & & & d_1 & & \\ & \ddots & & & \ddots & \\ & & c_n & & & d_n \end{bmatrix}.$$

Using the above as out notation for g_i, we see that $\langle g_i x_i, g_i y_i \rangle = \langle a_i x_i + c_i y_i, b_i x_i + d_i y_i \rangle = a_i d_i - b_i c_i = \det g_i$, so $\langle g_i v, g_i v' \rangle = (\det g_i)\langle v, v' \rangle$ for any $v, v' \in \operatorname{span}\{x_i, y_i\}$. It follows that we want to define the subgroup

$$\begin{aligned}\mathrm{GL}_2^{(n)}(k) &\coloneqq \mathrm{GL}_2^n(k)\cap\mathrm{GSp}_{2n}(k)\\ &= \{(g_1,\dots,g_n)\in\mathrm{GL}_2^n(k):\det g_1=\cdots=\det g_n\}.\end{aligned}$$

4.2. **Double Coset Computation.** In this subsection, we use the notation and conventions of the previous subsection. Because no confusion will arise, we will omit the field k when notating our groups. We will compute $\mathrm{GL}_2^{(n)}\backslash\mathrm{GSp}_{2n}/P_{2n}^{\mathrm{sp}}$ for $2n=6$. We start with $\mathrm{GSp}_{2n}/P_{2n}^{\mathrm{sp}}$.

Lemma 29. *Fix a symplectic vector space V of finite dimension. Then $\mathrm{GSp}(V)$ has a transitive left action on the set $\mathcal{X}(V)$ of maximal isotropic subspaces of V by left multiplication. In particular, if $P^{\mathrm{sp}}(V)$ is the subgroup fixing some maximal isotropic subspace X, then $\mathrm{GSp}(V)/P^{\mathrm{sp}}(V)$ is in natural bijection with $\mathcal{X}(V)$.*

Proof. Set $n\coloneqq\dim V$. Lemma 27 tells us that $\mathcal{X}(V)$ consists of isotropic subspaces of dimension $n/2$. Now, our left action is defined simply by translation: for $g\in\mathrm{GSp}(V)$ and $X\in\mathcal{X}(V)$, we set $g\cdot X\coloneqq gX$. Here are the checks on this action.

- Well-defined: note that gX is indeed an isotropic subspace because $g\in\mathrm{GSp}(V)$. Indeed, for any $x,x'\in X$, we see that $\langle gx,gx'\rangle=\lambda(g)\langle x,x'\rangle=0$ for some given constant $\lambda(g)\in k$. Further, $\dim gX=\dim X=\frac{1}{2}\dim V$ verifies that gX is maximal.
- Transitive: given $X,X'\in\mathcal{X}(V)$, we want $g\in\mathrm{GSp}(V)$ such that $X'=gX$. Well, via the modified Gram–Schmidt process of Lemma 24, we obtain can extend a basis of X to a basis $\{x_1,\dots,x_{n/2},y_1,\dots,y_{n/2}\}$ of V satisfying the conclusion of Lemma 24 and such that $X=\operatorname{span}\{x_1,\dots,x_{n/2}\}$. The same process for X' produces another basis $\{x_1',\dots,x_{n/2}',y_1',\dots,y_{n/2}'\}$ of V with the analogous conclusions.

  We now define $g\colon V\to V$ by $g\colon x_i\mapsto x_i'$ and $g\colon y_i\mapsto y_i'$. By construction, $gX=X'$, and we see that $g\in\mathrm{GSp}(V)$ by checking on the basis coming from X: note

  $$\langle gx_i,gx_j\rangle=\langle x_i',x_j'\rangle=0=\langle x_i,x_j\rangle,\qquad\langle gy_i,gy_j\rangle=\langle y_i',y_j'\rangle=0=\langle y_i,y_j\rangle$$

  and

  $$\langle gx_i,gy_j\rangle=\langle x_i',y_j'\rangle=1_{i=j}=\langle x_i,y_j\rangle$$

  for any indices i and j.

The above checks establish the second sentence of the lemma. The last sentence follows quickly from the Orbit–Stabilizer theorem: the bijection $\mathrm{GSp}(V)/P^{\mathrm{sp}}(V)\to\mathcal{X}(V)$ is given by $gP^{\mathrm{sp}}(V)\mapsto gX$. □

For the remainder of the subsection, even though it is not totally necessary, we will set $V\coloneqq k^{2n}$ to be a symplectic space with basis $\{x_1,\dots,x_n,y_1,\dots,y_n\}$ extracted by Lemma 24. This allows us to identify $\mathrm{GL}_2^{(n)}$ with a subgroup of GSp_n. For brevity, we set $V_i\coloneqq\operatorname{span}\{x_i,y_i\}$ so that $V=V_1\oplus\cdots\oplus V_n$; we also set $\mathcal{X}_{2n}\coloneqq\mathcal{X}(k^{2n})$.

In light of Lemma 29, we are interested in studying $\mathrm{GL}_2^{(n)}\backslash\mathcal{X}_{2n}$. The approach is to attach invariants to various isotropic subspaces in $\mathcal{X}_{2n}$ and use those to classify the orbits. Here are the relevant invariants.

Lemma 30. *Fix notation as above.*

(a) For any $g\in\mathrm{GL}_2^{(n)}$ and $X\in\mathcal{X}(V)$, we have $\dim(gX\cap V_i)=\dim(X\cap V_i)$.

(b) *For any $X \in \mathcal{X}(V)$, we have* $\dim(X \cap V_i) \in \{0, 1\}$.

Thus, $X \mapsto \dim(X \cap V_i)$ defines a function $\mathrm{GL}_2^{(n)} \backslash \mathcal{X}_n \to \{0, 1\}$.

Proof. To see (a), expand $g = (g_1, \ldots, g_n)$, and we compute

$$\dim(X \cap V_i) = \dim(gX \cap gV_i) = \dim(gX \cap \operatorname{span}\{g_i x_i, g_i y_i\}) = \dim(gX \cap V_i).$$

To see (b), we of course have $\dim(X \cap V_i) \geq 0$. On the other hand, note $\dim(X \cap V_i) \geq 2$ would imply that $V_i = (X \cap V_i) \subseteq X$, but this cannot occur because $\langle x_i, y_i \rangle = 1$ and X is isotropic. □

In light of Lemma 30, we define

$$\mathcal{X}_{(d_1,\ldots,d_n)} := \{X \in \mathcal{X}_n : \dim(X \cap V_i) = d_i \text{ for each } i\}.$$

Each $\mathcal{X}_{(d_1,\ldots,d_n)}$ provides a good candidate for an orbit when nonempty. Now, these invariants d_i are pleasant to work with because they allow an inductive process. Here is our "base case."

Lemma 31. *Fix notation as above with $2n = 2$. Then $\mathcal{X}_2 = \mathcal{X}_{(1)}$, and $\mathcal{X}_{(1)}$ is a $\mathrm{GL}_2^{(1)}$-orbit.*

Proof. Here, $V = V_1 = k^2$, so any maximal isotropic subspace $X \in \mathcal{X}_2$ will have $\dim(X \cap V_1) = \dim X = 1$, so $X \in \mathcal{X}_{(1)}$. It follows $\mathcal{X}_2 = \mathcal{X}_{(1)}$. Lastly, note that $\mathrm{GL}_2^{(1)} = \mathrm{GL}_2 = \mathrm{GSp}_2$, so the $\mathrm{GL}_2^{(1)}$-action on $\mathcal{X}_2$ is transitive by Lemma 29, so $\mathcal{X}_{(1)}$ is indeed a single orbit. □

Here is our "inductive step."

Lemma 32. *Let V be a symplectic space, and let $V = W \oplus W'$ be a decomposition of V into symplectic spaces. For any maximal isotropic subspace X of V, if $X \cap W$ is a maximal isotropic subspace of W, then*

$$X = (X \cap W) \oplus (X \cap W'),$$

and $X \cap W'$ is a maximal isotropic subspace of W'.

Proof. To see that $X = (X \cap W) \oplus (X \cap W')$, we must show that any $x \in X$ allows us to write $x = w + w'$ where $w \in X \cap W$ and $w' \in X \cap W'$. Well, we may at least write $x = w + w'$ for $w \in W$ and $w' \in W'$, and it remains to show $w, w' \in X$. Well, any $x_0 \in X \cap W$ has

$$\langle x_0, w \rangle = \langle x_0, w + w' \rangle = \langle x_0, x \rangle = 0,$$

where $\langle x_0, w' \rangle = 0$ because $V = W \oplus W'$ is a decomposition of symplectic spaces. Thus, $(X \cap W) \cup \operatorname{span}\{w\}$ is an isotropic subspace of W, so maximality assures us that $w \in X \cap W$. To finish off, we see $w' = x - w \in X$ as well.

It remains to show that $X \cap W'$ is a maximal isotropic subspace. If V is finite dimensional, one can see this by counting dimensions, but we will avoid this. Suppose $w'_0 \in W'$ satisfies $\langle w', w'_0 \rangle = 0$ for any $w' \in X \cap W'$; we must show $w'_0 \in X$. Well, for any $x \in X$, decompose $x = w + w'$ where $w \in W$ and $w' \in W'$. As above, we know $w \in X$, so $w' \in X$, so

$$\langle x, w'_0 \rangle = \langle w, w'_0 \rangle + \langle w', w'_0 \rangle = 0.$$

Thus, maximality of X implies $w'_0 \in X$, completing the proof. □

Lemma 33. *Fix notation as above with $n \geq 2$. Given some $X \in \mathcal{X}_{(d_1,\ldots,d_n)}$, if $d_n = 1$, then $X = (X \cap k^{n-2}) \oplus (X \cap V_n)$. Thus, in this case, $\mathcal{X}_{(d_1,\ldots,d_n)} \cong \mathcal{X}_{(d_1,\ldots,d_{n-1})} \times \mathcal{X}_{(1)}$ as $\mathrm{GL}_2^{(n)}$-sets.*

Proof. The second sentence follows from Lemma 32, where we are decomposing k^{2n} into $k^{2n-2} \oplus V_n$. The point is that $d_n = 1$ implies that $X \cap V_n$ is a maximal isotropic subspace of V_n.

It remains to prove the last sentence. Well, our bijection is given as follows.

$$\begin{array}{ccc} \mathcal{X}_{(d_1,\ldots,d_n)} & \cong & \mathcal{X}_{(d_1,\ldots,d_{n-1})} \times \mathcal{X}_{(1)} \\ X & \mapsto & (X \cap k^{2n-2} \ ,\ X \cap V_n) \\ X_1 \oplus X_2 & \leftarrow\!\shortmid & (X_1 \ ,\ X_2) \end{array}$$

The rightward map is well-defined by Lemma 32. Checking that the leftward map is well-defined and that the maps are inverse is direct from what we've already established.

Lastly, we must check that the bijection is an isomorphism of $\mathrm{GL}_2^{(n)}$-sets. It's enough to show that the leftward map is $\mathrm{GL}_2^{(n)}$-equivariant, for which we note

$$(g_1, \ldots, g_n)(X_1, X_2) = ((g_1, \ldots, g_{n-1}X_1, g_nX_2)$$

gets taken to $(g_1, \ldots, g_{n-1})X_1 \oplus g_nX_n$, which is indeed $(g_1, \ldots, g_n)(X_1 \oplus X_2)$. □

Corollary 34. *Fix notation as above. If $\mathcal{X}_{(d_1,\ldots,d_n)} \subseteq \mathcal{X}_{2n}$ is a nonempty $\mathrm{GL}_2^{(n)}$-orbit, then $\mathcal{X}_{(d_1,\ldots,d_n,1)} \subseteq \mathcal{X}_{2n+2}$ is a nonempty $\mathrm{GL}_2^{(n+1)}$-orbit.*

Proof. Lemma 33 grants us that

$$\mathcal{X}_{(d_1,\ldots,d_n,1)} \cong \mathcal{X}_{(d_1,\ldots,d_n)} \times \mathcal{X}_{(1)},$$

so we will show that the right-hand side is a transitive $\mathrm{GL}_2^{(n+1)}$-set. (Note the right-hand side is nonempty by hypothesis.) Well, for any two pairs (X_1, X_2) and (X_1', X_2') in $\mathcal{X}_{(d_1,\ldots,d_n)} \times \mathcal{X}_{(1)}$, we may find $g_1 \in \mathrm{GL}_2^{(n)}$ and $g_2 \in \mathrm{GL}_2$ so that $X_1' = g_1X_1$ and $X_2' = g_2X_2$. But now

$$g \coloneqq ((\det g_2)g_1, (\det g_1)g_2) \in \mathrm{GL}_2^{(n+1)}$$

has $(X_1', X_2') = g(X_1, X_2)$. □

As a starting step, we address $2n = 4$.

Proposition 35. *Fix notation as above with $2n = 4$. Then $\mathcal{X}_4 = \mathcal{X}_{(0,0)} \sqcup \mathcal{X}_{(1,1)}$, and these are $\mathrm{GL}_2^{(2)}$-orbits.*

Proof. Fix some $X \in \mathcal{X}_{(d_1,d_2)}$. We have two cases. Quickly, if $d_1 = 1$ or $d_2 = 1$, without loss of generality take $d_2 = 1$. Then Lemma 33 lets us decompose

$$\mathcal{X}_{(d_1,1)} \cong \mathcal{X}_{(d_1)} \times \mathcal{X}_{(1)},$$

but $\mathcal{X}_{(d_1)}$ must be $\mathcal{X}_{(1)}$ by Lemma 31. It follows that we are looking at $\mathcal{X}_{(1,1)}$, and $\mathcal{X}_{(1,1)}$ is an orbit by Corollary 34.

Lastly, we must deal with $\mathcal{X}_{(0,0)}$. Observe that this collection is nonempty because it contains $X_{(0,0)} \coloneqq \operatorname{span}\{x_1 - x_2, y_1 + y_2\}$, so it remains to show that it is an orbit. We will show that any $X \in \mathcal{X}_{(0,0,0)}$ is in the same orbit as $X_{(0,0)}$.

Because $X \in \mathcal{X}_{(0,0)}$, a nonzero element takes the form $v_1 + v_2$ where $v_1 \in V_1$ and $v_2 \in V_2$ are nonzero. Using an element of $\mathrm{GL}_2^{(2)}$ to move the lines $\operatorname{span}\{v_1\} \subseteq V_1$ and $\operatorname{span}\{v_2\} \subseteq V_2$ around, we may assume $c_1x_1 - c_2x_2 \in X$ for some nonzero $c_1, c_2 \in k$. Adjusting X by the element $\left(\begin{bmatrix} 1/c_1 & \\ & c_1 \end{bmatrix}, \begin{bmatrix} 1/c_2 & \\ & c_2 \end{bmatrix}\right)$, we may assume that $x_1 - x_2 \in X$.

Now, let $\{x_1 - x_2, (a_1x_1 + b_1y_1) + (a_2x_2 + b_2y_2)\}$ be a basis of X. Adjusting the second basis element by $x_1 - x_2$, we may assume that $a_2 = 0$. Checking that X is isotropic, we see that $b := b_1 = b_2$, which we see must be nonzero, so without loss of generality our basis looks like $\{x_1 - x_2, y_1 + y_2 - ax_2\}$. Adjusting X by $(\left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right])$ turns out basis into $\{x_1 - x_2, y_1 + y_2\}$, so X is in the same orbit as $X_{(0,0)}$. $\square$

We are now ready for $2n = 6$.

Proposition 36. *Fix notation as above with $2n = 6$. Then $\mathcal{X}_4 = \mathcal{X}_{(0,0,0)} \sqcup \mathcal{X}_{(1,0,0)} \sqcup \mathcal{X}_{(0,1,0)} \sqcup \mathcal{X}_{(0,0,1)} \sqcup \mathcal{X}_{(0,0,0)}$, and these are $\mathrm{GL}_2^{(2)}$-orbits.*

Proof. The argument is the same as in Proposition 35 but a little harder. Fix some $X \in \mathcal{X}_{(d_1,d_2,d_3)}$, and we have two cases. Quickly, if any of the d_i are 1, take $d_3 = 1$ without loss of generality. Then Lemma 33 lets us decompose into the $n = 4$ case, where we see

$$\mathcal{X}_{(d_1,d_2)} \in \{\mathcal{X}_{(0,0)}, \mathcal{X}_{(1,1)}\}$$

by Proposition 35. Thus, we either have $\mathcal{X}_{(0,0,1)}$ or $\mathcal{X}_{(1,1,1)}$, and each of these are orbits by Corollary 34.

It remains to deal with $\mathcal{X}_{(0,0,0)}$. Again, this is nonempty because it contains $X_{(0,0,0)} := \operatorname{span}\{x_1 - x_2, x_2 - x_3, y_1 + y_2 + y_3\}$, so it remains to show that it is an orbit. We will show that any $X \in \mathcal{X}_{(0,0,0)}$ is in the same orbit as $X_{(0,0,0)}$.

The key claim is that $\dim(X \cap (V_1 \oplus V_2)) \geq 1$. To see that the dimension is at least 1, let $\pi_3 \colon V \to V_3$ denote the projection. But then, $\dim X > \dim V_3$, so $\ker(\pi_3|_X) = X \cap (V_1 \oplus V_2)$ must be nonempty.

Thus, we may let $v_1 + v_2$ be a nonzero vector in $X \cap (V_1 \oplus V_2)$. Adjusting X by an element of $\mathrm{GL}_2^{(3)}$ as in Proposition 35, we may assume $x_1 - x_2 \in X$. A symmetric argument to the previous paragraph also allows us to let $w_2 + w_3$ be a nonzero vector in $X \cap (V_2 \oplus V_3)$. Adjusting X by an element of $\mathrm{GL}_2^{(3)}$ again to move v_3 around, we may assume our element has the form $(a_2x_2 + b_2y_2) - x_3$. However, we must have

$$\langle x_1 - x_2, (a_2x_2 + b_2y_2) - x_3 \rangle = 0,$$

so $b_2 = 0$ follows. Now, adjusting X by $\left(I_2, \left[\begin{smallmatrix} 1/a_2 & \\ & a_2 \end{smallmatrix}\right], I_2\right)$ grants $x_2 - x_3 \in X$.

Now, let a third basis vector of X be given by $v_1 + v_2 + v_3$. Adjusting this vector by $x_1 - x_2$ and $x_2 - x_3$ allows us to assume that it takes the form $c_3x_3 + d_1y_1 + d_2y_2 + d_3y_3$. Testing

$$\langle x_1 - x_2, c_3x_3 + d_1y_1 + d_2y_2 + d_3y_3 \rangle = \langle x_2 - x_3, c_3x_3 + d_1y_1 + d_2y_2 + d_3y_3 \rangle = 0$$

implies that $d := d_1 = d_2 = d_3$, and we must have $d \neq 0$ because $X \cap V_3 = \{0\}$. Thus, by scaling, we may take our vector to have the form $y_1 + y_2 + y_3 - c_3x_3$, whereupon adjusting X by $(I_2, I_2, \left[\begin{smallmatrix} 1 & c_3 \\ & 1 \end{smallmatrix}\right])$ grants $y_1 + y_3 + y_3 \in X$. It follows that $X = X_{(0,0,0)}$. $\square$

In the sequel, it will be helpful to have explicit representatives for $P_6^{\mathrm{sp}} \backslash \operatorname{GSp}_6 / \mathrm{GL}_2^{(3)}$. We follow [Ike89, Lemma 1.1].

Corollary 37. *We have the following representatives of $P_6^{\mathrm{sp}} \backslash \operatorname{GSp}_6 / \mathrm{GL}_2^{(3)}$.*

(a) *The element*

$$\eta_0 := \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

has $\eta_0^{-1}X \in \mathcal{X}_{(0,0,0)}$. *Further, the "stabilizer"* $\mathrm{GL}_2^{(3)} \cap \eta_0^{-1}P_6^{\mathrm{sp}}\eta_0$ *is the subgroup*

$$S(\eta_0) := \left\{ \left(\begin{bmatrix} a & b_1 \\ & d \end{bmatrix}, \begin{bmatrix} a & b_2 \\ & d \end{bmatrix}, \begin{bmatrix} a & b_3 \\ & d \end{bmatrix} \right) \in \mathrm{GL}_2^{(3)} \; : b_1 + b_2 + b_3 = 0 \right\}.$$

(b) *The element*

$$\eta_1 := \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

has $\eta_1^{-1}X \in \mathcal{X}_{(0,0,1)}$. *Further, the "stabilizer"* $\mathrm{GL}_2^{(3)} \cap \eta_1^{-1}P_6^{\mathrm{sp}}\eta_1$ *is the subgroup*

$$S(\eta_1) := \left\{ \left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_1 & -b_1 \\ -c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_3 & b_3 \\ & d_3 \end{bmatrix} \right) \in \mathrm{GL}_2^{(3)} \right\}.$$

Rearranging the rows and columns appropriately produces elements η_2 *and* η_3 *with* $\eta_2^{-1}X \in \mathcal{X}_{(0,1,0)}$ *and* $\eta_3^{-1}X \in \mathcal{X}_{(1,0,0)}$.

(c) *The element* $\eta_5 := I_6$ *has* $\eta_5^{-1}X \in \mathcal{X}_{(1,1,1)}$. *Further, the "stabilizer"* $\mathrm{GL}_2^{(3)} \cap \eta_5^{-1}P_6^{\mathrm{sp}}\eta_5$ *is the subgroup*

$$S(\eta_5) := \left\{ \left(\begin{bmatrix} a_1 & b_1 \\ & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ & d_2 \end{bmatrix}, \begin{bmatrix} a_3 & b_3 \\ & d_3 \end{bmatrix} \right) \in \mathrm{GL}_2^{(3)} \right\}.$$

Proof. We work with each class one at a time. We omit the checks that each η_i lives in GSp_6.

(a) Note

$$\eta_0^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

so $\eta_0^{-1}X$ is spanned by $\{-y_1 - y_2 + y_3, x_2 - x_1, x_3 + x_2\}$. Adjusting η_0^{-1} by $(I_2, I_2, -I_2)$ produces the element $X_{(0,0,0)}$ constructed in Proposition 36, so the first assertion follows.

For the stabilizer computation, we set $g := \left(\left[\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right]\right)$ and compute

$$\eta_0 g \eta_0^{-1} = \begin{bmatrix} d_1 & c_1 & 0 & -c_1 & 0 & 0 \\ -b_2 - b_3 & a_2 & a_2 - a_3 & 0 & b_2 + b_3 & -b_3 \\ b_3 & 0 & a_3 & 0 & -b_3 & b_3 \\ -b_1 - b_2 - b_3 & a_2 - a_1 & a_2 - a_3 & a_1 & b_2 + b_3 & -b_3 \\ d_1 - d_2 & c_1 + c_2 & c_2 & -c_1 & d_2 & 0 \\ d_3 - d_2 & c_2 & c_2 + c_3 & 0 & d_2 - d_3 & d_3 \end{bmatrix}. \tag{4.2.1}$$

Thus, $\eta_0 g \eta_0^{-1} \in P_6^{\mathrm{sp}}$ if and only if $g \in S(\eta_0)$.

(b) We will only prove the assertions involving η_1; the proofs of the others follow by rearranging the basis. Note

$$\eta_1^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

so $\eta_1^{-1}X$ is spanned by $\{-y_1 - y_2, -x_1 + x_2, x_3\}$, which we can see is $\operatorname{span}\{x_1 - x_2, y_1 + y_2\} \oplus \operatorname{span}\{x_3\}$ and thus in $\mathcal{X}_{(0,0,1)}$. (Note $\operatorname{span}\{x_1 - x_2, y_1 + y_2\} \in \mathcal{X}_{(0,0)}$ as in Proposition 35.)

For the stabilizer computation, we we set $g := \left(\left[\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right]\right)$ and compute

$$\eta_1 g \eta_1^{-1} = \begin{bmatrix} d_1 & c_1 & 0 & -c_1 & 0 & 0 \\ -b_2 & a_2 & 0 & 0 & b_2 & 0 \\ 0 & 0 & a_3 & 0 & 0 & b_3 \\ -b_1 - b_2 & -a_1 + a_2 & 0 & a_1 & b_2 & 0 \\ d_1 - d_2 & c_1 + c_2 & 0 & -c_1 & d_2 & 0 \\ 0 & 0 & c_3 & 0 & 0 & d_3 \end{bmatrix}. \tag{4.2.2}$$

Thus, $\eta_1 g \eta_1^{-1} \in P_6^{\mathrm{sp}}$ if and only if $g \in S(\eta_1)$.

(c) All assertions follow directly from the fact that η_5 is the identity. $\square$

Remark 38. Though it is not clear from the computation, we use the term "stabilizer" for $S(\eta_i)$ because $S(\eta_i)$ consists of the elements of $\mathrm{GL}_2^{(3)}$ fixing some isotropic subspace in the corresponding class of $\mathrm{GL}_2^{(3)} \backslash \mathcal{X}_6$. We have chosen a more explicit exposition because it will be helpful to have explicit matrices computed later on.

4.3. **Multiplicity One.** In this subsection, we prove a multiplicity one result which will become the functional equation. For the rest of this section, k will be a finite field, and π_1, π_2, π_3 are irreducible representations of GL_2; note that $\pi_1 \otimes \pi_2 \otimes \pi_3$ is a representation of GL_2^3 and hence of $\mathrm{GL}_2^{(3)}$ by restriction. For each i, we let ω_i denote the central character of π_i, and we set $\omega := \omega_1\omega_2\omega_3$. Using the decomposition $P_6^{\mathrm{sp}} = D_6^{\mathrm{sp}} U_6^{\mathrm{sp}}$, we define the characters χ_0 and χ_1 on P_6^{sp} by

$$\chi_0 \colon \begin{bmatrix} \lambda A & * \\ & A^\iota \end{bmatrix} \mapsto \lambda \qquad \text{and} \qquad \chi_1 \colon \begin{bmatrix} \lambda A & * \\ & A^\iota \end{bmatrix} \mapsto \det A.$$

(Notably, χ_0 is the restriction of the multiplier character $m\colon \mathrm{GSp}_6 \to \mathbb{C}^\times$.) We now set $\widetilde{\omega} := \omega \circ \chi_0\chi_1$ and $I(\omega) := \operatorname{Ind}_{P_6^{\mathrm{sp}}}^{\mathrm{GSp}_6}(\widetilde{\omega})$.

Example 39. The definition of $\widetilde{\omega}$ is perhaps a little strange. As an example computation, we note any $c \in k^\times$ yields

$$\widetilde{\omega}(cI_6) = \widetilde{\omega}\left(\begin{bmatrix} c^2(1/c \cdot I_3) & \\ & (1/c \cdot I_3)^\iota \end{bmatrix}\right) = \omega\left(c^2 \det(1/c \cdot I_3)\right) = \omega(c)^{-1}.$$

Computations of $\widetilde{\omega}$ (of which we will do many below) tend to look like this.

The goal of the present subsection is to prove the following result.

Theorem 40. *Fix notation as above. Suppose one of the following holds.*

- *Permutations of the following condition: π_1 is cuspidal, and $\pi_1 \ncong \pi_2^\vee$, and $\pi_1 \ncong \pi_3^\vee$.*
- *Permutations of the following condition: π_1 and π_2 are cuspidal, and $\pi_1 \ncong \pi_2^\vee$.*
- *Each π_i is cuspidal.*

Then

$$\dim \operatorname{Hom}_{\mathrm{GL}_2^{(3)}}(I(\omega) \otimes \pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) \le 1.$$

To begin, we make the following observation to allow us to use Frobenius reciprocity.

Lemma 41. *For any group G with subgroups H_1 and H_2, any representation ρ of H_1 has the decomposition*

$$\operatorname{Res}_{H_2}^G \operatorname{Ind}_{H_1}^G \rho \cong \bigoplus_{\eta \in H_1 \backslash G / H_2} \operatorname{Ind}_{H_2 \cap \eta^{-1} H_1 \eta}^{H_2} \rho_\eta,$$

where $\rho_\eta(g) := \rho\left(\eta g \eta^{-1}\right)$.

Proof. The forward map sends $f \in \operatorname{Ind}_{H_1}^G \rho$ to $(f_\eta)_\eta$ where $f_\eta(h_2) := f(\eta h_2)$ for any $h_2 \in H_2$. To see that this map is well-defined, note any $h \in H_2 \cap \eta^{-1} H_1 \eta$ has $f_\eta(hh_2) = f(\eta h h_2) = \rho\left(\eta h \eta^{-1}\right) f(\eta h_2) = \rho_\eta(h) f_\eta(h_2)$. All group actions are translation on the right, so this map is H_2-invariant as well.

Continuing, the backward map send $(f_\eta)_\eta$ to f defined by

$$f(h_1 \eta h_2) := \rho(h_1) f_\eta(h_2)$$

for any $h_1 \in H_1$ and $h_2 \in H_2$. To see that this is well-defined, note $h_1 \eta h_2 = h_1' \eta h_2'$ implies $\eta^{-1} h_1^{-1} h_1' \eta = h_2 (h_2')^{-1}$, so this element is in $H_2 \cap \eta^{-1} H_1 \eta$, so we see

$$\rho(h_1) f_\eta(h_2) = \rho(h_1) \rho_\eta\left(h_2 (h_2')^{-1}\right) f_\eta(h_2') = \rho(h_1') f_\eta(h_2').$$

Continuing, by construction, we see that $f \in \operatorname{Ind}_{H_1}^G \rho$, and this map is in fact inverse to the forward map, so we have exhibited the needed isomorphism. $\square$

The above lemma allows us to use Frobenius reciprocity to write

$$\begin{aligned}
&\operatorname{Hom}_{\mathrm{GL}_2^{(3)}}(I(\omega)\otimes\pi_1\otimes\pi_2\otimes\pi_3,\mathbb{C})\\
\cong&\bigoplus_{\eta\in P_6^{\mathrm{sp}}\backslash\,\mathrm{GSp}_6\,/\,\mathrm{GL}_2^{(3)}}\operatorname{Hom}_{\mathrm{GL}_2^{(3)}}\left(\operatorname{Ind}_{\mathrm{GL}_2^{(3)}\cap\eta^{-1}P_6^{\mathrm{sp}}\eta}^{\mathrm{GL}_2^{(3)}}\widetilde{\omega}_\eta\otimes\pi_1\otimes\pi_2\otimes\pi_3,\mathbb{C}\right)\\
\cong&\bigoplus_{\eta\in P_6^{\mathrm{sp}}\backslash\,\mathrm{GSp}_6\,/\,\mathrm{GL}_2^{(3)}}\operatorname{Hom}_{\mathrm{GL}_2^{(3)}\cap\eta^{-1}P_6^{\mathrm{sp}}\eta}\left(\widetilde{\omega}_\eta\otimes\pi_1\otimes\pi_2\otimes\pi_3,\mathbb{C}\right)\\
\cong&\bigoplus_{\eta\in P_6^{\mathrm{sp}}\backslash\,\mathrm{GSp}_6\,/\,\mathrm{GL}_2^{(3)}}\operatorname{Hom}_{\mathrm{GL}_2^{(3)}\cap\eta^{-1}P_6^{\mathrm{sp}}\eta}\left(\pi_1\otimes\pi_2\otimes\pi_3,\widetilde{\omega}_\eta^{-1}\right)\\
\cong&\bigoplus_{i=1}^{5}\operatorname{Hom}_{S(\eta_i)}\left(\pi_1\otimes\pi_2\otimes\pi_3,\widetilde{\omega}_{\eta_i}^{-1}\right).
\end{aligned}$$

We now go through and examine $\operatorname{Hom}_{S(\eta_i)}\left(\pi_1\otimes\pi_2\otimes\pi_3,\widetilde{\omega}_{\eta_i}^{-1}\right)$ for each η_i.

Lemma 42. *The representation $\pi:=\operatorname{Ind}_{U_2}^{P_2}\psi_2$ is naturally isomorphic to the vector space of functions $f\colon k^\times\to\mathbb{C}$ with P_2-action given by*

$$\left(\begin{bmatrix}a&b\\&1\end{bmatrix}f\right)(x)=\psi(bx)f(ax).$$

Proof. By definition of π, a function $f\in\pi$ is uniquely determined by its values $f\left(\left[\begin{smallmatrix}x&\\&1\end{smallmatrix}\right]\right)$ for $x\in k^\times$ because $f\left(\left[\begin{smallmatrix}a&b\\&1\end{smallmatrix}\right]\right)=\psi(b)f\left(\left[\begin{smallmatrix}a&\\&1\end{smallmatrix}\right]\right)$, so we may regard each $f\in\pi$ as a function on $k^\times$. To finish, we track through the P_2-action as

$$\left(\begin{bmatrix}a&b\\&1\end{bmatrix}f\right)(x)=f\left(\begin{bmatrix}x&\\&1\end{bmatrix}\begin{bmatrix}a&b\\&1\end{bmatrix}\right)=f\left(\begin{bmatrix}1&bx\\&1\end{bmatrix}\begin{bmatrix}ax&\\&1\end{bmatrix}\right)=\psi(bx)f(ax),$$

which completes the proof. ☐

Lemma 43. *Fix everything as above. Assume that at least one of the π_i is cuspidal. Then*

$$\dim\operatorname{Hom}_{S(\eta_0)}\left(\pi_1\otimes\pi_2\otimes\pi_3,\widetilde{\omega}_{\eta_0}^{-1}\right)\le 1.$$

Proof. Without loss of generality, say that π_1 is cuspidal. It will make no difference in the argument, so immediately restrict our attention to the subgroup

$$P:=\left\{\left(\begin{bmatrix}a&b_1\\&1\end{bmatrix},\begin{bmatrix}a&b_2\\&1\end{bmatrix},\begin{bmatrix}a&b_3\\&1\end{bmatrix}\right):b_1+b_2+b_3=0\right\}\subseteq S(\eta_0),$$

and in fact for much of the argument we will use

$$N:=\left\{\left(\begin{bmatrix}1&b_1\\&1\end{bmatrix},\begin{bmatrix}1&b_2\\&1\end{bmatrix},\begin{bmatrix}1&b_3\\&1\end{bmatrix}\right):b_1+b_2+b_3=0\right\}\subseteq P.$$

Using (4.2.1), we can compute that $\widetilde{\omega}_{\eta_0}$ vanishes on N. Now, because our representations π_i are higher-dimensional, we may write $\operatorname{Res}_{P_2}\pi_i=\pi\oplus J(\pi_i)$ where $\pi:=\operatorname{Ind}_{U_2}^{P_2}\psi_2$ and $J(\pi_i)$ is the Jacquet module of U_2-invariants; notably, $J(\pi_1)=0$. Thus, by expanding out the tensor product, we have the following cases.

- We show $\dim \operatorname{Hom}_P(\pi \otimes \pi \otimes \pi, \mathbb{C}) \leq 1$. We argue explicitly; namely, we claim that a P-linear map $T\colon \pi \otimes \pi \otimes \pi \to \mathbb{C}$ is uniquely determined by $T(1_1, 1_1, 1_1) \in \mathbb{C}$, where 1_1 denotes the 1-indicator. Here, we are using Lemma 42's description of π.

  By linearity, T is determined by its values on indicators $T(1_{a_1}, 1_{a_2}, 1_{a_3})$. Now, we see
$$\left(\begin{bmatrix} a & b \\ & 1 \end{bmatrix} 1_{a_i}\right)(x) = \psi(bx)1_{a_i}(ax) = \psi(bx)1_{a_i/a}(x) = \psi(ba_i/a)1_{a_i/a}(x).$$
  Thus, for example, if $a_1 \neq a_2$, we may find (b_1, b_2, b_3) such that $b_1 + b_2 + b_3 = 0$ while $\psi(a_1b_1 + a_2b_2 + a_3b_3) \neq 0$; explicitly, set $b_3 = 0$ and $b_2 = -a_1b_1/a_2$ while letting b_1 vary. Then the element $\left(\left[\begin{smallmatrix} 1 & b_1 \\ & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & b_2 \\ & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & b_3 \\ & 1 \end{smallmatrix}\right]\right)$ implies that $T(1_{a_1}, 1_{a_2}, 1_{a_3}) = 0$. An analogous argument shows that $a_2 \neq a_3$ forces $T(1_{a_1}, 1_{a_2}, 1_{a_3}) = 0$.

  Thus, T is determined by its values on $T(1_a, 1_a, 1_a)$. But the above work shows that
$$T(1_a, 1_a, 1_a) = T\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix} 1_a, \begin{bmatrix} a & \\ & 1 \end{bmatrix} 1_a, \begin{bmatrix} a & \\ & 1 \end{bmatrix} 1_a\right) = T(1_1, 1_1, 1_1),$$
  so T is indeed uniquely determined by $T(1_1, 1_1, 1_1)$.
- We show $\dim \operatorname{Hom}_N(\pi \otimes \pi \otimes J(\pi_3), \mathbb{C}) = 0$. Well, fix some N-linear map $T\colon \pi \otimes \pi \otimes J(\pi_3) \to \mathbb{C}$. Because $J(\pi_3)$ is U_2-invariant, we find
$$T\left(\begin{bmatrix} 1 & b_1 \\ & 1 \end{bmatrix} v_1, \begin{bmatrix} 1 & b_2 \\ & 1 \end{bmatrix} v_2, v_3\right) = T\left(\begin{bmatrix} 1 & b_1 \\ & 1 \end{bmatrix} v_1, \begin{bmatrix} 1 & b_2 \\ & 1 \end{bmatrix} v_2, \begin{bmatrix} 1 & -b_1 - b_2 \\ & 1 \end{bmatrix} v_3\right) = T(v_1, v_2, v_3)$$
  for any (v_1, v_2, v_3) and $b_1, b_2 \in k$. Thus, we can view T as a function $J(\pi_3) \to \operatorname{Hom}_{U_2 \times U_2}(\pi \otimes \pi, \mathbb{C})$. But this target is zero-dimensional: note
$$\operatorname{Res}^{P_2 \times P_2}_{U_2 \times U_2}(\pi \otimes \pi) = \operatorname{Res}^{P_2}_{U_2} \pi \otimes \operatorname{Res}^{P_2}_{U_2} \pi = \bigoplus_{\substack{\psi', \psi'' \in \widehat{k^\times} \\ \psi', \psi'' \neq 1}} (\psi' \otimes \psi'')$$
  by expanding out the tensor product. Thus, we see that there are no $(U_2 \times U_2)$-eigenvectors with eigenvalue 1, so $\dim \operatorname{Hom}_{U_2 \times U_2}(\mathbb{C}, \pi \otimes \pi) = 0$.
- We show $\dim \operatorname{Hom}_N(\pi \otimes J(\pi_2) \otimes J(\pi_3)) = 0$. Arguing as above, an N-linear map $T\colon \pi \otimes J(\pi_2) \otimes J(\pi_3) \to \mathbb{C}$ can be thought of as a map $J(\pi_1) \otimes J(\pi_2) \to \operatorname{Hom}_{U_2}(\pi, \mathbb{C})$. However, we see $\dim \operatorname{Hom}_{U_2}(\pi, \mathbb{C}) = 0$ from decomposing $\operatorname{Res}^{P_2}_{U_2} \pi = \bigoplus_{\psi' \in \widehat{k^\times}, \psi' \neq 1} \psi'$.

Summing the above cases (and their permutations) completes the proof. $\square$

Lemma 44. *Fix everything as above. Assume that one of the following conditions holds.*

- π_3 *is cuspidal.*
- $\pi_1 \not\cong \pi_2^\vee$.

Then $\dim \operatorname{Hom}_{S(\eta_1)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \widetilde{\omega}_{\eta_1}^{-1}) = 0$.

Proof. Quickly, we use (4.2.2) to compute $\widetilde{\omega}_{\eta_1}^{-1}$ on $S(\eta_1)$ to see that the symmetry condition on $T \in \operatorname{Hom}_{S(\eta_1)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \widetilde{\omega}_{\eta_1}^{-1})$ is
$$T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} v_1, \begin{bmatrix} a_1 & -b_1 \\ -c_1 & d_1 \end{bmatrix} v_2, \begin{bmatrix} a_3 & b_3 \\ & d_3 \end{bmatrix} v_3\right) = \omega(d_3)T(v_1, v_2, v_3).$$
We now argue each case independently.

- Suppose π_3 is cuspidal so that $\operatorname{Res}_{P_2}^{\mathrm{GL}_2}\pi_3 = \pi$. As above, any $S(\eta_1)$-linear map $T\colon \pi_1\otimes\pi_2\otimes\pi_3 \to \widetilde{\omega}_{\eta_1}^{-1}$ can be thought of as a map $\pi_1\otimes\pi_2 \to \operatorname{Hom}_{B_2}(\pi_3,(1,\omega))$, where $(1,\omega)\colon B_2\to\mathbb{C}$ is given by $(1,\omega)\left[\begin{smallmatrix} a_3 & b_3 \\ & d_3\end{smallmatrix}\right] = \omega(d_3)$.

  However, $\operatorname{Hom}_{B_2}(\pi_3,(1,\pi)) \subseteq \operatorname{Hom}_{P_2}(\pi_3,\mathbb{C}) = \operatorname{Hom}_{P_2}(\pi,\mathbb{C})$ is zero-dimensional because of the decomposition $\operatorname{Res}_{U_2}^{P_2}\pi = \bigoplus_{\psi'\in\widehat{k^\times},\psi'\neq 1}\psi'$.
- Suppose $\pi_1 \not\cong \pi_2^\vee$. As above, any $S(\eta_1)$-linear map $T\colon \pi_1\otimes\pi_2\otimes\pi_3\to\widetilde{\omega}_{\eta_1}^{-1}$ can be thought of as a map $\pi_3\to\operatorname{Hom}_{\mathrm{GL}_2}(\pi_1\otimes\pi_2,\mathbb{C})$ where GL_2 acts on $\pi_1\otimes\pi_2$ by

$$\begin{bmatrix} a & b \\ c & d\end{bmatrix}(v_1\otimes v_2) = \begin{bmatrix} a & b \\ c & d\end{bmatrix} v_1 \otimes \begin{bmatrix} a & -b \\ -c & d\end{bmatrix} v_2.$$

  Now, we claim $\operatorname{Hom}_{\mathrm{GL}_2}(\pi_1\otimes\pi_2,\mathbb{C}) = 0$, which will complete the argument. Well, set $w := \left[\begin{smallmatrix} -1 & \\ & 1\end{smallmatrix}\right]$, and the isomorphism $\pi_2\to\pi_2$ by $\pi_2(g)\mapsto\pi_2(wgw)$ sends the above GL_2-action on $\pi_1\otimes\pi_2$ to the diagonal action

$$\begin{bmatrix} a & b \\ c & d\end{bmatrix}(v_1\otimes v_2) = \begin{bmatrix} a & b \\ c & d\end{bmatrix} v_1 \otimes \begin{bmatrix} a & b \\ c & d\end{bmatrix} v_2.$$

  Thus, $\operatorname{Hom}_{\mathrm{GL}_2}(\pi_1\otimes\pi_2,\mathbb{C}) \cong \operatorname{Hom}_{\mathrm{GL}_2}(\pi_1,\pi_2^\vee)$, where everything has the standard GL_2-action. Because $\pi_1\not\cong\pi_2^\vee$, we see that $\operatorname{Hom}_{\mathrm{GL}_2}(\pi_1,\pi_2^\vee)$ vanishes, so we are done. $\square$

Lemma 45. *Fix everything as above. Assume that at least one of the π_i is cuspidal. Then*

$$\dim\operatorname{Hom}_{S(\eta_5)}\left(\pi_1\otimes\pi_2\otimes\pi_3,\widetilde{\omega}_{\eta_5}^{-1}\right) = 0.$$

Proof. Without loss of generality, suppose π_1 is cuspidal. We immediately restrict to the subgroup $U_2\times U_2\times U_2\subseteq S(\eta_5)$, upon which $\widetilde{\omega}_{\eta_5}^{-1}$ is trivial. Now, a $(U_2\times U_2\times U_2)$-linear map $T\colon\pi_1\otimes\pi_2\otimes\pi_3\to\mathbb{C}$ can be thought of as a map $T\colon\pi_2\otimes\pi_3\to\operatorname{Hom}_{U_2}(\pi_1,\mathbb{C})$. However, $\operatorname{Hom}_{U_2}(\pi_1,\mathbb{C}) = 0$ because of the decomposition $\operatorname{Res}_{U_2}^{\mathrm{GL}_2}\pi_1\cong\operatorname{Res}_{U_2}^{P_2}\pi\cong\bigoplus_{\psi'\in\widehat{k^\times},\psi'\neq1}\psi'$. $\square$

Combining Lemmas 43 to 45 (and their natural permutations) proves Theorem 40.

4.4. **Normalizing the Intertwining Operator.** Let $2n$ be a positive even integer. At this point, we recognize that $(M_{w_{2n}}\circ M_{w_{2n}})\colon I(\omega)\to I(\omega)$, so one might hope that this composite is a scalar and then to compute this scalar. However, there are cases (which we will discuss later on) where $I(\omega)$ fails to be irreducible, so we cannot expect $M_{w_{2n}}\circ M_{w_{2n}}$ to be a scalar. With that said, there is a reasonably large subrepresentation of $I(\omega)$ upon which $M_{w_{2n}}\circ M_{w_{2n}}$ is behaved.

Before going into the following statements and proofs, we define some notation. Given some finite-dimensional k-vector space V and operator $T\in\mathrm{GL}(V)$, we define the character $\psi_T\colon\operatorname{End}(V)\to k^\times$ by

$$\psi_T(A) := \psi(\operatorname{tr}(AT)).$$

In our application, T will be an inverible symmetric matrix in GL_n, and we will view ψ_T as a character of U_{2n}^{sp} by mapping $U_{2n}^{\mathrm{sp}}\to k^{n\times n}$ by $\left[\begin{smallmatrix} I_n & A \\ & I_n\end{smallmatrix}\right]\mapsto A$. Now, the main point of introducting ψ_T is to achieve a multiplicity-one result of eigenvectors with eigenvalue ψ_T. Before stating our multiplicity one result, it will be helpful to understand characters on P_{2n}^{sp}.

Lemma 46. *Let $\chi\colon P_{2n}^{\mathrm{sp}}\to\mathbb{C}^\times$ be a character. Then χ is trivial on the subgroup*

$$\left\{\begin{bmatrix} A & \\ & A^\iota\end{bmatrix}\begin{bmatrix} 1 & Z \\ & 1\end{bmatrix} : \det A = 1, Z\in\operatorname{Sym}_n(k)\right\}.$$

Proof. We will show that the subgroup above is contained in the commutator subgroup. We proceed in steps.

(1) We show that χ is trivial on U_{2n}^{sp}. Well, for any $u := \left[\begin{smallmatrix} 1 & Z \\ & 1 \end{smallmatrix}\right]$ in U_{2n}^{sp} where $Z \in \mathrm{Sym}_n(k)$, we use the fact that $2 \in k^\times$ to note that

$$\begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & Z \\ & 1 \end{bmatrix} \begin{bmatrix} 2 & \\ & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & Z \\ & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 2Z \\ & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -(1/2)Z \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & Z \\ & 1 \end{bmatrix},$$

so u is a commutator.

(2) We show that χ is trivial on the subgroup of matrices of the form $\left[\begin{smallmatrix} A & \\ & A^\iota \end{smallmatrix}\right]$ for $A \in \mathrm{SL}_n(k)$. Well, it is well-known that the commutator subgroup of $\mathrm{GL}_n(k)$ is $\mathrm{SL}_n(k)$ (in the case that, say, k has odd characteristic), so we can find $B, C \in \mathrm{GL}_n(k)$ such that $A = BCB^{-1}C^{-1}$. It follows that

$$\begin{bmatrix} A & \\ & A^\iota \end{bmatrix} = \begin{bmatrix} B & \\ & B^\iota \end{bmatrix} \begin{bmatrix} C & \\ & C^\iota \end{bmatrix} \begin{bmatrix} B & \\ & B^\iota \end{bmatrix}^{-1} \begin{bmatrix} C & \\ & C^\iota \end{bmatrix}^{-1},$$

so $\left[\begin{smallmatrix} A & \\ & A^\iota \end{smallmatrix}\right]$ is a commutator.

The above two cases complete the proof. □

Remark 47. Another way to state Lemma 46 is that any character $\chi\colon P_{2n}^{\mathrm{sp}} \to \mathbb{C}^\times$ factors through $(m, \chi_{\det})\colon P_{2n}^{\mathrm{sp}} \to \mathbb{F}_q^\times \times \mathbb{F}_q^\times$, where m is the multiplier character, and $\chi_{\det}$ is the "Siegel determinant" defined by

$$\chi_{\det}\left(\begin{bmatrix} \lambda A & \\ & A^\iota \end{bmatrix} \begin{bmatrix} 1 & Z \\ & 1 \end{bmatrix}\right) := \det A.$$

In other words, there are characters $\alpha_\chi, \beta_\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ such that $\chi = (\alpha_\chi \circ m)(\beta_\chi \circ \chi_{\det})$.

Example 48. Let $\chi\colon P_{2n}^{\mathrm{sp}} \to \mathbb{C}^\times$ be a character of the form $\chi = (\alpha_\chi \circ m)(\beta_\chi \circ \chi_{\det})$ where $\alpha_\chi, \beta_\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ are characters. Then we compute

$$\begin{aligned} {}^{(w_{2n})}\chi\left(\begin{bmatrix} \lambda A & \\ & A^\iota \end{bmatrix} \begin{bmatrix} 1 & Z \\ & 1 \end{bmatrix}\right) &= \chi\left(w_{2n} \begin{bmatrix} \lambda A & \\ & A^\iota \end{bmatrix} w_{2n}\right) \\ &= \chi\left(\begin{bmatrix} w_n A^\iota w_n & \\ & \lambda w_n A w_n \end{bmatrix}\right) \\ &= \alpha_\chi(\lambda)\beta_\chi(\lambda)^{-n}\beta_\chi(\det A)^{-1}. \end{aligned}$$

Thus, ${}^{(w_{2n})}\chi = \left(\alpha_\chi \beta_\chi^{-n} \circ m\right)\left(\beta_\chi^{-1} \circ \chi_{\det}\right)$.

We now state our multiplicity one result.

Proposition 49. *Fix notation as above. For any invertible symmetric matrix $T \in \mathrm{GL}_n$ and character $\chi\colon P_{2n}^{\mathrm{sp}} \to \mathbb{C}^\times$, we have*

$$\dim \mathrm{Hom}_{U_{2n}^{\mathrm{sp}}}\left(\mathrm{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi, \psi_T\right) = 1,$$

where ψ_T is a character on U_{2n}^{sp} as described above.

Proof. We use Mackey theory. To begin, we use Frobenius reciprocity and Lemma 41 to note

$$\begin{aligned}\operatorname{Hom}_{U_{2n}^{\mathrm{sp}}}\left(\operatorname{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}}\chi,\psi_T\right) &\cong \bigoplus_{\eta\in P_{2n}^{\mathrm{sp}}\backslash\mathrm{GSp}_{2n}/U_{2n}^{\mathrm{sp}}}\operatorname{Hom}_{U_{2n}^{\mathrm{sp}}}\left(\operatorname{Ind}_{U_{2n}^{\mathrm{sp}}\cap\eta^{-1}P_{2n}^{\mathrm{sp}}\eta}\chi_\eta,\psi_T\right)\\ &\cong \bigoplus_{\eta\in P_{2n}^{\mathrm{sp}}\backslash\mathrm{GSp}_{2n}/U_{2n}^{\mathrm{sp}}}\operatorname{Hom}_{U_{2n}^{\mathrm{sp}}\cap\eta^{-1}P_{2n}^{\mathrm{sp}}\eta}\left(\chi_\eta,\psi_T\right).\end{aligned}\tag{4.4.1}$$

To continue this argument, we need a rough idea what $P_{2n}^{\mathrm{sp}}\backslash\mathrm{GSp}_{2n}/U_{2n}^{\mathrm{sp}}$ is, for which we use the Bruhat decomposition

$$\mathrm{GSp}_{2n}=\bigsqcup_{w\in W(\mathrm{GSp}_{2n})}B_{2n}^{\mathrm{sp}}wB_{2n}^{\mathrm{sp}},$$

where $W(\mathrm{GSp}_{2n})$ is the Weyl group. In particular, we want to understand the Weyl group.

Lemma 50. *Let $2n$ be an even positive integer. Let Σ_{2n} be the set of permutations $\sigma\in S_{2n}$ such that $\sigma(i+n)\equiv\sigma(i)+n\pmod{2n}$ for each i.*

- *(a) For each w representing a class in $W(\mathrm{GSp}_{2n})$, there exists a unique permutation $\sigma\in\Sigma_{2n}$ such that $w=d\sigma$ for some diagonal matrix d.*
- *(b) For each $\sigma\in\Sigma_{2n}$, there exists some diagonal matrix d with entries in $\{\pm1\}$ such that $d\sigma\in\mathrm{GSp}_n$. In fact, $d=\operatorname{diag}(d_1,d_2,\ldots,d_{2n})$ is uniquely determined by the values $\{d_{\sigma(1)},\ldots,d_{\sigma(n)}\}$.*

Proof. We will show the parts independently.

(a) Recalling that the diagonal matrices of GSp_{2n} make up a maximal torus in B_{2n}^{sp}, we note that diagonal matrices are normalized by the semidirect product of permutation matrices and diagonal matrices (this is even true in GL_{2n}), so we can view elements of $W(\mathrm{GSp}_{2n})$ as permutation matrices with elements adjusted by a diagonal element to lie in GSp_{2n}.

In particular, we may write $w=d\sigma$ for some diagonal matrix d, and this σ is unique. It remains to show $\sigma\in\Sigma_{2n}$. Well, the main point is that $d\sigma\in\mathrm{GSp}_{2n}$ requires

$$d\sigma\widehat{w}_{2n}\sigma^{\intercal}d^{\intercal}=\widehat{w}_{2n}.$$

Setting $d\coloneqq\operatorname{diag}(d_1,\ldots,d_{2n})$, we now pass through a basis vector $e_{\sigma(i)}$ to compute

$$(-1)^{1_{i>n}}d_{\sigma(i+n)}d_{\sigma(i)}e_{\sigma(i+n)}=(-1)^{1_{\sigma(i)>n}}e_{\sigma(i)+n},\tag{4.4.2}$$

where indices live in $\{1,2,\ldots,2n\}$ but are considered $\pmod{2n}$. Because the diagonal elements of d are nonzero, we must have $\sigma(i+n)\equiv\sigma(i)+n\pmod{2n}$, meaning $\sigma\in\Sigma_{2n}$.

(b) We need a diagonal matrix $d=\operatorname{diag}(d_1,\ldots,d_{2n})$ such that $d\sigma\in\mathrm{GSp}_{2n}$, meaning $d\sigma\widehat{w}_{2n}\sigma^{\intercal}d^{\intercal}=\widehat{w}_{2n}$. Well, it suffices to check this on basis vectors $e_{\sigma(i)}$, for which we see it is enough (4.4.2). But because $\sigma\in\Sigma_{2n}$, it is equivalent to require

$$(-1)^{1_{i>n}}d_{\sigma(i)+n}d_{\sigma(i)}=(-1)^{1_{i>n}}d_{\sigma(i+n)}d_{\sigma(i)}=(-1)^{1_{\sigma(i)>n}}$$

for each index i. Observe $(-1)^{1_{(i+n)>n}}=-(-1)^{1_{i>n}}$ and $(-1)^{1_{\sigma(i+n)>n}}=-(-1)^{1_{\sigma(i)>n}}$ (indices are still taken $\pmod{2n}$), so if the above equation is satisfied at index i, then it is satisfied at index $i+n$.

As such, given signs $\{d_{\sigma(1)}, \ldots, d_{\sigma(n)}\}$, we must set $d_{\sigma(i)+n} := (-1)^{1_{\sigma(i)>n}} d_{\sigma(i)}$ for each $i \in \{1, 2, \ldots, 2\}$ to satisfy the equation at the indices $i \in \{1, 2, \ldots, n\}$, and this choice of signs will work. $\square$

In light of Lemma 50, we represent each $w \in W(\mathrm{GSp}_{2n})$ by $d_w \sigma_w$ where d_w is a diagonal matrix with entries in $\{\pm 1\}$ and $\sigma_w \in \Sigma_{2n}$; the permutation σ_w is determined by w.

The Weyl elements $W(\mathrm{GSp}_{2n})$ provide representatives for double cosets $B_{2n}^{\mathrm{sp}} \backslash \mathrm{GSp}_{2n} / B_{2n}^{\mathrm{sp}}$. It follows that each $g \in \mathrm{GSp}_{2n}$ can be expressed as $p\sigma_w d_w du$ where $p \in P_{2n}^{\mathrm{sp}}$ and $w \in W(\mathrm{GSp}_{2n})$ and $d \in D_{2n}^{\mathrm{sp}}$ and $u \in U_{2n}^{\mathrm{sp}}$. In other words, we have found that elements of $W(\mathrm{GSp}_{2n}) D_{2n}^{\mathrm{sp}}$ succeed in representing all double cosets in $P_{2n}^{\mathrm{sp}} \backslash \mathrm{GSp}_{2n} / U_{2n}^{\mathrm{sp}}$. It will be helpful later to have the following "normal" form for elements in Σ_{2n}.

Lemma 51. *Fix notation as above, and suppose $\sigma \in \Sigma_{2n}$.*

(a) There exists $\sigma' \in D_{2n}^{\mathrm{sp}} \cap \Sigma_{2n}$ such that $\sigma'\sigma(i) \equiv i \pmod n$ for each $i \in \{1, 2, \ldots, 2n\}$.
(b) For any $\sigma' \in D_{2n}^{\mathrm{sp}} \cap \Sigma_{2n}$,

$$\{i \in \{1, 2, \ldots, n\} : \sigma(i) \le n\} = \{i \in \{1, 2, \ldots, n\} : \sigma'\sigma(i) \le n\}.$$

Proof. We show the parts independently.

(a) The point is to "rearrange" the outputs of σ on $\{1, 2, \ldots, n\}$. Indeed, we define $\sigma'(\sigma(i))$ for $i \in \{1, 2, \ldots, n\}$ by

$$\sigma'(\sigma(i)) := \begin{cases} i & \text{if } \sigma(i) \le n, \\ i + n & \text{if } \sigma(i) > n. \end{cases}$$

Then, to have $\sigma' \in \Sigma_{2n}$, we must have $\sigma'(\sigma(i+n)) = \sigma'(\sigma(i)+n) = \sigma'(i)+n$ for each $i \in \{1, 2, \ldots, n\}$, so the above values have uniquely determined an element $\sigma' \in \Sigma_{2n}$.

Now, by construction, we have $\sigma'\sigma(i) \equiv i \pmod n$ for $i \in \{1, 2, \ldots, n\}$, and this extends to all $i \in \{1, 2, \ldots, 2n\}$ because $\sigma'\sigma \in \Sigma_{2n}$. Lastly, we see σ' maps $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$ and maps $\{n+1, n+2, \ldots, 2n\}$ to $\{n+1, n+2, \ldots, 2n\}$, so as a matrix σ' looks like

$$\sigma' = \begin{bmatrix} A & \\ & D \end{bmatrix}.$$

Here, A and D are permutation matrices, and $\sigma'(i+n) = \sigma'(i)+n$ implies $A = D = D^\iota$, so $\sigma' \in D_{2n}^{\mathrm{sp}}$.

(b) By hypothesis, σ' sends $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$ and sends $\{n+1, n+2, \ldots, 2n\}$ to $\{n+1, n+2, \ldots, 2n\}$, so $\sigma(i) \le n$ if and only if $\sigma'\sigma(i) \le n$. The equality follows. $\square$

Now, according to (4.4.1), we would like to understand χ_η and ψ_T on $H_\eta := U_{2n}^{\mathrm{sp}} \cap \eta^{-1} P_{2n}^{\mathrm{sp}} \eta$ for representatives η of our double cosets. We begin with χ_η.

Lemma 52. *Fix notation as above. Fix some $w \in W(\mathrm{GSp}_{2n})$, and set $\eta := \sigma_w d_w d$ where $d \in D_{2n}^{\mathrm{sp}}$. Then χ_η is trivial on $H_\eta := U_{2n}^{\mathrm{sp}} \cap \eta^{-1} P_{2n}^{\mathrm{sp}} \eta$.*

Proof. For any $u \in U_\eta$, we compute

$$\chi_\eta(u) = \chi\left(\eta u \eta^{-1}\right) = \chi\left(\sigma_w d_w \left(dud^{-1}\right) d_w^{-1} \sigma_w^{-1}\right) = \chi_{\sigma_w d_w}\left(dud^{-1}\right),$$

so it is enough to show that $\chi_{\sigma_w d_w}$ is trivial on $H_{\sigma_w d_w}$. (Notably, $dud^{-1} \in U_{2n}^{\mathrm{sp}}$ still.) In other words, we may assume that $d = I_{2n}$.

Now, fix any $u \in H_\eta$; we want to show $\chi\left(\sigma_w d_w u d_w^{-1}\sigma_w^{-1}\right) = 1$. By Lemma 46, it suffices to show that $\sigma_w d_w u d_w^{-1}\sigma_w^{-1}$ has multiplier 1 and has top-left quadrant with determinant 1. To begin, we recall $m\colon \mathrm{GSp}_{2n} \to \mathbb{F}_q^\times$ denotes the multiplier character and compute

$$m\left(\sigma_w d_w u d_w^{-1}\sigma_w^{-1}\right) = m(\sigma_w d_w)m(u)m(\sigma_w d_w)^{-1} = 1,$$

so we now want to show $\chi_1\left(\sigma_w d_w u d_w^{-1}\sigma_w^{-1}\right) = 1$. We will show this by Gaussian elimination. The following lemma will be useful.

Lemma 53. *Let k be a field, and let $z \in M_n(k)$ be called "sparse" if and only if $zv = 0$ or $v^\intercal z = 0$ for each $v \in k^n$. If z is sparse, then gzg^{-1} is sparse for any $g \in \mathrm{GL}_n(k)$ satisfying $g^{-1} = g^\intercal$.*

Proof. For any $v \in k^n$, we note either $zg^{-1}v = 0$ or $v^\intercal gz = \left(g^{-1}v\right)^\intercal z = 0$, which is what we wanted. ☐

To use Lemma 53, we note that $d_w u d_w^{-1} - I_{2n}$ is sparse: indeed, we may check being sparse on a basis, for which we note that any basis vector e_i has $d_w u d_w^{-1} e_i = e_i$. Thus, $\sigma_w d_w u d_w^{-1}\sigma_w^{-1} - I_{2n}$ is still sparse, so we write

$$\sigma_w d_w u d_w^{-1}\sigma_w^{-1} = \begin{bmatrix} A + I_n & B \\ 0 & D + I_n \end{bmatrix}.$$

Recall our end goal is to show $\chi_1\left(\sigma_w d_w u d_w^{-1}\sigma_w^{-1}\right) = 1$, so we want to show $\det(A + I_n) = 1$, which we now do Gaussian elimination to establish.

For each basis vector e_i with $1 \le i \le n$, we know that either $Ae_i = 0$ or $e_i^\intercal A = 0$, meaning that for each i, either the ith column of $A + I_n$ is e_i or the ith row of $A + I_n$ is $e_i^\intercal$. For example, if the ith column is e_i, then Gaussian elimination allows us to subtract this column from each other column, thus zeroing out the entire row while leaving the rest of the matrix unchanged. A similar process works for columns, from which we find $\det(A + I_n) = \det(I_n) = 1$, which is what we wanted. ☐

Combining Lemma 52 with (4.4.1), we want to count classes $\eta \in P_{2n}^{\mathrm{sp}}\backslash \mathrm{GSp}_{2n} / U_{2n}^{\mathrm{sp}}$ so that ψ_T is trivial on U_η. To complete the proof of the proposition, we thus must show that ψ_T is trivial on U_η for precisely one class η. To begin, we explain which class that is.

Lemma 54. *Fix notation as above.*

- *(a) ψ_T is trivial on $H_{\widehat{w}_{2n}} = U_{2n}^{\mathrm{sp}} \cap \widehat{w}_{2n}^{-1} P_{2n}^{\mathrm{sp}} \widehat{w}_{2n}$.*
- *(b) Fix some $w \in W(\mathrm{GSp}_{2n})$, and set $\eta = \sigma_w d_w d$ where $d \in D_{2n}^{\mathrm{sp}}$. If $\sigma_w(i) > n$ for each $i \in \{1, 2, \ldots, n\}$, then $P_{2n}^{\mathrm{sp}}\eta U_{2n}^{\mathrm{sp}} = P_{2n}^{\mathrm{sp}}\widehat{w}_{2n}U_{2n}^{\mathrm{sp}}$.*

Proof. We show the parts independently.

- (a) Suppose $u := \left[\begin{smallmatrix} I_n & Z \\ & I_n \end{smallmatrix}\right]$ lives in $H_{\widehat{w}_{2n}}$. Then $\widehat{w}_{2n} u \widehat{w}_{2n}^{-1} = u^\iota = \left[\begin{smallmatrix} I_n & \\ -Z & I_n \end{smallmatrix}\right]$ lives in P_{2n}^{sp}, so we must have $Z = 0$. Thus, $u = I_{2n}$, and it follows $\psi_T(u) = 1$.
- (b) We use Lemma 51, which provides $\sigma \in D_{2n}^{\mathrm{sp}} \cap \Sigma_{2n}$ such that $\sigma\sigma_w(i) \equiv i \pmod n$ for each $i \in \{1, 2, \ldots, n\}$. However, $\sigma_w(i) > n$ for each $i \in \{1, 2, \ldots, n\}$, so Lemma 51 enforces
$$\sigma\sigma_w(i) = i + n$$
for each $i \in \{1, 2, \ldots, n\}$. Because $\sigma\sigma_w \in \Sigma_{2n}$, this extends to $\sigma\sigma_w(i) = i + n$ for each $i \in \{1, 2, \ldots, 2n\}$, where indices are taken $\pmod n$ as usual.

Continuing, we define the diagonal matrix d_σ so that $\sigma d_\sigma \sigma_w d_w = \widehat{w}_{2n}$; more precisely, we may do this by the uniqueness of (b) in Lemma 50. Now, we see that $\sigma d_\sigma \in \mathrm{GSp}_{2n}$, but σ maps $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and $\{n+1, n+2, \ldots, 2n\} \to \{n+1, n+2, \ldots, 2n\}$, so $\sigma d_\sigma \in D_{2n}^{\mathrm{sp}}$.

All that remains is computation. We see

$$P_{2n}^{\mathrm{sp}} \eta U_{2n}^{\mathrm{sp}} = P_{2n}^{\mathrm{sp}} \sigma d_\sigma \sigma_w d_w d U_{2n}^{\mathrm{sp}} = P_{2n}^{\mathrm{sp}} \widehat{w}_{2n} d U_{2n}^{\mathrm{sp}} = P_{2n}^{\mathrm{sp}} d^\iota \widehat{w}_{2n} U_{2n}^{\mathrm{sp}} = P_{2n}^{\mathrm{sp}} \widehat{w}_{2n} U_{2n}^{\mathrm{sp}},$$

which completes the proof. $\square$

Thus, to complete the proof, we want to show that ψ_T is nontrivial for each double coset $\eta \in P_{2n}^{\mathrm{sp}} \eta U_{2n}^{\mathrm{sp}}$ distinct from $P_{2n}^{\mathrm{sp}} \widehat{w}_{2n} U_{2n}^{\mathrm{sp}}$.

Lemma 55. *Fix notation as above. Fix some $w \in W(\mathrm{GSp}_{2n})$, and set $\eta := \sigma_w d_w d$ where $d \in D_{2n}^{\mathrm{sp}}$. If $P_{2n}^{\mathrm{sp}} \eta U_{2n}^{\mathrm{sp}} \neq P_{2n}^{\mathrm{sp}} \widehat{w}_{2n} U_n^{\mathrm{sp}}$, then ψ_T is nontrivial on $H_\eta = U_{2n}^{\mathrm{sp}} \cap \eta^{-1} P_{2n}^{\mathrm{sp}} \eta$.*

Proof. Quickly, we claim that $H_\eta = H_{\sigma_w d_w}$. Indeed, if $u \in H_\eta$, then $d^{-1}(\sigma_w d_w)^{-1} u (\sigma_w d_w) d \in P_n^{\mathrm{sp}}$, but $d \in P_n^{\mathrm{sp}}$ implies $(\sigma_w d_w)^{-1} u (\sigma_w d_w) \in P_n^{\mathrm{sp}}$, so $u \in H_{\sigma_w d_w}$. A symmetric argument establishes the other inclusion.

Thus, we may assume that $d = I_{2n}$. Now, by Lemma 54, $P_{2n}^{\mathrm{sp}} \eta U_n^{\mathrm{sp}} \neq P_{2n}^{\mathrm{sp}} \widehat{w}_{2n} U_{2n}^{\mathrm{sp}}$ implies that $\sigma_w(i) \leq n$ for some $i \in \{1, 2, \ldots, n\}$; without loss of generality, assume $\sigma_w(1) \leq n$. Now, by adjusting σ_w by a permutation in $D_{2n}^{\mathrm{sp}} \cap \Sigma_{2n}$ via Lemma 51, we may assume that $\sigma_w(i) \equiv i \pmod{n}$ for each $i \in \{1, 2, \ldots, 2n\}$. In particular, $\sigma_w(1) = 1$.

We are now ready to compute H_η. Fix $u := \left[\begin{smallmatrix} I_n & Z \\ & I_n \end{smallmatrix}\right]$ for $Z \in \mathrm{Sym}_n$, and we test for $\eta^{-1} u \eta \in P_{2n}^{\mathrm{sp}}$. Fix indices $i, j \in \{1, 2, \ldots, n\}$, and we want to compute

$$e_{i+n}^\intercal d_w^{-1} \sigma_w^{-1} u \sigma_w d_w e_j = \pm (\sigma_w e_{i+n})^\intercal u (\sigma_w e_j) = \pm e_{\sigma_w(i)+n}^\intercal u e_{\sigma_w(j)}.$$

We have the following cases.

- If $\sigma_w(i) = i$ and $\sigma_w(j) = j$, then we are looking at $\pm e_{i+n}^\intercal u e_j = \pm e_{i+n}^\intercal e_j = 0$ because $j \leq n < i + n$.
- If $\sigma_w(i) = i$ and $\sigma_w(j) = j + n$, then we are looking at $\pm e_{i+n}^\intercal u e_{j+n} = \pm 1_{i=j} = 0$, where $i \neq j$ because $\sigma_w(i) = i$ while $\sigma_w(j) \neq j$.
- If $\sigma_w(i) = i + n$ and $\sigma_w(j) = j$, then we are looking at $\pm e_i^\intercal u e_j = \pm 1_{i=j} = 0$, where $i \neq j$ as in the previous case.
- Lastly, if $\sigma_w(i) = i + n$ and $\sigma_w(j) = j + n$, then we are looking at $\pm e_{i+n}^\intercal u e_j = \pm u_{j,i+n}$.

Thus, we see

$$H_\eta = \{u \in U_{2n}^{\mathrm{sp}} : u_{i+n,j} = 0 \text{ if } \sigma_w(i) = i + n \text{ and } \sigma_w(j) = j + n\}.$$

Because $\sigma_w(1) = 1$, we thus see that

$$\left\{ \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{1n} & 0 & \cdots & 0 \end{bmatrix} : u_{11}, u_{12}, \ldots, u_{1n} \in k \right\} \subseteq H_\eta,$$

where we have identified Sym_n with U_{2n}^{sp} in the usual way. However, for any invertible symmetric $T \in \mathrm{Sym}_n^\times$, we see that ψ_T is nontrivial on the above subgroup, so we are done. $\square$

The above lemma completes the proof of Proposition 49. $\square$

Now, one way to think about Proposition 49 is that we have shown $\operatorname{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi$ has a single U_{2n}^{sp}-eigenvector with eigenvalue ψ_T. However, it is not so difficult to write one down. The following is a finite-field analogue of a "spherical vector."

Lemma 56. *Fix notation as above, and fix some $T \in \mathrm{Sym}_n^\times$ and character $\chi\colon P_{2n}^{\mathrm{sp}} \to \mathbb{C}^\times$. Define $f_{T,\chi}\colon \mathrm{GSp}_{2n} \to \mathbb{C}$ by*

$$f_{T,\chi}(g) := \begin{cases} \chi(p)\psi_T(u) & \text{if } g = p\widehat{w}_{2n}u \text{ for } p \in P_n^{\mathrm{sp}}, u \in U_n^{\mathrm{sp}}, \\ 0 & \text{else.} \end{cases}$$

Then $f_{T,\chi}$ is well-defined, nonzero, and lives in $\operatorname{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi$. Further, $f_{T,\omega}$ is a U_{2n}^{sp}-eigenvector with eigenvalue ψ_T.

Proof. We begin by checking that f is well-defined; note $f_{T,\chi} \neq 0$ follows quickly because $f_{T,\chi}(\widehat{w}_n) = 1$. Well, suppose we have $p_1, p_2 \in P_{2n}^{\mathrm{sp}}$ and $u_1, u_2 \in U_{2n}^{\mathrm{sp}}$ such that $p_1\widehat{w}_{2n}u_1 = p_2\widehat{w}_{2n}u_2$; we claim $p_1 = p_2$ and $u_2 = u_2$, from which $\widetilde{\omega}(p_1)\psi_T(u_1) = \widetilde{\omega}(p_2)\psi_T(u_2)$ follows immediately. Well, set $p := p_2^{-1}p_1$ and $u := u_2u_1^{-1}$, and we want to show that $p = u = 1$. For this, we observe

$$p = \widehat{w}_{2n}u\widehat{w}_{2n}^{-1} = u^\iota.$$

Setting $u := \left[\begin{smallmatrix} 1 & Z \\ & 1 \end{smallmatrix}\right]$, we note $u^\iota = \left[\begin{smallmatrix} 1 & \\ -Z & 1 \end{smallmatrix}\right]$ lives in P_{2n}^{sp} if and only if $Z = 0$, which means $u = p = I_{2n}$.

Next up, we show $f_{T,\chi} \in \operatorname{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi$. Well, fix $g_0 \in \mathrm{GSp}_6$ and $p \in P_{2n}^{\mathrm{sp}}$, and we want to show $f_{T,\chi}(pg_0) = \chi(p)f_{T,\chi}(g_0)$. This follows directly from the definitions. For example, if g_0 does take the form $p_0\widehat{w}_{2n}u_0$, then $pg_0 = pp_0\widehat{w}_{2n}u_0$, and

$$f_{T,\chi}(pg_0) = \chi(pp_0)\psi_T(u_0) = \chi(p)f_{T,\chi}(g_0)$$

Otherwise, g_0 does not take the form $p_0w_{2n}u_0$, so pg_0 also does not live in the double coset $P_{2n}^{\mathrm{sp}}\widehat{w}_{2n}U_{2n}^{\mathrm{sp}}$, so $f_{T,\chi}(pg_0) = 0 = \chi(p)f_{T,\chi}(g_0)$.

Lastly, we show that $f_{T,\chi}$ is a U_{2n}^{sp}-eigenvector with eigenvalue ψ_T. This again follows directly from the definitions. Fix $g_0 \in \mathrm{GSp}_{2n}$ and $u \in U_{2n}^{\mathrm{sp}}$, and we want to show that $f_{T,\chi}(g_0u) = \psi_T(u)f_{T,\chi}(g_0)$. Indeed, an identical argument to the above but switching ps with us (and direction of multiplication) establishes the claim. $\square$

Now, using $f_{T,\chi}$ written above as a concrete U_{2n}^{sp}-eignvector of $\operatorname{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi$, we can use the multiplicity-one result of Proposition 49 to achieve the following result.

Proposition 57. *Fix notation as above, and let $\chi\colon P_{2n}^{\mathrm{sp}} \to \mathbb{C}^\times$ be a character of the form $\chi = (\alpha_\chi \circ m)(\beta_\chi \circ \chi_{\det})$ where $\alpha_\chi, \beta_\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ are characters. Then*

$$M_{w_{2n}}f_{T,\chi} = \alpha_\chi(-1)\beta_\chi(-1)^{n(n-1)/2} \cdot g_n(\beta_\chi, \psi, T)f_{T,{}^{(w_{2n})}\chi}.$$

Here, ${}^{(w_n)}\chi$ is a character on P_{2n}^{sp} given by ${}^{(w_{2n})}\chi(du) = \chi(w_{2n}dw_{2n})$ for any $d \in D_{2n}^{\mathrm{sp}}$ and $u \in U_{2n}^{\mathrm{sp}}$. Additionally, $g_n(\beta_\chi, \psi, T)$ is the Gauss sum considered in Appendix B.

Proof. Quickly, we check that ${}^{(w_{2n})}\chi$ is in fact a character: for any $d_1, d_2 \in D_{2n}^{\mathrm{sp}}$ and $u_1, u_2 \in U_{2n}^{\mathrm{sp}}$, we see

$$\begin{aligned}
{}^{(w_{2n})}\chi(d_1u_1d_2u_2) &= {}^{(w_{2n})}\chi\left(d_1d_2 \cdot d_2^{-1}u_1d_2u_2\right) \\
&= \chi(w_{2n}d_1d_2w_{2n}) \\
&= \chi(w_{2n}d_1w_{2n})\chi(w_{2n}d_2w_{2n}) \\
&= {}^{(w_{2n})}\chi(d_1u_1)\,{}^{(w_{2n})}\chi(d_2u_2).
\end{aligned}$$

Now, we note that $M_{w_{2n}}$ is a G-invariant operator, so because $f_{T,\chi}$ is an U_{2n}^{sp}-eigenvector with eigenvalue ψ_T, we see

$$u \cdot M_{w_{2n}} f_{T,\chi} = M_{w_{2n}}(u \cdot f_{T,\chi}) = \psi_T(u) \cdot M_{w_{2n}} f_{T,\chi},$$

so $M_{w_{2n}} f_{T,\chi}$ continues to be a U_{2n}^{sp}-eigenvector with eigenvalue ψ_T but in the space $\operatorname{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}}\,{}^{(w_{2n})}\chi$. By Proposition 49, the space of such eigenvectors is one-dimensional, and Lemma 56 grants us a nonzero eigenvector $f_{T,{}^{(w_{2n})}\chi}$. Thus, there is a (unique) constant c such that

$$M_{w_{2n}} f_{T,\chi} = c f_{T,{}^{(w_{2n})}\chi}.$$

It remains to compute the constant c. Well, plugging in $\widehat{w}_{2n}$, we see that

$$\begin{aligned}
c &= c f_{T,{}^{(w_{2n})}\chi}(\widehat{w}_n) \\
&= M_{w_{2n}} f_{T,\chi}(\widehat{w}_{2n}) \\
&= \sum_{u \in U_{2n}^{\mathrm{sp}}} f_{T,\chi}(w_{2n}u\widehat{w}_{2n}).
\end{aligned}$$

Now, $f_{T,\chi}$ is supported on $P_{2n}^{\mathrm{sp}}\widehat{w}_{2n}U_{2n}^{\mathrm{sp}}$, so to have $f_{T,\chi}(w_{2n}u\widehat{w}_{2n}) \neq 0$, there must exist $v \in U_{2n}^{\mathrm{sp}}$ such that $w_{2n}u\widehat{w}_{2n}v^{-1}\widehat{w}_{2n}^{-1} \in P_{2n}^{\mathrm{sp}}$. Writing $u := \left[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right]$ and $v := \left[\begin{smallmatrix} 1 & Y \\ & 1 \end{smallmatrix}\right]$, we compute

$$w_{2n}u\widehat{w}_{2n}v^{-1}\widehat{w}_{2n}^{-1} = w_{2n}uv^{\intercal} = \begin{bmatrix} & w_n \\ w_n & \end{bmatrix}\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ Y & 1 \end{bmatrix} = \begin{bmatrix} w_nY & w_n \\ w_n(1+XY) & w_nX \end{bmatrix}.$$

This lives in P_{2n}^{sp} if and only if X is invertible and $Y = -X^{-1}$. So in the case where X is invertible, we note that the above work gives the decomposition

$$w_{2n}\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}\widehat{w}_{2n} = \begin{bmatrix} -w_nX^{-1} & w_n \\ & w_nX \end{bmatrix}\widehat{w}_n\begin{bmatrix} 1 & -X^{-1} \\ & 1 \end{bmatrix}.$$

It follows that

$$\begin{aligned}
c &= \sum_{X \in \mathrm{Sym}_n^{\times}(k)} f_{T,\chi}\left(\begin{bmatrix} -w_nX^{-1} & w_n \\ & w_nX \end{bmatrix}\widehat{w}_n\begin{bmatrix} 1 & -X^{-1} \\ & 1 \end{bmatrix}\right) \\
&= \sum_{A \in \mathrm{Sym}_n^{\times}(k)} \chi\left(\begin{bmatrix} w_nA & w_n \\ & -w_nA^{-1} \end{bmatrix}\right)\psi_T(A) \\
&= \alpha_\chi(-1)\beta_\chi(-1)^{n(n-1)/2} \sum_{A \in \mathrm{Sym}_n^{\times}(k)} \beta_\chi(\det A)\psi_T(A),
\end{aligned}$$

as desired. Notably, $\det w_n = (-1)^{n(n-1)/2}$. ∎

Corollary 58. *Fix notation as above, and let $\chi\colon P_{2n}^{\mathrm{sp}} \to \mathbb{C}^\times$ be a character of the form $\chi = (\alpha_\chi \circ m)(\beta_\chi \circ \chi_{\det})$ where $\alpha_\chi, \beta_\chi \colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ are characters. Then*

$$(M_{w_{2n}} \circ M_{w_{2n}})(f_{T,\chi}) = g_n\left(\beta_\chi, \psi, T\right) g_n\left(\beta_\chi^{-1}, \psi^{-1}, T\right) f_{T,\chi}.$$

Proof. The main point is that plugging Example 48 into Proposition 57 implies

$$\begin{aligned} M_{w_{2n}} f_{T,(w_{2n})\chi} &= \alpha_\chi(-1)\beta_\chi(-1)^{n(n-1)/2}\beta_\chi(-1)^n \sum_{A\in\mathrm{Sym}_n^\times(k)} \beta_\chi(\det A)^{-1}\psi_T(A) \\ &= \alpha_\chi(-1)\beta_\chi(-1)^{n(n-1)/2} \sum_{A\in\mathrm{Sym}_n^\times(k)} \beta_\chi(\det A)^{-1}\psi_T(A)^{-1} \\ &= \alpha_\chi(-1)\beta_\chi(-1)^{n(n-1)/2} g_n\left(\beta_\chi^{-1}, \psi^{-1}, T\right). \end{aligned}$$

Combining with Proposition 57 completes the proof. □

Example 59. Take $\chi = \widetilde{\omega}$ so that $\alpha_\chi = \beta_\chi = \omega$. If $\omega^2 \neq 1$, then Theorem 89 implies that

$$g_n\left(\beta_\chi, \psi, T\right) g_n\left(\beta_\chi^{-1}, \psi^{-1}, T\right) = q^{\frac{1}{2}\binom{n+1}{2}}.$$

Corollary 58 tells us that $M_{w_{2n}} \circ M_{w_{2n}}$ behaves as a scalar on the particular vector $f_{T,\chi}$. To extend this to all of $\mathrm{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi$, we check when $\mathrm{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi$ is irreducible.

Proposition 60. *Fix notation as above, and let $\chi\colon P_{2n}^{\mathrm{sp}} \to \mathbb{C}^\times$ be a character of the form $\chi = (\alpha_\chi \circ m)(\beta_\chi \circ \chi_{\det})$ where $\alpha_\chi, \beta_\chi \colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ are characters. Then*

$$\dim \mathrm{End}_{\mathrm{GSp}_{2n}} \mathrm{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi = \begin{cases} n+1 & \text{if } \beta_\chi = 1, \\ \lfloor (n+1)/2 \rfloor & \text{if } \beta_\chi \neq 1 \text{ and } \beta_\chi^2 = 1, \\ 1 & \text{if } \beta_\chi^2 \neq 1. \end{cases}$$

In particular, if $\omega^2 \neq 1$, then $I(\omega)$ is irreducible.

Proof. We compute $\dim \mathrm{End}_{\mathrm{GSp}_{2n}} \mathrm{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi$ using Mackey theory. The proof uses many of the same tools as Proposition 49. Using Frobenius reciprocity and Lemma 41, we see

$$\begin{aligned} \mathrm{End}_{\mathrm{GSp}_{2n}} \mathrm{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi &\cong \mathrm{Hom}_{P_{2n}^{\mathrm{sp}}}\left(\mathrm{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi, \omega\right) \\ &\cong \bigoplus_{\eta\in P_{2n}^{\mathrm{sp}}\backslash \mathrm{GSp}_{2n}/P_{2n}^{\mathrm{sp}}} \mathrm{Hom}_{P_{2n}^{\mathrm{sp}}\cap \eta^{-1}P_{2n}^{\mathrm{sp}}\eta}(\chi_\eta, \chi). \end{aligned}$$

Thus, we are interested in studying double cosets $P_{2n}^{\mathrm{sp}}\backslash \mathrm{GSp}_{2n}/P_{2n}^{\mathrm{sp}}$. As in Proposition 49, we use the Bruhat decomposition, which tells us that double cosets in $B_{2n}^{\mathrm{sp}}\backslash \mathrm{GSp}_{2n}/B_{2n}^{\mathrm{sp}}$ are uniquely represented by the Weyl elements $\{\sigma_w d_w : w \in W(\mathrm{GSp}_6)\}$, so these Weyl elements also provide representatives of the double cosets in $P_{2n}^{\mathrm{sp}}\backslash \mathrm{GSp}_{2n}/P_{2n}^{\mathrm{sp}}$. As in Lemma 51, we want to provide a "normal form" for our Weyl elements.

Lemma 61. *Fix notation as above, and fix $\sigma_1 d_1, \sigma_2 d_2$ representing Weyl elements $w_1, w_2 \in W(\mathrm{GSp}_{2n})$. Then*

$$\#\{i \in \{1,2,\ldots,n\} : \sigma_1(i) > n\} = \#\{i \in \{1,2,\ldots,n\} : \sigma_2(i) > n\}$$

if and only if $P_{2n}^{\mathrm{sp}} w_1 P_{2n}^{\mathrm{sp}} = P_{2n}^{\mathrm{sp}} w_2 P_{2n}^{\mathrm{sp}}$.

Proof. For brevity, define

$$r(w) \coloneqq \#\{i \in \{1, 2, \dots, n\} : \sigma_w(i) > n\}.$$

We want to show that r descends to an injective map $P_{2n}^{\mathrm{sp}} \backslash \operatorname{GSp}_{2n} / P_{2n}^{\mathrm{sp}} \to \mathbb{Z}$.

To begin, we show that the map is well-defined. Let X be the maximal isotropic subspace of k^{2n} spanned by $\{e_1, \dots, e_n\}$, and we let Y be the maximal isotropic subspace spanned by $\{e_{n+1}, \dots, e_{2n}\}$; we then let $\pi_Y \colon k^{2n} \to Y$ denote the projection. Now, we begin by claiming

$$r(w) \stackrel{?}{=} \dim \pi_Y(wX).$$

Indeed, $\pi_Y(wX)$ is spanned by the vectors $\pi_Y(we_i)$ for $i \in \{1, 2, \dots, n\}$ and hence by the vectors $e_{\sigma_w(i)}$ where $w(i) > n$. The equality follows.

Now, we thus see that $\pi_Y(wpX) = \pi_Y(wX)$ for any $p \in P_{2n}^{\mathrm{sp}}$, so r is well-defined on $\operatorname{GSp}_{2n} / P_{2n}^{\mathrm{sp}}$. Furthermore, we note that $\dim \pi_Y(pW) = \dim p\pi_Y(W) = \dim \pi_Y(W)$ for any subspace $W \subseteq k^{2n}$, so it follows that r further descends to a function on $P_{2n}^{\mathrm{sp}} \backslash \operatorname{GSp}_{2n} / P_{2n}^{\mathrm{sp}}$.

Lastly, we must show that r is injective. Well, fix some $r \in \{0, 1, \dots, n\}$, and we will show that any $w \in W(\operatorname{GSp}_6)$ with $r(w) = r$ is in the same double coset as some fixed Weyl element. To begin, we may choose a permutation σ of $\{1, 2, \dots, n\}$ so that

$$\{i \in \{1, 2, \dots, n\} : \sigma_w \sigma(i) > n\} = \{1, 2, \dots, r\}.$$

Then σ may be extended to a permutation in Σ_{2n} as in Lemma 51, and we can see that $\sigma \in D_{2n}^{\mathrm{sp}}$. Thus, replacing σ_1 with σ and doing similarly for w_2, we may assume that

$$\{i \in \{1, 2, \dots, n\} : \sigma_w(i) > n\} = \{1, 2, \dots, r\}$$

on the nose. From here, Lemma 51 grants us another $\sigma' \in D_{2n}^{\mathrm{sp}} \cap \Sigma_{2n}$ so that $\sigma'\sigma(i) \equiv i \pmod{n}$ while also preserving $\{i \in \{1, 2, \dots, n\} : \sigma_w(i) > n\}$, so by adjusting σ_w by this σ', we may assume that

$$\sigma_w(i) = \begin{cases} i & \text{if } 1 \le i \le r, \\ i + n & \text{if } r < i \le n. \end{cases}$$

These data uniquely determine $\sigma_w \in \Sigma_{2n}$ and hence the Weyl element w. This completes the proof of injectivity. $\square$

The proof of Lemma 61 implies that there are $n+1$ double cosets in $P_{2n}^{\mathrm{sp}} \backslash \operatorname{GSp}_{2n} / P_{2n}^{\mathrm{sp}}$, which we can compute are given by

$$\eta_r \coloneqq \begin{bmatrix} I_{n-r} & & & \\ & & & -I_r \\ & & I_{n-r} & \\ & I_r & & \end{bmatrix}$$

where $0 \le r \le n$. Notably, $\eta_r^{-1} = \eta_r^\intercal$. Now, for each $\eta \in P_6^{\mathrm{sp}} \backslash \operatorname{GSp}_6 / P_6^{\mathrm{sp}}$, we set $P_\eta \coloneqq P_{2n}^{\mathrm{sp}} \cap \eta^{-1} P_{2n}^{\mathrm{sp}} \eta$. We want to check when $\chi_{\eta_r} = \chi$. To begin, we compute P_{η_r}: writing out some $g \in P_{2n}^{\mathrm{sp}}$ as a block matrix, we compute

$$\begin{aligned}\eta_r g \eta_r^{-1} &= \begin{bmatrix} I_{n-r} & & & \\ & & & -I_r \\ & & I_{n-r} & \\ & I_r & & \end{bmatrix} \begin{bmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ & & D_1 & D_2 \\ & & D_3 & D_4 \end{bmatrix} \begin{bmatrix} I_{n-r} & & & \\ & & & -I_r \\ & & I_{n-r} & \\ & I_r & & \end{bmatrix}^{-1} \\ &= \begin{bmatrix} A_1 & -B_2 & B_1 & A_2 \\ & D_4 & -D_3 & \\ & -D_2 & D_1 & \\ A_3 & -B_4 & B_3 & A_4 \end{bmatrix},\end{aligned}$$

which live in P_{2n}^{sp} if and only if $A_3 = B_4 = D_2 = 0$. Thus, $\chi_{\eta_r} = \chi$ if and only if we always have

$$\chi\left(\begin{bmatrix} A_1 & -B_2 & B_1 & A_2 \\ & D_4 & -D_3 & \\ & & D_1 & \\ & & B_3 & A_4 \end{bmatrix}\right) = \chi\left(\begin{bmatrix} A_1 & A_2 & B_1 & B_2 \\ & A_4 & B_3 & \\ & & D_1 & \\ & & D_3 & D_4 \end{bmatrix}\right)$$

The multiplier of the left-hand side is $m\left(\eta_r g \eta_r^{-1}\right) = m(g)$, which is also the multiplier of the right-hand side. Thus, we no longer care about α_χ. It remains to look at β_χ, where we see we require

$$\beta_\chi(\det D_1 \cdot \det A_4)^{-1} = \beta_\chi(\det D_1 \cdot \det D_4)^{-1},$$

where we take the convention that the "empty" matrix has determinant 1. Equivalently, we are asking to always have $\beta_\chi(\det A_4) = \beta_\chi(\det D_4)$. Now, for $g \in P_{2n}^{\mathrm{sp}}$, we see that $A_4 = m(g) D_4^\iota$, so we are asking for

$$\beta_\chi(\det A_4)^2 = \beta_\chi(m(g))^r.$$

We have the following cases.

- If $\beta_\chi^2 = 1$ and r is even, then both sides are 1.
- If $r = 0$, then A_4 is the empty matrix, so both sides are 1.
- Suppose r is odd and in particular nonzero; we claim $\chi_{\eta_r} = \chi$ if and only if $\beta_\chi = 1$. Here, A_4 is an arbitrary nonempty invertible $r \times r$ matrix, so $\det A_4$ is an arbitrary element of $\mathbb{F}_q^\times$; the same holds for $m(g)$. Thus, we are basically asking for

$$\beta_\chi(x)^2 = \beta_\chi(y)^r$$

  for any $x, y \in \mathbb{F}_q^\times$. Setting $y = 1$ forces $\beta_\chi^2 = 1$, and setting $x = 1$ forces $\beta_\chi^r = 1$. Because r is odd, this is equivalent to $\beta_\chi = 1$.

Synthesizing the above cases completes the proof. $\square$

Remark 62. Adjusting the tallying portion of the above argument correctly, we find that

$$\dim \operatorname{End}_{\mathrm{Sp}_{2n}} \operatorname{Ind}_P^{\mathrm{Sp}_{2n}} \chi = \begin{cases} n+1 & \text{if } \chi^2 = 1, \\ 1 & \text{if } \chi^2 \neq 1. \end{cases}$$

Here, $P \subseteq \mathrm{Sp}_{2n}$ is the Siegel parabolic $\mathrm{Sp}_{2n} \cap P_{2n}^{\mathrm{sp}}$.

Proposition 60 now assures us that in the case where $\chi^2 \neq 1$, the composite $(M_{w_6} \circ M_{w_6})\colon I(\omega) \to I(\omega)$ must be a scalar, and in fact this scalar is q^3 by Example 59. When

$\chi^2 = 1$, the composite no longer need to be a scalar, but we can understand it. The idea is to build a reasonably nice basis.

Lemma 63. *Fix notation as above, and let $\chi\colon P_{2n}^{\mathrm{sp}} \to \mathbb{C}^\times$ be a character. For each irreducible subrepresentation $\pi \subseteq \operatorname{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi$, there is a vector $v \in \pi$ which is a χ-eigenvector.*

Proof. Observe that

$$\operatorname{Hom}_{P_{2n}^{\mathrm{sp}}}(\operatorname{Res}\pi, \chi) = \operatorname{Hom}_{\mathrm{GSp}_{2n}}\left(\pi, \operatorname{Ind}_{P_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}} \chi\right) \geq 1,$$

so we are done. □

Thus, if we want to understand how M_{w_6} acts on its various irreducible subrepresentations, we are allowed to only look at the χ-eigenvectors. The computation of Proposition 60 explains that $\chi = (\alpha_\chi \circ m)(\beta_\chi \circ \chi_{\det})$ with $\beta_\chi = 1$ makes this space four-dimensional, and $\beta_\chi^2 = 1$ while $\beta_\chi \neq 1$ makes this space two-dimensional. Explicitly, such an eigenvector can be reduced to a function on $P_6^{\mathrm{sp}} \backslash \mathrm{GSp}_6 / P_6^{\mathrm{sp}}$, which a prior has four representatives $\eta_0, \eta_1, \eta_2, \eta_3$, but η_1 and η_3 do not contribute in the quadratic case. Letting f_0, f_1, f_2, f_3 denote the corresponding basis of eigenvectors (with $f_1 = f_3 = 0$ in the quadratic case), we are able to write M_{w_6} as a 4×4 matrix.

Example 64. In the case of $\chi = 1$ and $n = 3$, we have $M_{w_6}\colon I(1) \to I(1)$ can be written as the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1/q & (q-1)/q \\ 0 & 1/q^3 & \left(q^2-1\right)/q^3 & (q-1)/q \\ 1/q^6 & \left(q^3-1\right)/q^6 & \left(q^3-1\right)/q^4 & \left(q^4-q^3-q+1\right)/q^4 \end{bmatrix}.$$

This diagonalizes and has all nonzero eigenvalues. A similar computation can be done in the quadratic case to understand the composite $(M_{w_6} \circ M_{w_6})$.

Remark 65. It is our expectation that the eigenvalues of $M_{w_{2n}}$ can all be understood as (possibly signed) explicit powers of q even in the cases where $\chi^2 = 1$, but we have not been able to prove this.

4.5. **The Zeta Function.** To define our zeta function, we begin by defining the subgroups $Z \coloneqq \{cI_6 : c \in k^\times\}$ and

$$N \coloneqq \left\{ \left(\begin{bmatrix} 1 & b_1 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & b_2 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & b_3 \\ & 1 \end{bmatrix} \right) : b_1 + b_2 + b_3 = 0 \right\} \subseteq \mathrm{GL}_2^{(3)}.$$

Note that $ZN \subseteq S(\eta_0)$, so $\eta_0 ZN \eta_0^{-1} \subseteq P_6^{\mathrm{sp}}$, so define for brevity $S(\omega) \coloneqq \operatorname{Ind}_{\eta_0 ZN \eta_0^{-1}}^{\mathrm{GSp}_6} \omega^{-1}$, where ω^{-1} is considered to be a character by its behavior on Z, which is canonically isomorphic to $k^\times$. We have the following definition.

Definition 66. We define $Z\colon S(\omega) \otimes \left(\operatorname{Ind}_{U_2}^{\mathrm{GL}_2} \psi_2\right)^{\otimes 3} \to \mathbb{C}$ by

$$Z(f, W_1, W_2, W_3) \coloneqq \sum_{g \in ZN \backslash \mathrm{GL}_2^{(3)}} f(\eta_0 g) W_1(g_1) W_2(g_2) W_3(g_3),$$

where $g = (g_1, g_2, g_3)$. In the future, we may abbreviate $W_1(g_1)W_2(g_2)W_3(g_3)$ to $W(g)$.

Indeed, Z is linear in each coordinate, so its definition on the tensor product is well-founded. To see that the summands are N-invariant, the important check is that, by construction of $f \in S(\omega)$, any $cn \in ZN$ has

$$f(\eta_0 cng) = \omega^{-1}(c)f(\eta_0 g)$$

while $W_i(cn_1g) = \omega_i(c)\psi_2(n_1)W_i(g)$, so all the added terms cancel. (Notably, $\omega = \omega_1\omega_2\omega_3$ and $\psi_2(n_1n_2n_3) = 1$ by construction of N.)

We would like to combine Z with our multiplicity-one result Theorem 40. For this, we need the following two checks.

Lemma 67. *Fix three irreducible representations π_1, π_2, and π_3 of GL_2 of Whittaker type. Then Z restricts to a $\mathrm{GL}_2^{(3)}$-linear map*

$$I(\omega) \otimes \mathcal{W}(\pi_1, \psi) \otimes \mathcal{W}(\pi_2, \psi) \otimes \mathcal{W}(\pi_3, \psi) \to \mathbb{C}.$$

Proof. A direct computation shows that $I(\omega) \subseteq S(\omega)$: indeed, for $f \in I(\omega)$, we need to check that

$$f\left(\eta_0 zn\eta_0^{-1}g\right) \stackrel{?}{=} \omega(z)^{-1}f(g)$$

for any $\eta_0 zn\eta_0^{-1} \in \eta_0 ZN\eta_0^{-1}$, but this can be done by directly computing $\widetilde{\omega}\left(\eta_0 zn\eta_0^{-1}\right)$. Thus, it does make sense to say that Z restricts to $I(\omega) \otimes \mathcal{W}(\pi_1, \psi) \otimes \mathcal{W}(\pi_2, \psi) \otimes \mathcal{W}(\pi_3, \psi)$.

It remains to see that Z is $\mathrm{GL}_2^{(3)}$-linear. This is also a direct computation: we see that

$$\begin{aligned} Z(g_0f \otimes g_0W) &= \sum_{g \in ZN\backslash \mathrm{GL}_2^{(3)}} (g_0f)(g)(g_0W)(g) \\ &= \sum_{g \in ZN\backslash \mathrm{GL}_2^{(3)}} f(gg_0)W(gg_0) \\ &= \sum_{g \in ZN\backslash \mathrm{GL}_2^{(3)}} f(g)W(g) \\ &= Z(f \otimes W), \end{aligned}$$

as desired. $\square$

Lemma 68. *Fix three irreducible representations π_1, π_2, and π_3 of GL_2 of Whittaker type. Then the restriction*

$$Z\colon I(\omega) \otimes \mathcal{W}(\pi_1, \psi) \otimes \mathcal{W}(\pi_2, \psi) \otimes \mathcal{W}(\pi_3, \psi) \to \mathbb{C}$$

is nonzero.

Proof. We must find an input on which Z is nonzero. For this, we use Bessel functions combined with the function $f \in I(\omega)$ defined by

$$f(g) := \begin{cases} \widetilde{\omega}(p) & \text{if } g = p\eta_0 \text{ for some } p \in P_6^{\mathrm{sp}} \\ 0 & \text{else.} \end{cases}$$

Note that $f \in I(\omega)$ by construction. We now compute

$$Z(f \otimes \mathcal{J}_{\pi_1,\psi} \otimes \mathcal{J}_{\pi_2,\psi} \otimes \mathcal{J}_{\pi_3,\psi}) = \sum_{g \in ZN\backslash \mathrm{GL}_2^{(3)}} f(\eta_0 g)\mathcal{J}_{\pi_1,\psi}(g_1)\mathcal{J}_{\pi_2,\psi}(g_2)\mathcal{J}_{\pi_3,\psi}(g_3),$$

where $g = (g_1, g_2, g_3)$ as usual. Now, $f(\eta_0 g) \neq 0$ requires $\eta_0 g \eta_0^{-1} \in P_6^{\mathrm{sp}}$, so $\eta_0 \in S(\eta_0)$, so we may write g as

$$g = \left(\begin{bmatrix} a & b_1 \\ & d \end{bmatrix}, \begin{bmatrix} a & b_2 \\ & d \end{bmatrix}, \begin{bmatrix} a & b_3 \\ & d \end{bmatrix} \right),$$

where $b_1 + b_2 + b_3 = 0$. Using the fact that we only care about the coset ZNg, we may assume that $b_1 = b_2 = b_3 = 0$ and that $d = 1$. Indeed, modding out by Z allows us to assume that $d = 1$, and then we see

$$\left(\begin{bmatrix} a & b_1 \\ & 1 \end{bmatrix}, \begin{bmatrix} a & b_2 \\ & 1 \end{bmatrix}, \begin{bmatrix} a & b_3 \\ & 1 \end{bmatrix} \right) = \left(\begin{bmatrix} 1 & b_1 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & b_2 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & b_3 \\ & 1 \end{bmatrix} \right) \left(\begin{bmatrix} a & 0 \\ & 1 \end{bmatrix}, \begin{bmatrix} a & 0 \\ & 1 \end{bmatrix}, \begin{bmatrix} a & 0 \\ & 1 \end{bmatrix} \right),$$

so modding out by N gets rid of the left term. Now, by Proposition 6, the only time we can have $\mathcal{J}_{\pi_\bullet,\psi}\left(\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right]\right) \neq 0$ is for $a = 1$. In total, we must have $g = (I_2, I_2, I_2)$, which is a single coset. Thus, we find

$$Z(f \otimes \mathcal{J}_{\pi_1,\psi} \otimes \mathcal{J}_{\pi_2,\psi} \otimes \mathcal{J}_{\pi_3,\psi}) = f(\eta_0)\mathcal{J}_{\pi_1,\psi}(I_2)\mathcal{J}_{\pi_2,\psi}(I_2)\mathcal{J}_{\pi_3,\psi}(I_2) = 1,$$

which is indeed nonzero. □

4.6. **The Functional Equation.** We now combine Theorem 40 with the zeta function constructed in the previous subsection to define our gamma factor.

Lemma 69. *Fix three cuspidal irreducible representations π_1, π_2, and π_3 of GL_2. Then there is a unique constant $\gamma \in \mathbb{C}^\times$ such that*

$$Z(M_{w_6} f, W) = \gamma Z(f, W)$$

for any $f \in I(\omega)$ and $W \in \mathcal{W}(\pi_1, \psi) \otimes \mathcal{W}(\pi_2, \psi) \otimes \mathcal{W}(\pi_3, \psi)$.

Proof. Because M_{w_6} is GSp_6-invariant, we see that $(f, W) \mapsto Z(M_{w_6} f, W)$ is also $\mathrm{GL}_2^{(3)}$-invariant by Lemma 67. However, the space

$$\mathrm{Hom}_{\mathrm{GL}_2^{(3)}}(I(\omega) \otimes \pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C})$$

is one-dimensional by Proposition 11, so existence of the needed constant γ exists because Z is a nonzero element of the above space by Lemma 68 and hence a basis. □

Definition 70. Fix three cuspidal irreducible representations π_1, π_2, and π_3 of GL_2. Then the *γ-factor* is the unique $\Gamma(\pi_1 \times \pi_2 \times \pi_3, \psi)$ such that

$$Z(M_{w_6} f, W) = \Gamma(\pi_1 \times \pi_2 \times \pi_3, \psi) Z(f, W)$$

for any $f \in I(\omega)$ and $W \in \mathcal{W}(\pi_1, \psi) \otimes \mathcal{W}(\pi_2, \psi) \otimes \mathcal{W}(\pi_3, \psi)$.

Here are some immediate corollaries of our definition.

Corollary 71. *Fix three cuspidal irreducible representations π_1, π_2, and π_3 of GL_2. Then*

$$\Gamma(\pi_1 \times \pi_2 \times \pi_3, \psi) = \frac{1}{\#(ZN)} \sum_{\substack{g = (g_1, g_2, g_3) \in \mathrm{GL}_2^{(3)} \\ \eta_0 g \eta_0^{-1} = p w_6 u \\ p \in P_6^{\mathrm{sp}}, u \in U_6^{\mathrm{sp}}}} \widetilde{\omega}(p) \mathcal{J}_{\pi_1,\psi}(g_1) \mathcal{J}_{\pi_2,\psi}(g_2) \mathcal{J}_{\pi_3,\psi}(g_3).$$

Proof. Let $f_0 \in I(\omega)$ be the vector supported on $P_6^{\mathrm{sp}}\eta_0$ and defined by $f(p\eta_0) = \widetilde{\omega}(p)$ for each $p \in P_6^{\mathrm{sp}}$. Then the proof of Lemma 14 implies that $Z(f_0, \mathcal{J}) = 1$ where $\mathcal{J} = \mathcal{J}_{\pi_1,\psi} \otimes \mathcal{J}_{\pi_2,\psi} \otimes \mathcal{J}_{\pi_3,\psi}$. Thus, by definition, we find

$$\begin{aligned}\gamma(\pi_1 \times \pi_2 \times \pi_3, \psi) &= \sum_{g \in ZN \backslash \mathrm{GL}_2^{(3)}} M_{w_6} f_0(\eta_0 g) \mathcal{J}(g) \\ &= \sum_{g \in ZN \backslash \mathrm{GL}_2^{(3)}} \left(\sum_{u \in U_6^{\mathrm{sp}}} f_0(w_6 u \eta_0 g) \right) \mathcal{J}(g).\end{aligned}$$

Now, $f_0(w_6 u g) \neq 0$ if and only if we can write $w_6 u \eta_0 g = p\eta_0$ for some $p \in P_6^{\mathrm{sp}}$, which only happens when $\eta_0 g \eta_0^{-1} = u^{-1} w_6 p$. Such a decomposition of $\eta_0 g \eta_0^{-1}$ in $U_6^{\mathrm{sp}} w_6 P_6^{\mathrm{sp}}$ is unique (see, for example, the proof of Lemma 56), so the result follows upon writing out the definition of f_0. ◻

Corollary 72. *Fix three cuspidal irreducible representations π_1, π_2, and π_3 of GL_2. Then*

$$\Gamma\left(\pi_1^\vee \times \pi_2^\vee \times \pi_3^\vee, \psi^{-1}\right) = \overline{\Gamma(\pi_1 \times \pi_2 \times \pi_3, \psi)}.$$

Proof. This follows immediately from taking the conjugate of both sides of Corollary 71 and noting that

$$\overline{\mathcal{J}_{\pi,\psi}(g)} = \mathcal{J}_{\pi,\psi}\left(g^{-1}\right) = \mathcal{J}_{\pi^\vee,\psi^{-1}}(g)$$

by [Nie14, Propositions 3.5]. ◻

Remark 73. One can use Corollary 72 to compute the magnitude of Γ by using the functional equation twice, but doing this requires adjusting the functional equation somewhat. In particular, one is able to show that Γ is nonzero upon checking that none of the eigenvalues of $M_{w_6} \circ M_{w_6}$ are zero as done in Example 64.

5. Comparison with Local Field Scenario

The Local Langlands Correspondence gives us access to gamma factor on the Galois side, which are better understood as in §6, from the local p-adic scenario. In order to gain access to the Galois side from our current scenario over finite fields, we will demonstrate a way to lift our functional equation over finite fields to one over local p-adic fields, hence relating their respective gamma factors.

In this section, K is a local p-adic field, $\mathcal{O}_K$ is the ring of integers of K, $\mathfrak{p} \subset \mathcal{O}_K$ is the prime ideal of $\mathcal{O}_K$, $k := \mathcal{O}_K/\mathfrak{p}$ is the residue field of K, $q = |k|$, and $\nu \colon \mathcal{O}_K \to k$ is the valuation map. We will also denote ν as the valuation map on $\mathrm{GL}_n(\mathcal{O}_K)$, where the valuation is taken entry-wise. Additionally, we define once and for all an additive character $\psi \colon K \to \mathbb{C}^\times$ with conductor $\mathfrak{p}$. As such, the restriction of ψ to $\mathcal{O}_K$ induces a character on k, which by abuse of notation we will also label as ψ.

By convention, we set Haar measure dx on K so that $\mathrm{vol}(\mathfrak{p}) = 1$, and we set Haar measure $d^\times x$ on $K^\times$ by $d^\times x := dx/|x|$. Later on we will also want a Haar measure on $\mathrm{SL}_2(\mathcal{O}_K)$, which we normalize so that

$$\mathrm{vol}\left(\left\{\begin{bmatrix} 1+a & b \\ c & 1+d \end{bmatrix} \in \mathrm{SL}_2(\mathcal{O}_K) : a, b, c, d \in \mathfrak{p}\right\}\right) = 1.$$

All the listed groups are unimodular (in particular, either abelian or compact), so their left and right Haar measures align.

5.1. **Review of Level Zero Representations.** Given an irreducible cuspidal representation of $\mathrm{GL}_n(k)$ and some nonzero complex number $z \in \mathbb{C}^\times$, we can produce an irreducible supercuspidal representation of $\mathrm{GL}_n(K)$. Such representations of $\mathrm{GL}_n(K)$ are called of *level zero*.

Definition 74 (Level Zero Representation)**.** A representation π of $\mathrm{GL}_n(K)$ is of level zero if there exists an irreducible cuspidal representation σ of $\mathrm{GL}_n(k)$ and a representation Λ of $K^\times \cdot \mathrm{GL}_n(\mathcal{O})$ such that $\Lambda|_{\mathrm{GL}_n(\mathcal{O})} = \sigma \circ \nu$ and

$$\pi \cong \mathrm{ind}_{K^\times \mathrm{GL}_n(\mathcal{O})}^{\mathrm{GL}_n(K)} \Lambda,$$

where ind is smooth compact induction.

The representation Λ, and hence the representation π, can be recovered given just σ, which determines Λ on $\mathrm{GL}_n(\mathcal{O})$, and the central character ω_Λ of Λ. Likewise, ω_Λ is determined by the central character of σ and $s \coloneqq \omega_\Lambda(\varpi) \in \mathbb{C}^\times$. By [BK93, Theorem 8.4.1], this map $(\sigma, s) \mapsto \pi$ is a bijection to level zero representations of $\mathrm{GL}_n(K)$. The fact that π is irreducible supercuspidal also comes from [BK93].

Given this bijection, we may unambiguously denote a level zero representation as (σ, s) or σ_s, where σ is an irreducible cuspidal representation of $\mathrm{GL}_n(k)$ and $s \in \mathbb{C}^\times$.

5.2. **Lifting the Zeta Sum.** Throughout this subsection, we fix cuspidal representations π_1, π_2, π_3 of $\mathrm{GL}_2(k)$ which lift to level-zero supercuspidal representations Π_1, Π_2, Π_3 of $\mathrm{GL}_2(K)$ as described in section 5.1. By convention, we write $\lambda_i \coloneqq \omega_{\Pi_i}(\varpi)$, so the pair (π_i, λ_i) uniquely determines Π_i.

In this subsection, we examine how one can relate the finite-field Z-sum defined in section 4.5 with its local counterpart. This lifting process must be done in steps: we must know how to lift Whittaker functions, we must know to lift elements of $I(\omega)$, and lastly we must compare the Z-sum with the Z-integral.

To begin, we describe how to lift Whittaker functions.

Proposition 75 ([YZ20, Proposition 3.9])**.** *Let Π be a level-zero supercuspidal representation of $\mathrm{GL}_2(K)$ arising from the cuspidal representation π of $\mathrm{GL}_2(k)$ and $\lambda \coloneqq \omega_\Pi(\varpi)$. For any Whittaker function $W \in \mathcal{W}(\pi, \psi)$, there is a Whittaker function $\mathcal{L}W \in \mathcal{W}(\Pi, \psi)$ supported on $U_2(K)K^\times \mathrm{GL}_2(\mathcal{O}_K)$ such that*

$$\mathcal{L}W(uzg) = \psi(u)\omega_\Pi(z)W(\nu(g))$$

for any $u \in U_2(K)$ and $z \in K^\times$ and $g \in \mathrm{GL}_2(\mathcal{O}_K)$.

Next up, we must lift $f \in I(\omega_\pi)$ to $\mathcal{L}f \in I(\omega_\Pi, s, t)$. For brevity, given complex numbers $s, t \in \mathbb{C}$, we define $I(\omega_\Pi, s, t)$ as containing right $\mathrm{GSp}_6(\mathcal{O}_K)$-finite functions $f\colon\ \mathrm{GSp}_6(K) \to \mathbb{C}$ such that

$$f\left(\begin{bmatrix} \lambda A & * \\ & A^\iota \end{bmatrix} g\right) = \omega_\Pi(\lambda \det A)|\lambda|^s \, |\det A|^t \, f(g).$$

For brevity, we define

$$\omega_{\Pi,s,t}\left(\begin{bmatrix}\lambda A & * \\ & A^{\iota}\end{bmatrix}\right) := \omega_\Pi(\lambda \det A)|\lambda|^s \left|\det A\right|^t.$$

We are now ready to state our lifting result.

Proposition 76. *Fix notation as above, and let $f \in I(\omega_\pi)$. Then there is a function $\mathcal{L}_{s,t}f \in I(\omega_\Pi, s, t)$ supported on $P_6^{\mathrm{sp}}(K)\,\mathrm{GSp}_6(\mathcal{O}_K)$ such that*

$$\mathcal{L}_{s,t}f\left(\begin{bmatrix}\lambda A & * \\ & A^{\iota}\end{bmatrix} g\right) = \omega_\Pi(\lambda \det A)|\lambda|^s \left|\det A\right|^t f(\nu(g))$$

for any $\left[\begin{smallmatrix}\lambda A & * \\ & A^{\iota}\end{smallmatrix}\right] \in P_6^{\mathrm{sp}}(K)$ and $g \in \mathrm{GSp}_6(\mathcal{O}_K)$.

Proof. To begin, define a function $f_0\colon \mathrm{GSp}_6(K) \to \mathbb{C}$ by

$$f_0(g) := \begin{cases} f_0(\nu(g)) & \text{if } g \in \mathrm{GSp}_6(\mathcal{O}_K), \\ 0 & \text{if } g \notin \mathrm{GSp}_6(\mathcal{O}_K). \end{cases}$$

By construction, $f_0(hg) = \omega_\pi(\nu(h))f_0(g)$ for any $h \in P_6^{\mathrm{sp}}(\mathcal{O}_K)$ and $g \in \mathrm{GSp}_6(K)$. This allows us to define

$$\mathcal{L}_{s,t}f(g) := \int_{P_6^{\mathrm{sp}}(\mathcal{O}_K)\backslash P_6^{\mathrm{sp}}(K)} \omega_{\Pi,s,t}(h)^{-1} f_0(hg)\,dh.$$

The left-invariance property of f_0 by $P_6^{\mathrm{sp}}(\mathcal{O}_K)$ implies that this integral is well-defined (i.e., the integrand does not depend on choice of representative of $P_6^{\mathrm{sp}}(\mathcal{O}_K)\backslash P_6^{\mathrm{sp}}(K)$). It remains to check that $\mathcal{L}f$ satisfies the needed symmetry conditions. To begin, for $h_0 \in P_6^{\mathrm{sp}}$ and $g \in \mathrm{GSp}_6(K)$, we compute

$$\begin{aligned}\mathcal{L}_{s,t}f(h_0 g) &= \int_{P_6^{\mathrm{sp}}(\mathcal{O}_K)\backslash P_6^{\mathrm{sp}}(K)} \omega_{\Pi,s,t}(h)^{-1} f_0(hh_0 g)\,dh \\ &= \int_{P_6^{\mathrm{sp}}(\mathcal{O}_K)\backslash P_6^{\mathrm{sp}}(K)} \omega_{\Pi,s,t}\left(hh_0^{-1}\right)^{-1} f_0(hg)\,dh \\ &= \omega_{\Pi,s,t}(h_0)\mathcal{L}_{s,t}f(g).\end{aligned}$$

Additionally, we see that $\mathcal{L}_{s,t}f(g) = f_0(g) = f(\nu(g))$ for any $g \in \mathrm{GSp}_6(\mathcal{O}_K)$, which when combined with $\mathcal{L}_{s,t}f \in I(\omega, s)$ completes the proof. ☐

Remark 77. By the Iwasawa decomposition, $\mathrm{GSp}_6(K) = P_6^{\mathrm{sp}}(K)\,\mathrm{GSp}_6(\mathcal{O}_K)$, so the support of $\mathcal{L}_{s,t}f$ is not hindered by this condition.

The last piece we need before our theorem is to recall the definition of the Z-integral. For brevity, let $\mathcal{W}(\Pi,\psi)$ consist of functions of the form $W := W_1 \otimes W_2 \otimes W_3$ where $W_i \in \mathcal{W}(\Pi_i, \psi)$ for $i \in \{1,2,3\}$; then define $\mathcal{L}W$ to be the corresponding lift. Then for $f \in I(\omega_\Pi, s, t)$ and $W \in \mathcal{W}(\Pi,\psi)$, one defines

$$Z(f,W) := \int_{Z(K)N_0(K)\backslash\,\mathrm{GL}_2^{(3)}(K)} f(\eta_0 g) W(g)\,dg.$$

This integral absolutely converges for $\operatorname{Re} s, t \gg 0$. We are now ready to prove our result.

Theorem 78. *Fix notation and Haar measures as above. Then for* $\operatorname{Re} s, t \gg 0$ *and any* $f \in I(\omega)$ *and* $W \in \mathcal{W}(\pi, \psi)$*, we have*

$$Z(\mathcal{L}_{s,t} f, \mathcal{L} W) = (q-1) Z(f, W).$$

Proof. We compute $Z(\mathcal{L}_{s,t} f, \mathcal{L} W)$ directly. As in [Ike89, section 3.1], we note that each element of $Z(K) N_0(K) \backslash \operatorname{GL}_2^{(3)}(K)$ is represented by a matrix of the form

$$g \coloneqq \left(\begin{bmatrix} a & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} \begin{bmatrix} x & \\ & x^{-1} \end{bmatrix} g_1, \begin{bmatrix} a & \\ & 1 \end{bmatrix} g_2, \begin{bmatrix} a & \\ & 1 \end{bmatrix} \begin{bmatrix} y & \\ & y^{-1} \end{bmatrix} g_3 \right)$$

where $a, x, y \in K^\times$ and $b \in K$ and $g_1, g_2, g_3 \in \operatorname{SL}_2(\mathcal{O}_K)$. In this case, the Haar measure dg becomes $|axy|^{-2} \, d^\times a \, d^\times x \, d^\times y \, db \, dg_1 \, dg_2 \, dg_3$. Now, to have $W(g) \neq 0$, we claim that $a, x, y \in \mathcal{O}_K^\times$. We take this in cases.

- We show that $a \in \mathcal{O}_K^\times$. The main point is that we need $\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right] g_2$ to live in the support of W_2, which is $U_2(K) K^\times \operatorname{GL}_2(\mathcal{O}_K)$, so we must have

$$z \begin{bmatrix} 1 & u \\ & 1 \end{bmatrix} \begin{bmatrix} a & \\ & 1 \end{bmatrix} \in \operatorname{GL}_2(\mathcal{O}_K)$$

  for some $z, u \in K$. The matrix is upper-triangular, and the diagonal entries are az and z, so we see that $z \in \mathcal{O}_K^\times$ and then $a \in \mathcal{O}_K^\times$ are forced.
- We show $x \in \mathcal{O}_K^\times$; the argument that $y \in \mathcal{O}_K^\times$ is similar. Once again, the main point is that $\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} x & \\ & x^{-1} \end{smallmatrix}\right] g_1$ to live in the support of W_1, so we must have

$$z \begin{bmatrix} 1 & u \\ & 1 \end{bmatrix} \begin{bmatrix} a & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} \begin{bmatrix} x & \\ & x^{-1} \end{bmatrix} \in \operatorname{GL}_2(\mathcal{O}_K)$$

  for some $z \in K^\times$ and $u \in K$. Again, the diagonal entries of this matrix are zax and zx^{-1}; because $a \in \mathcal{O}_K^\times$, we see that $zx, zx^{-1} \in \mathcal{O}_K^\times$. Thus,

$$|x| = \sqrt{|zx| \, / \, |zx^{-1}|} = 1,$$

  so $x \in \mathcal{O}_K^\times$ follows.

We now apply the variable substitution $\left[\begin{smallmatrix} x & \\ & x^{-1} \end{smallmatrix}\right] g_1 \mapsto g_1$ and $\left[\begin{smallmatrix} y & \\ & y^{-1} \end{smallmatrix}\right] g_3 \mapsto g_3$; we do not pick up a modular character from this substitution because $\operatorname{SL}_2(\mathcal{O}_K)$ is compact and hence unimodular. At this point, we see $Z(\mathcal{L}_{s,t} f, \mathcal{L} W)$ equals

$$\left(\int_{\mathcal{O}_K^\times} d^\times x \right)^2 \int_K \int_{\mathcal{O}_K^\times} \int_{\operatorname{SL}_2(\mathcal{O}_K)^3} \mathcal{L}_{s,t} f \left(\eta_0 \left(\begin{bmatrix} a & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} g_1, \begin{bmatrix} a & \\ & 1 \end{bmatrix} g_2, \begin{bmatrix} a & \\ & 1 \end{bmatrix} g_3 \right) \right)$$
$$\mathcal{L} W_1 \left(\begin{bmatrix} a & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} g_1 \right) \mathcal{L} W_2 \left(\begin{bmatrix} a & \\ & 1 \end{bmatrix} g_2 \right) \mathcal{L} W_3 \left(\begin{bmatrix} a & \\ & 1 \end{bmatrix} g_3 \right) dg_1 \, dg_2 \, dg_3 \, d^\times a \, db.$$

The outside integral evaluates to $(q-1)^2$ because of how we have chosen to normalize our Haar measures. We now separate the computation into two cases.

- We integrate over $b \in \mathcal{O}_K$; label the relevant contribution $Z_{\mathcal{O}_K}(\mathcal{L}_{s,t} f, \mathcal{L} W)$. In this case, all matrices live in $\operatorname{GL}_d(\mathcal{O}_K)$ for suitable dimension d, so the constructions of

$\mathcal{L}_{s,t}f$ and $\mathcal{L}W_i$ immediately makes $Z_{\mathcal{O}_K}(\mathcal{L}_{s,t}f, \mathcal{L}W)$ equal

$$(q-1)^2 \sum_{\substack{b\in k, a\in k^\times \\ g_1,g_2,g_3\in \mathrm{SL}_2(k)}} f\left(\eta_0\left(\begin{bmatrix} a & \\ & 1\end{bmatrix}\begin{bmatrix} 1 & b \\ & 1\end{bmatrix} g_1, \begin{bmatrix} a & \\ & 1\end{bmatrix} g_2, \begin{bmatrix} a & \\ & 1\end{bmatrix} g_3\right)\right)$$

$$W_1\left(\begin{bmatrix} a & \\ & 1\end{bmatrix}\begin{bmatrix} 1 & b \\ & 1\end{bmatrix} g_1\right) W_2\left(\begin{bmatrix} a & \\ & 1\end{bmatrix} g_2\right) W_3\left(\begin{bmatrix} a & \\ & 1\end{bmatrix} g_3\right).$$

Notably, the Haar measure $d^\times a$ is simply da on $\mathcal{O}_K^\times$, and dg on $\mathrm{SL}_2(\mathcal{O}_K)$ was constructed to make this integration work as above. Anyway, rearranging by sending $\left[\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right] g_1 \mapsto g_1$, we see that $Z_{\mathcal{O}_K}(\mathcal{L}_{s,t}f, \mathcal{L}W)$ equals

$$q(q-1)^2 \sum_{\substack{a\in k^\times \\ g_1,g_2,g_3\in \mathrm{SL}_2(k)}} f\left(\eta_0\left(\begin{bmatrix} a & \\ & 1\end{bmatrix} g_1, \begin{bmatrix} a & \\ & 1\end{bmatrix} g_2, \begin{bmatrix} a & \\ & 1\end{bmatrix} g_3\right)\right)$$

$$W_1\left(\begin{bmatrix} a & \\ & 1\end{bmatrix} g_1\right) W_2\left(\begin{bmatrix} a & \\ & 1\end{bmatrix} g_2\right) W_3\left(\begin{bmatrix} a & \\ & 1\end{bmatrix} g_3\right)$$

At this point, we recognize that we have the following bijections.

$$\begin{array}{ccccccccc} k^\times & \times & \mathrm{SL}_2(k) & \times & \mathrm{SL}_2(k) & \times & \mathrm{SL}_2(k) & \cong & \mathrm{GL}_2^{(3)}(k) \\ (a & , & g_1 & , & g_2 & , & g_3) & \mapsto & \left(\left[\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right] g_1, \left[\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right] g_2, \left[\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right] g_3\right) \\ (\det g_1 & , & \left[\begin{smallmatrix} 1/\deg g_1 & \\ & 1\end{smallmatrix}\right] g_1 & , & \left[\begin{smallmatrix} 1/\deg g_1 & \\ & 1\end{smallmatrix}\right] g_2 & , & \left[\begin{smallmatrix} 1/\deg g_1 & \\ & 1\end{smallmatrix}\right] g_3) & \leftarrow\!\shortmid & (g_1, g_2, g_3) \end{array}$$

Thus, we see that $Z_{\mathcal{O}_K}(\mathcal{L}_{s,t}f, \mathcal{L}W)$ equals

$$q(q-1)^2 \sum_{g\in \mathrm{GL}_2^{(3)}(k)} f(\eta_0 g) W(g).$$

Modding out by $Z(k)N_0(k)$, which has magnitude $q(q-1)$, we see that this equals $(q-1)Z(f, W)$.

- We integrate over $b \notin \mathcal{O}_K$; label the relevant contribution $Z_{K\setminus\mathcal{O}_K}(\mathcal{L}_{s,t}f, \mathcal{L}W)$. We would like to move b as far as out as possible to gain better control of it. As such, we note

$$\begin{bmatrix} a & \\ & 1\end{bmatrix}\begin{bmatrix} 1 & b \\ & 1\end{bmatrix} = \begin{bmatrix} 1 & ab \\ & 1\end{bmatrix}\begin{bmatrix} a & \\ & 1\end{bmatrix}$$

and so apply the substitution $ab \mapsto b$; note $|a| = 1$, so db does not change. Doing so tells us that $Z_{K\setminus\mathcal{O}_K}(\mathcal{L}_{s,t}f, \mathcal{L}W)$ equals

$$(q-1)^2 \int_{K\setminus\mathcal{O}_K}\int_{\mathcal{O}_K^\times}\int_{\mathrm{SL}_2(\mathcal{O}_K)^3} \mathcal{L}_{s,t}f\left(\eta_0\left(\begin{bmatrix} 1 & b \\ & 1\end{bmatrix}\begin{bmatrix} a & \\ & 1\end{bmatrix} g_1, \begin{bmatrix} a & \\ & 1\end{bmatrix} g_2, \begin{bmatrix} a & \\ & 1\end{bmatrix} g_3\right)\right)$$

$$\psi(b)\mathcal{L}W_1\left(\begin{bmatrix} a & \\ & 1\end{bmatrix} g_1\right) \mathcal{L}W_2\left(\begin{bmatrix} a & \\ & 1\end{bmatrix} g_2\right) \mathcal{L}W_3\left(\begin{bmatrix} a & \\ & 1\end{bmatrix} g_3\right) dg_1\, dg_2\, dg_3\, d^\times a\, db.$$

We now apply an Iwasawa decomposition to move the b outside $\mathcal{L}_{s,t}f$. Explicitly, we find

$$\eta_0\left(\begin{bmatrix} 1 & b \\ & 1\end{bmatrix}, 1, 1\right) = \left(\begin{bmatrix} 1/b & -1 \\ & b\end{bmatrix}, 1, 1\right)\left(\begin{bmatrix} & 1 \\ -1 & 1/b\end{bmatrix}, 1, 1\right)\eta_0,$$

so $Z_{K\setminus\mathcal{O}_K}(\mathcal{L}_{s,t}f,\mathcal{L}W)$ equals

$$(q-1)^2\int_{K\setminus\mathcal{O}_K}\int_{\mathcal{O}_K^\times}\int_{\mathrm{SL}_2(\mathcal{O}_K)^3}\omega_{\Pi,s,t}\left(\left(\begin{bmatrix}1/b & -1\\ & b\end{bmatrix},1,1\right)\right)\psi(b)$$
$$\mathcal{L}_{s,t}f\left(\left(\begin{bmatrix} & 1\\ -1 & 1/b\end{bmatrix},1,1\right)\eta_0\left(\begin{bmatrix}a & \\ & 1\end{bmatrix}g_1,\begin{bmatrix}a & \\ & 1\end{bmatrix}g_2,\begin{bmatrix}a & \\ & 1\end{bmatrix}g_3\right)\right)$$
$$\mathcal{L}W_1\left(\begin{bmatrix}a & \\ & 1\end{bmatrix}g_1\right)\mathcal{L}W_2\left(\begin{bmatrix}a & \\ & 1\end{bmatrix}g_2\right)\mathcal{L}W_3\left(\begin{bmatrix}a & \\ & 1\end{bmatrix}g_3\right)\,dg_1\,dg_2\,dg_3\,d^\times a\,db.$$

Now, $1/b\in\mathfrak{p}$, so b does not impact any of the last two lines above, so factoring out the integral on b and arguing as in the previous case, we see that $Z_{K\setminus\mathcal{O}_K}(\mathcal{L}_{s,t}f,\mathcal{L}W)$ equals

$$(q-1)^2\underbrace{\int_{K\setminus\mathcal{O}_K}\omega_{\Pi,s,t}\left(\left(\begin{bmatrix}1/b & -1\\ & b\end{bmatrix},1,1\right)\right)\psi(b)\,db}_{I:=}\sum_{g\in\mathrm{GL}_2^{(3)}(k)}f\left(\left(\begin{bmatrix} & 1\\ -1 & \end{bmatrix},1,1\right)\eta_0 g\right)W(g).$$

It remains to compute the integral I. This computation is somewhat technical. Evaluating $\omega_{\Pi,s,t}$ and "stratifying" by $|b|$, this integral is

$$\begin{aligned}I&=\int_{K\setminus\mathcal{O}_K}\omega_\Pi(1/b)\,|1/b|^t\,\psi(b)\,db\\&=\int_{K\setminus\mathcal{O}_K}\omega_\Pi(b)^{-1}\,|b|^{-t-1}\,\psi(b)\,d^\times b\\&=\sum_{r=1}^\infty\int_{|b|=q^r}\omega_\Pi(b)^{-1}\,|b|^{-t-1}\,\psi(b)\,d^\times b.\end{aligned}$$

Setting $b\mapsto b\varpi^{-r}$ and then noting $d^\times b=db$ on $\mathcal{O}_K^\times$ yields

$$\begin{aligned}I&=\sum_{r=1}^\infty\left(\omega_\Pi(\varpi)^r q^{-r(t+1)}\int_{\mathcal{O}_K^\times}\omega_\Pi(b)^{-1}\psi(b\varpi^{-r})\,db\right)\\&=\sum_{r=1}^\infty\left(\omega_\Pi(\varpi)^r q^{-r(t+1)}\sum_{x\in k^\times}\int_{x+\mathfrak{p}}\omega_\Pi(b)^{-1}\psi(b\varpi^{-r})\,db\right)\\&=\sum_{r=1}^\infty\left(\omega_\Pi(\varpi)^r q^{-r(t+1)}\sum_{x\in k^\times}\omega_\pi(x)^{-1}\psi(x\varpi^{-r})\int_{\mathfrak{p}}\psi(b\varpi^{-r})\,db\right).\end{aligned}$$

Now, ψ is a nontrivial character on $\mathcal{O}_K$, so for $r\geq 1$, we see that $b\mapsto\psi(b\varpi^{-r})$ is a nontrivial character on $\mathfrak{p}^r=\varpi^r\mathcal{O}_K$ and hence on $\mathfrak{p}$. Thus, the integral vanishes, meaning we have no contribution in this case.

Tallying the contributions from the above cases completes the proof. ☐

5.3. **Lifting the Intertwining Operator.** Using notation from section 4.1, recall U_{2n}^+ is the unipotent radical, U_{2n}^- its analogue for lower triangular matrices, and $U_w^-:=U_{2n}^+\cap wU_{2n}^-w^{-1}$ for any Weyl group element $w\in W(\mathrm{GSp}_{2n})$. We have defined an intertwining operator

$M_{w_{2n}}\colon \operatorname{Ind}_{B_{2n}^{\mathrm{sp}}(k)}^{\mathrm{GSp}_{2n}(k)}\chi \to \operatorname{Ind}_{B_{2n}^{\mathrm{sp}}(k)}^{\mathrm{GSp}_{2n}(k)}{}^{w_{2n}}\chi$ using the long Weyl element, namely

$$(M_{w_{2n}}f)(g) \coloneqq \sum_{u \in U^-_{w_{2n}}(k)} f(w_{2n}ug),$$

where ${}^{w}\chi(g) \coloneqq \chi(w^{-1}gw)$. The analogous intertwining operator in the local p-adic case is an operator $\widetilde{M}^{\chi}_{w_{2n}} : \operatorname{ind}_{B_{2n}^{\mathrm{sp}}(K)}^{\mathrm{GSp}_{2n}(K)}\chi \to \operatorname{ind}_{B_{2n}^{\mathrm{sp}}(K)}^{\mathrm{GSp}_{2n}(K)}{}^{w_{2n}}\chi$ given by

$$(\widetilde{M}^{\chi}_{w_{2n}}f)(g) \coloneqq \int_{U^-_{w_{2n}}(K)} f(w_{2n}ug)\,du.$$

We may similarly define an intertwining operator for any Weyl group element $w \in W(\mathrm{GSp}_{2n})$, but we use the operator associated to w_{2n} in our functional equation producing the triple product gamma factor.

Our present objective is to relate $M_{w_{2n}}\colon \operatorname{Ind}_{B_{2n}^{\mathrm{sp}}(k)}^{\mathrm{GSp}_{2n}(k)}\chi \to \operatorname{Ind}_{B_{2n}^{\mathrm{sp}}(k)}^{\mathrm{GSp}_{2n}(k)}{}^{w_{2n}}\chi$ over finite field with the intertwining operator $\widetilde{M}^{\chi}_{w_{2n}}$ for $n=3$ and $\chi = \omega_{\Pi,s,t}$. We will first work for general χ, then determine what happens for $\chi = \omega_{\Pi,s,t}$.

Following the convention of [Cas75], if $\sigma : B_{2n}^{\mathrm{sp}} \to \mathrm{GL}(W)$ is a representation, we will define smooth compact induction as

$$\operatorname{ind}_{B_{2n}^{\mathrm{sp}}}^{\mathrm{GSp}_{2n}}\sigma \coloneqq \{f : \mathrm{GSp}_{2n} \to W \text{ locally compact} \mid f(bg) = \sigma(b)\delta_B^{1/2}(b)f(g)\,\forall\, b \in B_{2n}^{\mathrm{sp}}, g \in \mathrm{GSp}_{2n}\}.$$

For all practical purposes, $\sigma = \chi$ will be a character on B_{2n}^{sp}. One can routinely compute the modular quasicharacter δ_B of B_6^{sp} to be

$$\delta_B\left(\begin{bmatrix} \lambda x_1 & & & & & \\ & \lambda x_2 & & & * & \\ & & \lambda x_3 & & & \\ & & & x_1^{-1} & & \\ & & & & x_2^{-1} & \\ & & & & & x_3^{-1} \end{bmatrix}\right) = |\lambda|^6|x_1|^6|x_2|^4|x_3|^2.$$

It is difficult to interface with our intertwining operator $\widetilde{M}^{\chi}_{w_{2n}}$ directly. Instead, the name of the game will be to decompose $\widetilde{M}^{\chi}_{w_{2n}}$ into intertwining operators $\widetilde{M}^{\chi}_{s_i}$ associated to simple reflections s_i, and then track our lifted test function through each simple reflection.

The Weyl group $W(\mathrm{GSp}_6)$ is of Cartan type C_3 and thus has three simple reflections s_1, s_2, s_3, defined below.

$$s_1 \coloneqq \begin{bmatrix} & -1 & & & & \\ 1 & & & & & \\ & & 1 & & & \\ & & & & -1 & \\ & & & 1 & & \\ & & & & & 1 \end{bmatrix}, \quad s_2 \coloneqq \begin{bmatrix} 1 & & & & & \\ & & -1 & & & \\ & 1 & & & & \\ & & & 1 & & \\ & & & & & -1 \\ & & & & 1 & \end{bmatrix}, \quad s_3 \coloneqq \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & & & & -1 \\ & & & 1 & & \\ & & & & 1 & \\ & & 1 & & & \end{bmatrix}$$

Since we wish to work with each $\widetilde{M}^{\chi}_{s_i}$, which is an integral over $U^-_{s_i}$, it is useful to provide these sets explicitly. Following the definition $U^-_{s_i} \coloneqq U^+_6 \cap s_i U^-_6 s_i^{-1}$, we can compute

$$U^-_{s_1} = \left\{ \begin{bmatrix} 1 & -x & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & x & 1 & \\ & & & & & 1 \end{bmatrix} : x \in K \right\}$$

$$U^-_{s_2} = \left\{ \begin{bmatrix} 1 & & & & & \\ & 1 & -x & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & x & 1 \end{bmatrix} : x \in K \right\}$$

$$U^-_{s_3} = \left\{ \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & -x \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} : x \in K \right\}.$$

As Weyl group elements, one can compute $w_6 = s_3 s_2 s_1 s_3 s_2 s_3$. (As elements of GSp_6, the two sides differ by a maximal torus element.) This decomposition of w_6 in $W(\mathrm{GSp}_6)$ translates to a decomposition of $\widetilde{M}^{\chi}_{w_6}$ in the Hecke algebra, namely

$$\widetilde{M}^{\chi}_{w_6}\mathcal{L}f = \widetilde{M}^{s_2 s_1 s_3 s_2 s_3 \chi}_{s_3} \widetilde{M}^{s_1 s_3 s_2 s_3 \chi}_{s_2} \widetilde{M}^{s_3 s_2 s_3 \chi}_{s_1} \widetilde{M}^{s_2 s_3 \chi}_{s_3} \widetilde{M}^{s_3 \chi}_{s_2} \widetilde{M}^{\chi}_{s_3} \mathcal{L}f.$$

We will describe these twisted characters. Fix any Borel character $\chi \coloneqq \tau_{z_m} \otimes \alpha_{z_1} \otimes \beta_{z_2} \otimes \mu_{z_3}$, where

$$\chi\left(\begin{bmatrix} \lambda x_1 & & & & & \\ & \lambda x_2 & & & * & \\ & & \lambda x_3 & & & \\ & & & x_1^{-1} & & \\ & & & & x_2^{-1} & \\ & & & & & x_3^{-1} \end{bmatrix}\right) = \tau(\lambda)\alpha(x_1)\beta(x_2)\mu(x_3)|\lambda|^{z_m}|x_1|^{z_1}|x_2|^{z_2}|x_3|^{z_3}.$$

(In fact, all characters of the Borel subgroup are induced from characters on the maximal split torus, hence are of this form.) Then, we may describe the twisted characters on each component:

$$\begin{aligned} {}^{s_3}\chi &= \tau\mu^{-1}_{z_m - z_3} \otimes \alpha_{z_1} \otimes \beta_{z_2} \otimes \mu^{-1}_{-z_3} \\ {}^{s_2 s_3}\chi &= \tau\mu^{-1}_{z_m - z_3} \otimes \alpha_{z_1} \otimes \mu^{-1}_{-z_3} \otimes \beta_{z_2} \\ {}^{s_2 s_3 s_3}\chi &= \tau\beta^{-1}\mu^{-1}_{z_m - z_2 - z_3} \otimes \alpha_{z_1} \otimes \mu^{-1}_{-z_3} \otimes \beta^{-1}_{-z_2} \\ {}^{s_1 s_3 s_2 s_3}\chi &= \tau\beta^{-1}\mu^{-1}_{z_m - z_2 - z_3} \otimes \mu^{-1}_{-z_3} \otimes \alpha_{z_1} \otimes \beta^{-1}_{-z_2} \\ {}^{s_2 s_1 s_3 s_2 s_3}\chi &= \tau\beta^{-1}\mu^{-1}_{z_m - z_2 - z_3} \otimes \mu^{-1}_{-z_3} \otimes \beta^{-1}_{-z_2} \otimes \alpha_{z_1} \end{aligned}$$

$$ {}^{w_6}\chi = \tau\alpha^{-1}\beta^{-1}\mu^{-1}_{z_m-z_1-z_2-z_3} \otimes \mu^{-1}_{-z_3} \otimes \beta^{-1}_{-z_2} \otimes \alpha^{-1}_{-z_1}. $$

For a representation $\sigma : B^{\mathrm{sp}}_{2n}(K) \to \mathrm{GL}(W)$, smooth compact induction is defined as

$$ \mathrm{ind}_{B^{\mathrm{sp}}_{2n}(K)}^{\mathrm{GSp}_{2n}(K)} \sigma \coloneqq \{f : \mathrm{GSp}_{2n} \to W \mid f(bg) = \delta_B^{1/2}(b)\sigma(b)f(g) \,\forall\, b \in B^{\mathrm{sp}}_{2n}\}. $$

Given a character $\chi_0 = \tau\otimes\alpha_1\otimes\cdots\otimes\alpha_n$ of $B^{\mathrm{sp}}_{2n}(k)$, some lifting $\chi = \tau_{z_m}\otimes(\alpha_1)_{z_1}\otimes\cdots\otimes(\alpha_n)_{z_n}$ character of $B^{\mathrm{sp}}_{2n}(K)$, a function $f \in \mathrm{Ind}_{B^{\mathrm{sp}}_{2n}(k)}^{\mathrm{GSp}_{2n}(k)} \chi_0$, and its corresponding inflation $f_0 : \mathrm{GSp}_{2n}(K) \to \mathbb{C}$ as defined in the proof of Proposition 76, we see that the lifting of functions given by

$$ \mathcal{L}f(g) \coloneqq \int_{B^{\mathrm{sp}}_{2n}(\mathcal{O}_K)\backslash B^{\mathrm{sp}}_{2n}(K)} \delta_B^{-1/2}\chi^{-1}(b)f_0(bg)\,db $$

satisfies $\mathcal{L}f \in \mathrm{ind}_{B^{\mathrm{sp}}_{2n}(K)}^{\mathrm{GSp}_{2n}(K)} \chi$ and $\mathcal{L}f|_{\mathrm{GSp}_{2n}(\mathcal{O}_K)} = f_0|_{\mathrm{GSp}_{2n}(\mathcal{O}_K)}$.

We are now ready to relate each $\widetilde{M}^{\chi}_{s_i}\mathcal{L}f$ to its corresponding intertwining operator $M_{s_i}f$ over finite fields. To establish notation, let $\chi_0 = \tau\otimes\alpha\otimes\beta\otimes\mu$, lift it to $\chi = \tau_{z_m}\otimes\alpha_{z_1}\otimes\beta_{z_2}\otimes\mu_{z_3}$, and fix $f \in \mathrm{Ind}_{B^{\mathrm{sp}}_6(k)}^{\mathrm{GSp}_6(k)}$. For the proceeding computations, we will use the equalities

$$ \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & -x \\ & 1 \end{bmatrix} = \begin{bmatrix} 1/x & 1 \\ & x \end{bmatrix}\begin{bmatrix} 1 & \\ -1/x & 1 \end{bmatrix} $$
$$ \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & \\ x & 1 \end{bmatrix} = \begin{bmatrix} x & \\ -1 & 1/x \end{bmatrix}\begin{bmatrix} 1 & 1/x \\ & 1 \end{bmatrix}. $$

This is an Iwasawa decomposition when $|x| > 1$.

For the following, we may assume our support of $\widetilde{M}^{\chi}_{s_i}\mathcal{L}f$ to be $\mathrm{GSp}_6(\mathcal{O}_K)$, as for any element $g \in \mathrm{GSp}_6(K)$, we can find an Iwasawa decomposition $g = bk$ and factor out the Borel term b using our definition of compact induction. Thus, we can fairly assume $g \in \mathrm{GSp}_6(\mathcal{O}_K)$. We have

$$ \left(\widetilde{M}^{\chi}_{s_1}\mathcal{L}f\right)(g) = \int_{U^-_{s_1}} \mathcal{L}f(s_1^{-1}ug)\,du = \int_K \mathcal{L}\left(s_1^{-1}\begin{bmatrix} 1 & -x & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & x & 1 & \\ & & & & & 1 \end{bmatrix} g \right) dx $$

$$ = \int_{\mathcal{O}_K} \mathcal{L}f\left(s_1^{-1}\begin{bmatrix} 1 & -x & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & x & 1 & \\ & & & & & 1 \end{bmatrix} g \right) dx $$

$$ + \int_{|x|>1} \mathcal{L}f\left(\begin{bmatrix} 1/x & 1 & & & & \\ & x & & & & \\ & & 1 & & & \\ & & & x & & \\ & & & -1 & 1/x & \\ & & & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & & & \\ -1/x & 1 & & & & \\ & & 1 & & & \\ & & & 1 & 1/x & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} g \right) dx $$

$$= \sum_{x \in k} f\left(s_1^{-1} \begin{bmatrix} 1 & -x & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & x & 1 & \\ & & & & & 1 \end{bmatrix} \nu(g) \right)$$

$$+ f(\nu(g)) \int_{|x|>1} \delta_B^{1/2} \cdot \chi \left(\begin{bmatrix} 1/x & 1 & & & & \\ & x & & & & \\ & & 1 & & & \\ & & & x & & \\ & & & -1 & 1/x & \\ & & & & & 1 \end{bmatrix} \right) dx$$

$$= \left(M_{s_1} f\right)(\nu(g)) + f(\nu(g)) \int_{|x|>1} \alpha(x)^{-1} |x|^{-z_1} \beta(x) |x|^{z_2} \delta_B^{1/2} \left(\begin{bmatrix} 1/x & 1 & \\ & x & \\ & & 1 \end{bmatrix} \right) |x| \, d^\times x$$

$$= \mathcal{L}\left(M_{s_1} f\right)(g) + \mathcal{L}f(g) \int_{|x|>1} \beta\alpha^{-1}(x) |x|^{z_2 - z_1} \cdot \left| \frac{1}{|x|^2} \right|^{1/2} \cdot |x| \, d^\times x$$

$$= \mathcal{L}\left(M_{s_1} f\right)(g) + \mathcal{L}f(g) \int_{|x|<1} \alpha\beta^{-1}(x) |x|^{z_1 - z_2} \, d^\times x$$

$$= \mathcal{L}\left(M_{s_1} f\right)(g) + \mathcal{L}f(g) \sum_{r \geq 1} q^{-r(z_1 - z_2)} \int_{\mathcal{O}_K^\times} \alpha\beta^{-1}(x) \, d^\times x$$

$$= \mathcal{L}\left(M_{s_1} f\right)(g) + \mathcal{L}f(g) \cdot \delta_{\alpha,\beta} (q-1) \frac{q^{z_2 - z_1}}{1 - q^{z_2 - z_1}},$$

so we conclude

$$\widetilde{M}_{s_1}^{\chi} \mathcal{L} f = \mathcal{L}\left(M_{s_1} f\right) + \delta_{\alpha,\beta} \cdot (q-1) \frac{q^{z_2 - z_1}}{1 - q^{z_2 - z_1}} \mathcal{L} f.$$

The procedure for $\widetilde{M}_{s_2}^{\chi} \mathcal{L} f$ follows a similar story:

$$\left(\widetilde{M}_{s_2}^{\chi} \mathcal{L} f\right)(g) = \int_{U_{s_2}^-} \mathcal{L} f(s_2^{-1} u g) \, du = \int_K \mathcal{L} f \left(s_2^{-1} \begin{bmatrix} 1 & & & & & \\ & 1 & -x & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & x & 1 \end{bmatrix} g \right) dx$$

$$= \int_{\mathcal{O}_K} \mathcal{L} f \left(s_2^{-1} \begin{bmatrix} 1 & & & & & \\ & 1 & -x & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & x & 1 \end{bmatrix} g \right) dx$$

$$
+\int_{|x|>1}\mathcal{L}f\left(\begin{bmatrix}1&&&&&\\&1/x&1&&&\\&&x&&&\\&&&1&&\\&&&&x&\\&&&&-1&1/x\end{bmatrix}\begin{bmatrix}1&&&&&\\&1&&&&\\&-1/x&1&&&\\&&&1&&\\&&&&1&1/x\\&&&&&1\end{bmatrix}g\right)dx
$$

$$
=\sum_{x\in k}f\left(s_2^{-1}\begin{bmatrix}1&&&&&\\&1&-x&&&\\&&1&&&\\&&&1&&\\&&&&1&\\&&&&x&1\end{bmatrix}\nu(g)\right)
$$

$$
+f(\nu(g))\int_{|x|>1}\delta_B^{1/2}\cdot\chi\left(\begin{bmatrix}1&&&&&\\&1/x&1&&&\\&&x&&&\\&&&1&&\\&&&&x&\\&&&&-1&1/x\end{bmatrix}\right)dx
$$

$$
=(M_{s_2}f)(\nu(g))+f(\nu(g))\int_{|x|>1}\beta(x)^{-1}|x|^{-z_2}\mu(x)|x|^{z_3}\delta_B^{1/2}\left(\begin{bmatrix}1&&\\&1/x&1\\&&x\end{bmatrix}\right)|x|\,d^\times x
$$

$$
=\mathcal{L}(M_{s_2}f)(g)+\mathcal{L}f(g)\int_{|x|>1}\beta^{-1}\mu(x)|x|^{z_3-z_2}\left|\frac{1}{x^2}\right|^{1/2}\cdot|x|\,d^\times x
$$

$$
=\mathcal{L}(M_{s_2}f)(g)+\mathcal{L}f(g)\int_{|x|<1}\beta\mu^{-1}(x)|x|^{z_2-z_3}\,d^\times x
$$

$$
=\mathcal{L}(M_{s_2}f)(g)+\mathcal{L}f(g)\sum_{r\geq 1}q^{r(z_3-z_2)}\int_{\mathcal{O}_K^\times}\beta\mu^{-1}(x)\,d^\times x
$$

$$
=\mathcal{L}(M_{s_2}f)(g)+\mathcal{L}f(g)\cdot\delta_{\beta,\mu}(q-1)\frac{q^{z_3-z_2}}{1-q^{z_3-z_2}},
$$

so

$$
\widetilde{M}^\chi_{s_2}\mathcal{L}f=\mathcal{L}(M_{s_2}f)+\delta_{\beta,\mu}\cdot(q-1)\frac{q^{z_3-z_2}}{1-q^{z_3-z_2}}\mathcal{L}f.
$$

Finally, we run these simplifications on the intertwining operator for s_3.

$$
\left(\widetilde{M}^\chi_{s_3}\mathcal{L}f\right)(g)=\int_{U^-_{s_3}}\mathcal{L}f(s_3^{-1}ug)\,du=\int_K\mathcal{L}f\left(s_3^{-1}\begin{bmatrix}1&&&&&\\&1&&&&\\&&1&&&-x\\&&&1&&\\&&&&1&\\&&&&&1\end{bmatrix}g\right)dx
$$

$$
\begin{aligned}
&= \int_{\mathcal{O}_K} \mathcal{L}f\left(s_3^{-1} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & -x \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} g \right) dx \\
&\qquad + \int_{|x|>1} \mathcal{L}f\left(\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1/x & & & 1 \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & x \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & -1/x & & & 1 \end{bmatrix} g \right) dx \\
&= \sum_{x \in k} f\left(s_3^{-1} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & -x \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \nu(g) \right) \\
&\qquad + f(\nu(g)) \int_{|x|>1} \delta_B^{1/2} \cdot \chi \left(\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1/x & & & 1 \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & x \end{bmatrix} \right) dx \\
&= (M_{s_3} f)(\nu(g)) + f(\nu(g)) \int_{|x|>1} \mu\left(\frac{1}{x}\right) \left|\frac{1}{x}\right|^{z_3} (|x|^{-2})^{1/2} |x| \, d^\times x \\
&= \mathcal{L}(M_{s_3} f)(g) + \mathcal{L}f(g) \int_{|x|<1} \mu(x) |x|^{z_3} \, d^\times x \\
&= \mathcal{L}(M_{s_3} f)(g) + \mathcal{L}f(g) \sum_{r \geq 1} q^{r(-z_3)} \int_{\mathcal{O}_K^\times} \mu(x) \, d^\times x \\
&= \mathcal{L}(M_{s_3} f)(g) + \mathcal{L}f(g) \cdot \mathbb{1}_{\mu=1} (q-1) \frac{q^{-z_3}}{1-q^{-z_3}}.
\end{aligned}
$$

To summarize, given a fixed character $\chi = \tau_{z_m} \otimes \alpha_{z_1} \otimes \beta_{z_2} \otimes \mu_{z_3}$ of B_6^{sp} and $f \in \operatorname{Ind}_{B_6^{\mathrm{sp}}(k)}^{\mathrm{GSp}_6(k)} \chi_0$,

$$
\begin{aligned}
\widetilde{M}_{s_1}^{\chi} \mathcal{L}f &= \mathcal{L}\left(M_{s_1} f\right) + \delta_{\alpha,\beta} \cdot (q-1) \frac{q^{z_2 - z_1}}{1 - q^{z_2 - z_1}} \mathcal{L}f \\
\widetilde{M}_{s_2}^{\chi} \mathcal{L}f &= \mathcal{L}\left(M_{s_2} f\right) + \delta_{\beta,\mu} \cdot (q-1) \frac{q^{z_3 - z_2}}{1 - q^{z_3 - z_2}} \mathcal{L}f \\
\widetilde{M}_{s_3}^{\chi} \mathcal{L}f &= \mathcal{L}\left(M_{s_3} f\right) + \mathbb{1}_{\mu=1} \cdot (q-1) \frac{q^{-z_3}}{1 - q^{-z_3}} \mathcal{L}f
\end{aligned}
$$

We can now carefully expand $\widetilde{M}_{w_6}^{\chi} \mathcal{L}f = \widetilde{M}_{s_3}^{s_2 s_1 s_3 s_2 s_3 \chi} \widetilde{M}_{s_2}^{s_1 s_3 s_2 s_3 \chi} \widetilde{M}_{s_1}^{s_3 s_2 s_3 \chi} \widetilde{M}_{s_3}^{s_2 s_3 \chi} \widetilde{M}_{s_2}^{s_3 \chi} \widetilde{M}_{s_3}^{\chi} \mathcal{L}f$ using the above three equations. We will also use the relation $M_{s_3}^2 = (q-1) M_{s_3} + q$, which comes from M_{s_3} being a generator of the Iwahori-Hecke algebra of $W(\mathrm{GSp}_6)$. Since we hope for χ to come from the product of central characters ω_π, we will enforce $\alpha = \beta = \mu = \omega_\Pi$. Assuming $\omega_\Pi \neq 1$,

$$\begin{aligned}
\widetilde{M}^{\chi}_{w_6}\mathcal{L}f = \mathcal{L}(M_{w_6}f) + \mathbb{1}_{\omega_\Pi^2=1} \cdot (q-1) \Bigg(&\frac{q^{-z_2-z_1}}{1-q^{-z_2-z_1}}\mathcal{L}(M_{s_3}M_{s_1}M_{s_3}M_{s_2}M_{s_3}f \\
&+ \frac{q^{-z_3-z_1}}{1-q^{-z_3-z_1}}\mathcal{L}(M_{s_3}M_{s_2}M_{s_3}M_{s_2}M_{s_3}f) + (q-1)\frac{q^{-z_3-z_2}}{1-q^{-z_3-z_2}}\mathcal{L}(M_{s_3}M_{s_2}M_{s_1}M_{s_3}f) \\
&+ (q-1)^2\frac{q^{-z_3-z_1}}{1-q^{-z_3-z_1}}\left(\frac{q^{-z_2-z_1}}{1-q^{-z_2-z_1}} + \frac{q^{-z_3-z_2}}{1-q^{-z_3-z_2}}\right)\mathcal{L}(M_{s_3}M_{s_2}M_{s_3}f) \\
&+ (q-1)^2\frac{(q^{-z_3-z_2})(q^{-z_2-z_1})}{(1-q^{-z_3-z_2})(1-q^{-z_2-z_1})}\mathcal{L}(M_{s_3}M_{s_1}M_{s_3}f) \\
&+ q\frac{q^{-z_3-z_2}}{1-q^{-z_3-z_2}}\mathcal{L}(M_{s_3}M_{s_2}M_{s_1}f) \\
&+ (q-1)q\frac{(q^{-z_3-z_1})(q^{-z_2-z_1})}{(1-q^{-z_3-z_1})(1-q^{-z_2-z_1})}\mathcal{L}(M_{s_2}M_{s_3}f) \\
&+ (q-1)q\frac{(q^{-z_3-z_2})(q^{-z_2-z_1})}{(1-q^{-z_3-z_2})(1-q^{-z_2-z_1})}\mathcal{L}(M_{s_3}M_{s_1}f) \\
&+ (q-1)q\frac{(q^{-z_3-z_2})(q^{-z_3-z_1})}{(1-q^{-z_3-z_2})(1-q^{-z_3-z_1})}\mathcal{L}(M_{s_3}M_{s_2}f) \\
&+ (q-1)^2((q-1)^2+q)\frac{q^{-z_3-z_2}q^{-z_3-z_1}q^{-z_2-z_1}}{(1-q^{-z_3-z_2})(1-q^{-z_3-z_1})(1-q^{-z_2-z_1})}\mathcal{L}(M_{s_3}f) \\
&+ (q-1)^3q\frac{q^{-z_3-z_2}q^{-z_3-z_1}q^{-z_2-z_1}}{(1-q^{-z_3-z_2})(1-q^{-z_3-z_1})(1-q^{-z_2-z_1})}\mathcal{L}f\Bigg).
\end{aligned}$$

Although we declared $\omega_\Pi \neq 1$ to manage the number of error terms, if we were to relieve this assumption (i.e., if we kept track of the error terms contributed by $\widetilde{M}^{\chi}_{s_3}$), then one can show the error term would have common denominator $(1-q^{2-z_1})(1-q^{2-z_2})(1-q^{2-z_3})$. We will use this shortly to identify the poles of the lifted intertwining operator.

Now we choose appropriate values for z_m, z_1, z_2, z_3. Following [Ike99, p.303-4], we normalize our intertwining operator as

$$\left(\widetilde{M}^{\chi}_{w_6}\right)^* := \gamma(2s-2,\chi,\psi)\gamma(4s-3,\chi^2,\psi)\widetilde{M}^{\chi}_{w_6},$$

where the gamma factor is defined by

$$\gamma(s,\chi,\psi) := \varepsilon(s,\chi,\psi)\frac{L(\chi^{-1},1-s)}{L(\chi,s)} = \varepsilon(s,\chi,\psi)\frac{1-q^{-s}\chi(\varpi)}{1-q^{-(1-s)}\chi(\varpi)^{-1}}.$$

The normalizing factors $\gamma(2s-2,\chi,\psi)$ and $\gamma(4s-3,\chi^2,\psi)$ suggest that $\widetilde{M}^{\chi}_{w_6}$ has a pole at $4s-3=0$ when $\chi^2=1$, and an additional pole at $2s-2=0$ when $\chi=1$.

Again following [Ike99, p.303], we define the degenerate principal series $I(\omega,s)$ as the space of functions $f : \mathrm{GSp}_6(K) \to \mathbb{C}$ such that for any $p = \begin{bmatrix} \lambda A & * \\ & A^\iota \end{bmatrix}$ and $g \in \mathrm{GSp}_6(K)$, we have

$$f(pg) = \omega(\lambda \det A)|\lambda|^{3s+3/2}|\det A|^{2s+1}f(g) = \omega_{\Pi,3s+3/2,2s+1}f(g).$$

We will want to apply our intertwining operator on $I(\omega'_\Pi, s)$ for some twisting $\omega'_\Pi := (\omega_\Pi)_{z_m} \otimes (\omega_\Pi)_{z_1} \otimes (\omega_\Pi)_{z_2} \otimes (\omega_\Pi)_{z_3}$ of ω_Π. By definition of compact induction, $I(\omega'_\Pi, s) \subseteq \mathrm{ind}_{B_6^{\mathrm{sp}}(K)}^{\mathrm{GSp}_6(K)} \chi$ for a character χ such that $\delta_B^{1/2} \cdot \chi = \omega'_{\Pi,3s+3/2,2s+1}$. Using our previous computation of δ_B, we have

$$\chi\left(\begin{bmatrix} \lambda A & * \\ & A^\iota \end{bmatrix}\right) = \omega_\Pi(\lambda \det A)|\lambda|^{3s-3/2+z_m}|x_1|^{2s-2+z_1}|x_2|^{2s-1+z_2}|x_3|^{2s+z_3},$$

where A has diagonal entries x_1, x_2, x_3. Let $y_1(s) := 2s - 2 + z_1$, $y_2(s) := 2s - 1 + z_2$, $y_3(s) := 2s + z_3$. The poles for the error term occurring when $\omega_\Pi^2 = 1$ happen at each of the following three equalities:

$$\begin{cases} y_2(s) = -y_3(s) \\ y_2(s) = -y_1(s) \\ y_3(s) = -y_1(s). \end{cases}$$

We expect there to be only one pole at $s = 3/4$, so these equalities must all be true for $s = 3/4$. In particular, this yields $y_1(3/4) = -y_2(3/4) = y_3(3/4) = -1/2$. Solving for the z_i's accordingly, we have

$$z_1 = \frac{1}{2}, \qquad z_2 = -\frac{1}{2}, \qquad z_3 = -\frac{3}{2},$$

so these are our values for z_1, z_2, z_3. Our choice for z_m is inconsequential, so we may freely set $z_m = 0$.

6. Gamma Factors from the Galois Side

Although this project aims to perform all computations strictly on the representation theory side, certain powerful correspondences between the representation/automorphic side and the Galois side allow us to "preview" our results from the Galois side.

To elaborate, let K be a local p-adic field and k its residue field, which we know to be finite. There is a way to lift a representation of $\mathrm{GL}_n(k)$ to a certain class of representations of $\mathrm{GL}_n(K)$, and the Local Langlands Correspondence gives us a bijection between irreducible admissible complex representations of $\mathrm{GL}_n(K)$ and certain representations of the Weil group, called Weil-Deligne representations. These are our central objects of interest on the Galois side, which we introduce first.

Along each correspondence, we have explicit relations between the epsilon factors. Thus, computations of epsilon factors of Weil-Deligne representations will inform us on the epsilon factor of representations of $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$, up to sign and power of $q = |k|$.

We will adopt the same notation as in the previous section. Additionally, let k_n be the (unique) degree-n field extension of k in $\overline{k}$.

6.1. **Weil-Deligne Representations.**

As these are representations of the Weil group, it makes sense to begin by defining the Weil group.

We recall some results from local class field theory. Since k_n/k is a finite extension of a finite field, the extension is Galois and it is cyclic of order n, with generator the Frobenius element $\Phi_k : x \mapsto x^q$. Furthermore, for $m < n$, we have natural projection maps

$\mathrm{Gal}(k_n/k) \to \mathrm{Gal}(k_m/k)$ where any $\sigma \in \mathrm{Gal}(k_n/k)$ is restricted to an automorphism of k_m. Thus, $\{\mathrm{Gal}(k_n/k)\}_n$ forms a directed system, so we can write

$$\mathrm{Gal}(\overline{k}/k) = \varprojlim_n \mathrm{Gal}(k_n/k) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}},$$

where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$. Note that $\widehat{\mathbb{Z}}$ comes equipped with a natural profinite topology, and the Frobenius element Φ_K topologically generates $\mathrm{Gal}(\overline{k}/k)$, i.e. $\langle \Phi_k \rangle \subset \mathrm{Gal}(\overline{k}/k)$ is dense.

We also have a natural map $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(\overline{k}/k)$. Any $\sigma \in \mathrm{Gal}(\overline{K}/K)$ restricts to an automorphism on $\overline{\mathcal{O}}$, which in turn induces an automorphism on $\overline{k} = \overline{\mathcal{O}}/\mathfrak{P}$ (here, $\mathfrak{P}$ is the prime of $\overline{\mathcal{O}}$). Since $\mathfrak{P}$ is a prime lying over $\mathfrak{p}$, this induced automorphism of $\overline{k}$ fixes k, so we have produced an element of $\mathrm{Gal}(\overline{k}/k)$. Call this map $\pi : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(\overline{k}/k)$. We now have a short exact sequence

$$1 \to I_K \to \mathrm{Gal}(\overline{K}/K) \xrightarrow{\pi} \mathrm{Gal}(\overline{k}/k) \simeq \widehat{\mathbb{Z}} \to 1,$$

where I_K is the inertia group of K.

Definition 79 (Weil group)**.** The **Weil group** W_K of K is a topological group, where as a group, $W_K := \pi^{-1}(\langle \Phi_K \rangle) \subset \mathrm{Gal}(\overline{K}/K)$, and its topology is such that $\pi : W_K \to \langle \Phi_K \rangle \simeq \mathbb{Z}$ is continuous ($\mathbb{Z}$ has the discrete topology) and the induced subspace topology on I_K coincides with the induced subspace topology from $I_K \subset \mathrm{Gal}(\overline{K}/K)$.

Equivalently, we could define the Weil group as the pullback of the following:

$$\begin{tikzcd} 1 \arrow[r] & I_K \arrow[r] \arrow[d] & W_K \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & I_K \arrow[r] & \mathrm{Gal}(\overline{K}/K) \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & 1 \end{tikzcd}$$

One can observe that we can express W_K as the semidirect product $W_K = I_K \rtimes \langle \Phi_K \rangle$.

While we are still in class field theory, we will define the wild inertia group, as it will appear briefly later.

Definition 80 (Wild Inertia Group)**.** Let K^{tr} be the maximal tamely ramified extension of K. Then, the **wild inertia group** is $P_K := \mathrm{Gal}(\overline{K}/K^{\mathrm{tr}})$. Equivalently, P_K is the first ramification group of $\overline{K}/K$.

Now we may define Weil-Deligne representations.

Definition 81 (Weil-Deligne Representation)**.** A **Weil-Deligne representation** is a pair $\phi = (\rho, N)$ such that:

(1) $\rho : W_K \to \mathrm{GL}(V_\rho)$ is a finite dimensional representation such that $\rho(w)$ is semisimple for every $w \in W_K$ and $\ker \rho$ contains an open subgroup of I_K,
(2) $N \in \mathrm{End}(V_\rho)$ is nilpotent, satisfying $\rho(w) N \rho(w)^{-1} = \|w\| \cdot N$ for all $w \in W_K$.

Equivalence of Weil-Deligne representations comes naturally: we say two Weil-Deligne representations $\phi = (\rho, N)$ and $\phi' = (\rho', N')$ are equivalent if there exists a linear isomorphism $\alpha : V \to V'$ such that for all $w \in W_K$, both diagrams commute:

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V' \\ {\scriptstyle\rho(w)}\downarrow & & \downarrow{\scriptstyle\rho'(w)} \\ V & \xrightarrow[\alpha]{} & V' \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\alpha} & V' \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N'} \\ V & \xrightarrow[\alpha]{} & V' \end{array}$$

We say the two Weil-Deligne representations are I_K**-equivalent** if the above diagrams commute with ρ, ρ' replaced by their respective restrictions to I_K.

Finally, we say a Weil-Deligne representation is **tamely ramified** if it is trivial on the wild inertia group, i.e. (ρ, N) is tamely ramified if $\rho(P_K) = 1$.

6.2. **Macdonald's Correspondence.** Let $\Phi_I^t(\mathrm{GL}_n)$ be the set of I_K-equivalence classes of n-dimensional tamely ramified Weil-Deligne representations of W_K, and let $\Pi(\mathrm{GL}_n(k))$ be the set of isomorphism classes of irreducible representations of $\mathrm{GL}_n(k)$. We now provide a way to identify $\Pi(\mathrm{GL}_n(k))$ with $\Phi_I^t(\mathrm{GL}_n)$, which will allow us to obtain information about epsilon factors of irreducible representations of $\mathrm{GL}_n(k)$ from computations of epsilon factors for Weil-Deligne representations, which we are able to do.

This correspondence, called the Macdonald's Correspondence, between $\Pi(\mathrm{GL}_n(k))$ and $\Phi_I^t(\mathrm{GL}_n)$ is parameterized by a certain class of partition-valued functions, which we now construct.

Let $\mathcal{P}_n$ be the set of partitions of n, and $\mathcal{P} := \bigcup_{n\geq 0} \mathcal{P}_n$. For a partition $p \in \mathcal{P}$, we have $p \in \mathcal{P}_n$ for some integer n; we define $|p| = n$. Denote Γ_n as the character group $\widehat{k_n^\times}$. We have natural norm maps $N_{n,m} : k_n^\times \to k_m^\times$ for $m \mid n$; these induce maps on their character groups $N_{n,m} : \Gamma_m \to \Gamma_n$. These maps turn $\{\Gamma_n\}$ into a directed system, so we may define $\Gamma := \varinjlim_n \Gamma_n$.

Denote $\Phi := \Phi_K$ as the Frobenius element. It acts on Γ via $\Phi \cdot \gamma = \gamma^q$ for $\gamma \in \Gamma$. Denote the set of Φ-orbits in Γ as $\Phi\backslash\Gamma$. Given a Φ-orbit f, we define the **degree** of f as $d(f) := |f|$.

Definition 82. Define $P_n(\Gamma)$ as the set of partition-valued functions $\lambda : \Gamma \to \mathcal{P}$ such that

(1) $\lambda \circ \Phi = \lambda$, i.e. λ is constant on Φ-orbits;
(2) $\sum_{\gamma\in\Gamma} |\lambda(\gamma)| = n$.

This set is the central force behind Macdonald's Correspondence. We have bijections from $P_n(\Gamma)$ to both $\Pi(\mathrm{GL}_n(k))$ and $\Phi_I^t(\mathrm{GL}_n)$, so we can identify π_λ with $\phi_\lambda = (\rho_\lambda, N_\lambda)$, where $\pi_\lambda \in \Pi(\mathrm{GL}_n(k))$ and $\phi_\lambda \in \Phi_I^t(\mathrm{GL}_n)$ are the respective corresponding representations to $\lambda \in P_n(\Gamma)$.

Note from (1), since any λ is invariant on Φ-orbits in Γ, for any $f \in \Phi\backslash\Gamma$, we may unambiguously define $\lambda(f) := \lambda(\gamma)$ for any $\gamma \in f$. Even better, Gauss sums are invariant on Φ-orbits. Fix an additive character $\psi \in \widehat{k^+}$, and define $\psi_n = \psi \circ \mathrm{trace}_{k_n/k}$. Let $\gamma \in \Gamma_n$. Then, we compute

$$\begin{aligned} \tau(\Phi\cdot\gamma, \psi_n) &= -\sum_{x\in k_n^\times} \Phi\cdot\gamma(x^{-1})\psi_n(x) \\ &= -\sum_{x\in k_n^\times} \gamma(x^{-q})\psi_n(x) \\ &= -\sum_{x\in k_n^\times} \gamma(x^{-q})\psi_n(x^q) \end{aligned}$$

$$= -\sum_{x \in k_n^\times} \gamma(x^{-1})\psi_n(x)$$
$$= \tau(\gamma, \psi_n),$$

where the third equality follows from x, x^q being Galois conjugates and the fourth equality follows from the Frobenius map $x \mapsto x^q$ being an automorphism. Thus, like with $\lambda \in P_n(\Gamma)$, we can unambiguously define for any $f \in \Phi\backslash\Gamma$, $\tau(f, \psi) := \tau(\gamma, \psi_n)$ for any $\gamma \in f$.

6.3. **Epsilon Factors for** $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$**.** We know that the epsilon factor on the Galois side is multiplicative, so we have the equation

$$\varepsilon_0(\pi, \psi) = \prod_{f_i \in \Phi\backslash\Gamma} \varepsilon_0(\pi_{f_1} \otimes \pi_{f_2} \otimes \pi_{f_3}, \psi)$$

where f_1 is the indicator partition-valued function such that $f_1(\gamma) = (1)$ if $\gamma \in f_1$ and $()$ (the empty partition) otherwise. Let $\rho := (\rho_1 \otimes \rho_2 \otimes \rho_3)_I$ be a Weil-Deligne representation, where under the Macdonald Correspondence, each ρ_i corresponds to some $\pi_i \in \Pi(\mathrm{GL}_n(k))$. In turn, each π_i is represented by a matrix of the form $\begin{pmatrix} \alpha_i & \\ & \alpha_i^q \end{pmatrix}$ for some $\alpha \in k_2^\times \backslash k^\times$. Then, ρ is represented by a diagonal matrix whose diagonal entries are representations of the Galois orbits of $\alpha_1\alpha_2\alpha_3$, which are $\{\alpha_1\alpha_2\alpha_3, \alpha_1\alpha_2\alpha_3^q, \alpha_1\alpha_2^q\alpha_3, \alpha_1^q\alpha_2, \alpha_3, \alpha_1\alpha_2^q\alpha_3^q, \alpha_1^q\alpha_2\alpha_3^q, \alpha_1^q\alpha_2^q\alpha_3, \alpha_1^q\alpha_2^q\alpha_3^q\}$. Using the multiplicativity of ε_0 and choose a representative for each Φ-orbit, we conclude from the Galois side

$$\varepsilon_0(\rho, \psi) := q^{-4}\tau(\alpha_1\alpha_2\alpha_3, \psi_2)\tau(\alpha_1\alpha_2\alpha_3^q, \psi_2)\tau(\alpha_1\alpha_2^q\alpha_3, \psi_2)\tau(\alpha_1\alpha_2^q\alpha_3^q, \psi).$$

6.4. **Product of Gauss Sums as Norm Sum.** Having a product of Gauss sums is a bit clunky. Luckily, there is a clever way to write our product as a single sum, iterating over the units of a tensor product.

Consider the tensor product $k_n \otimes_k k_n \otimes_k k_n$. We have $k_n \simeq k[\theta]$ for some $\theta \in k_n$ where the minimal polynomial $p(X) \in k[X]$ of θ has degree n. Thus, we can write $k[\theta] \simeq k[X]/(p(X))$. This gives us a series of isomorphisms

$$\begin{aligned} k_n \otimes_k k_n \otimes_k k_n &\simeq k_n \otimes_k k[X]/(p(X)) \otimes_k k[Y]/(p(Y)) \\ &\simeq k_n \otimes_k k[X,Y]/(p(X), p(Y)) \\ &\simeq k_n[X,Y]/(p(X), p(Y)). \end{aligned}$$

Expanding $p(X) = a_0 + a_1 X + \cdots + X^n$, we see that

$$\begin{aligned} p\left(\theta^{1/q^r}\right)^{q^r} &= \left(a_0 + a_1\theta^{1/q^r} + \cdots + \theta^{n/q^r}\right)^{q^r} \\ &= a_0 + a_1\theta + \cdots + \theta^n = 0, \end{aligned}$$

so $p\left(\theta^{1/q^r}\right) = 0$ for every $r < n$. It follows that we can factor

$$p(X) = \prod_{r=1}^{n} (X - \theta^{1/q^{r-1}}).$$

Thus, by Chinese Remainder Theorem, we have a final isomorphism

$$\begin{aligned} k_n[X,Y]/(p(X),p(Y)) &\simeq k_n^{\oplus n^2} \\ [f(X,Y)] &\mapsto \left(f(\theta^{1/q^{i-1}},\theta^{1/q^{j-1}})\right)_{1\le i,j\le n}. \end{aligned}$$

This isomorphism $k_n \otimes_k k_n \otimes_k k_n^{\oplus n^2}$ provides three distinct actions of k_n on $k_n^{\oplus n^2}$, depending on which component k_n is acting on in the tensor product. For the following, fix $\alpha \in k_n$.

The first component of the tensor product is simply multiplied as a scalar to the polynomial via the isomorphisms, so α acts on $(x_1,\ldots,x_{n^2})$ by scalar multiplication, i.e. the first action is defined as

$$\alpha \cdot_1 (x_1,\ldots,x_{n^2}) = (\alpha x_1,\ldots,\alpha x_{n^2}).$$

The second component contributes a polynomial in X. Let $\alpha = a_0 + a_1\theta + \cdots + a_m\theta^m$; this corresponds to the polynomial $\alpha(X) = \alpha(X,Y) = a_0 + a_1X + \cdots + a_mX^m$. We wish to write $\alpha(\theta^{1/q^r})$ in terms of $\alpha(\theta)$. We have

$$\begin{aligned} \alpha(\theta^{1/q^r})^{q^r} &= (a_0 + a_1\theta^{1/q^r} + \cdots + a_m\theta^{m/q^r})^{q^r} \\ &= a_0 + a_1\theta + \cdots + a_m\theta^m = \alpha(\theta), \end{aligned}$$

where the second equality follows because $a_i \in k$ and $\operatorname{char} k \mid |k| = q$. Thus, for any $f(X,Y) \in k_n[X,Y]/(p(X),p(Y))$ and $1 \le i,j \le n$, we have

$$\begin{aligned} \alpha(\theta^{1/q^{i-1}},\theta^{1/q^{j-1}})f(\theta^{1/q^{i-1}},\theta^{1/q^{j-1}}) &= \alpha(\theta^{1/q^{i-1}})f(\theta^{1/q^{i-1}},\theta^{1/q^{j-1}}) \\ &= \alpha(\theta)^{1/q^{i-1}}f(\theta^{1/q^{i-1}},\theta^{1/q^{j-1}}) \\ &= \alpha^{1/q^{i-1}}f(\theta^{1/q^{i-1}},\theta^{1/q^{j-1}}), \end{aligned}$$

so the second action is given by

$$\begin{aligned} \alpha \cdot_2 (x_1,\ldots,x_{n^2}) &= \left(\alpha^{q^{-\lfloor \frac{j-1}{n} \rfloor}} x_j\right)_{1\le j\le n^2} \\ &= (\alpha x_1,\ldots,\alpha x_n,\alpha^{1/q}x_{n+1},\ldots,\alpha^{1/q^{n-1}}x_{n^2}). \end{aligned}$$

Finally, the last component contributes a polynomial in Y, so the action is similar to the above in the sense that we have

$$\alpha(\theta^{1/q^{i-1}},\theta^{1/q^{j-1}})f(\theta^{1/q^{i-1}},\theta^{1/q^{j-1}}) = \alpha^{1/q^{j-1}}f(\theta^{1/q^{i-1}},\theta^{1/q^{j-1}}),$$

which provides a third action given by

$$\begin{aligned} \alpha \cdot_3 (x_1,\ldots,x_{n^2}) &= \left(\alpha^{q^{-(j \bmod n)}} x_j\right)_{1\le j\le n^2} \\ &= (\alpha x_1,\alpha^{1/q}x_2,\ldots,\alpha^{1/q^{n+1}}x_n,\alpha x_{n+1},\ldots,\alpha^{1/q^{n+1}}x_{n^2}). \end{aligned}$$

Let $c_j = \left\lfloor \frac{j-1}{n} \right\rfloor$ and $d_j = j \pmod n$. Consider the map $T_{(x_1,\ldots,x_{n^2})} : k_n^{\oplus n^2} \to k_n^{\oplus n^2}$ given by $(y_1,\ldots,y_{n^2}) \mapsto (x_1y_1,\ldots,x_{n^2}y_{n^2})$, where multiplication in each component is standard multiplication in k_n. We can write this tuple in terms of each of our three actions:

$$\begin{aligned} T_{(x_1,\ldots,x_{n^2})}(y_1,\ldots,y_{n^2}) &= (x_1 \cdot_1 y_1, \ldots, x_{n^2} \cdot_1 y_{n^2}) \\ &= (x_j^{q^{-c_j}} \cdot_2 y_j)_{1\le j\le n^2} \\ &= (x_j^{q^{-d_j}} \cdot_3 y_j)_{1\le j\le n^2}. \end{aligned}$$

We can define the trace of $T_{(x_1,\ldots,x_{n^2})}$, viewed as a k-linear map. Taking the standard basis of $k_n^{\oplus n^2}$, this amounts to just the sum of all the components, i.e. $\operatorname{trace} T_{(x_1,\ldots,x_{n^2})} = \sum_{i=1}^{n^2} x_i$.

We can also define three distinct norm functions, one for each action, which takes the determinant of $T_{(x_1,\ldots,x_{n^2})}$ with respect to the standard basis on $k_n^{\oplus n^2}$ and the given action. Restricting our interest to the case $n = 2$, we have

$$\begin{aligned} T_{(x_1,x_2,x_3,x_4)}(y_1,y_2,y_3,y_4) &= (x_1 \cdot_1 y_1, x_2 \cdot_1 y_2, x_3 \cdot_1 y_3, x_4 \cdot_1 y_4) \\ &= (x_1 \cdot_2 y_1, x_2 \cdot_2 y_2, x_3^{1/q} \cdot_2 y_3, x_4^{1/q} \cdot_2 y_4) \\ &= (x_1 \cdot_3 y_1, x_2^{1/q} \cdot_3 y_2, x_3 \cdot_3 y_3, x_4^{1/q} \cdot_3 y_4). \end{aligned}$$

Letting N_1, N_2, N_3 be the norm functions with respect to $\cdot_1, \cdot_2, \cdot_3$, respectively, we can compute

$$\begin{aligned} N_1(x_1,x_2,x_3,x_4) &= x_1x_2x_3x_4 \\ N_2(x_1,x_2,x_3,x_4) &= x_1x_2x_3^qx_4^q \\ N_3(x_1,x_2,x_3,x_4) &= x_1x_2^qx_3x_4^q. \end{aligned}$$

Via the isomorphism $k_2 \otimes_k k_2 \otimes_k k_2 \simeq k_2^{\oplus 4}$, we see that the diagonal multiplication rule on the left agrees with component-wise multiplication on the right. Thus, the units in $k_2 \otimes_k k_2 \otimes_k k_2$ are isomorphic to $(k_2^\times)^{\oplus 4}$, so we write

$$\begin{aligned} \varepsilon_0(\rho,\psi) &= q^{-4}\tau(\alpha_1\alpha_2\alpha_3,\psi_2)\tau(\alpha_1\alpha_2\alpha_3^q,\psi_2)\tau(\alpha_1\alpha_2^q\alpha_3,\psi_2)\tau(\alpha_1\alpha_2^q\alpha_3^q,\psi) \\ &= q^{-4}\prod_{j=0}^{3}\sum_{x\in k_2^\times} \alpha_1^{-1}(x)\alpha_2^{-1}\left(x^{q^{\lfloor j/2\rfloor}}\right)\alpha_3^{-1}\left(x^{q^{j \bmod 2}}\right)\psi_2(x) \\ &= \sum_{\vec{x}\in(k_2^\times)^{\oplus 4}} \alpha_1^{-1}(N_1(\vec{x}))\alpha_2^{-1}(N_2(\vec{x}))\alpha_3^{-1}(N_3(\vec{x}))\psi(\operatorname{tr}\vec{x}) \\ &= \sum_{\xi\in(k_2\otimes_k k_2\otimes_k k_2)^\times} \alpha_1^{-1}(N_1(\xi))\alpha_2^{-1}(N_2(\xi))\alpha_3^{-1}(N_3(\xi))\psi(\operatorname{tr}\xi). \end{aligned}$$

Appendix A. Computation of $c(1, I_6, \psi)$

Throughout this section, $\mathbb{F}_q$ denotes a finite field with q elements, where q is an odd prime-power, and $\mathrm{Sym}_3^\times$ denotes the set of invertible 3×3 matrices with entries in $\mathbb{F}_q$. The goal of the present section is to prove the following result.

Theorem 83. *We have*

$$\sum_{A\in\mathrm{Sym}_3^\times} \psi(\operatorname{tr} A) = q^2.$$

We will prove this by using combinatorics and elementary number theory in order to compute the number of invertible symmetric matrices with given diagonal entries. For brevity, given $(a_{11}, a_{22}, a_{33}) \in \mathbb{F}_q^3$, we say that $A \in \mathrm{Sym}_3^\times$ has "type (a_{11}, a_{22}, a_{33}) if and only if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

We now examine each type individually, in ascending levels of difficulty.

Lemma 84. *There are* $(q-1)^3$ *matrices in* $\mathrm{Sym}_3^\times$ *of type* $(0,0,0)$.

Proof. Our matrices take the form

$$A := \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & 0 & a_{23} \\ a_{13} & a_{23} & 0 \end{bmatrix},$$

which has determinant $\det A = 2a_{12}a_{13}a_{23}$. As such, this matrix is invertible if and only if each a_{12}, a_{13}, a_{23} is nonzero, totaling to $(q-1)^3$ matrices. $\square$

Lemma 85. *For any* $a \in \mathbb{F}_q^\times$, *there are* $q^3 - q^2 - (q-1)^2$ *matrices in* $\mathrm{Sym}_3^\times$ *of type* $(a,0,0)$. *The same statement holds for permutations of* $(a,0,0)$.

Proof. Our matrices take the form

$$A := \begin{bmatrix} a & a_{12} & a_{13} \\ a_{12} & 0 & a_{23} \\ a_{13} & a_{23} & 0 \end{bmatrix},$$

which has determinant $\det A = 2a_{12}a_{13}a_{23} - aa_{23}^2$. By counting the complement, we would like to show that there are $q^2 + (q-1)^2$ solutions $(x,y,z) \in \mathbb{F}_q^3$ to $2xyz - ax^2 = 0$. There are two cases.

- If $x = 0$, then any $(y,z) \in \mathbb{F}_q^2$ will work, totaling to q^2 matrices here.
- If $x \neq 0$, then we see $2yz = ax \neq 0$. Thus, there are $q-1$ choices for $y \in \mathbb{F}_q^\times$, from which z is forced. Counting over all $x \in \mathbb{F}_q^\times$, there are $(q-1)^2$ matrices here.

Summing completes the proof. $\square$

Lemma 86. *For any* $a, b \in \mathbb{F}_q^\times$, *there are* $q^3 - q - (q-1)^2$ *matrices in* $\mathrm{Sym}_3^\times$ *of type* $(a,b,0)$. *The statement holds for permutations of* $(a,b,0)$.

Proof. Our matrices take the form

$$A := \begin{bmatrix} a & a_{12} & a_{13} \\ a_{12} & b & a_{23} \\ a_{13} & a_{23} & 0 \end{bmatrix},$$

which has determinant $\det A = 2a_{12}a_{13}a_{23} - aa_{23}^2 - ba_{13}^2$. By counting the complement, we would like to show that there are $q + (q-1)^2$ solutions $(x,y,z) \in \mathbb{F}_q^3$ to $2xyz = ax^2 + by^2$. We have two cases.

- If $x = 0$, then we must have $y = 0$, from which any $z \in \mathbb{F}_q$ will do. There are q matrices here.

- If $x \neq 0$ and $y \neq 0$, then $z := \left(ax^2 + by^2\right)/(2xy)$ is forced. Totaling, there are $(q-1)^2$ matrices here.

Summing completes the proof. $\square$

Lemma 87. *For any $a, b, c \in \mathbb{F}_q^\times$, there are $q^3 - (q^2+1)$ matrices in $\mathrm{Sym}_3^\times$ of type (a,b,c).*

Proof. Our matrices take the form

$$A := \begin{bmatrix} a & a_{12} & a_{13} \\ a_{12} & b & a_{23} \\ a_{13} & a_{23} & c \end{bmatrix}.$$

Scaling will not change invertibility, so for psychological reasons we replace A with $a^{-1}A$ so that we may assume $a = 1$. Then $\det A = 2a_{12}a_{13}a_{23} - ba_{12}^2 - ca_{12}^2 - a_{23}^2 + bc$. By counting the complement, we would like to show that there are $q^2 + 1$ solutions $(x, y, z) \in \mathbb{F}_q^3$ to $x^2 - 2xyz = -cy^2 - bz^2 + bc$. Sending $x \mapsto x + yz$, we are counting solutions to

$$x^2 = y^2z^2 - cy^2 - bz^2 + bc = \left(y^2 - b\right)\left(z^2 - c\right).$$

We now do casework on what elements on the right-hand side are squares. This requires the following lemma.

Lemma 88. *Fix $a \in \mathbb{F}_q^\times$. The number of $x \in \mathbb{F}_q$ such that $x^2 - a$ is a square is*

$$\begin{cases} \frac{q-1}{2} & \text{if } a \text{ is not a square,} \\ \frac{q+1}{2} & \text{if } a \text{ is a square.} \end{cases}$$

Proof. We are counting the number of $x \in \mathbb{F}_q$ for which there is a solution $y \in \mathbb{F}_q$ to the equation $x^2 - a = y^2$. This rearranges to

$$(x+y)(x-y) = a.$$

Setting $s := \frac{x+y}{2}$ and $d := \frac{x-y}{2}$, we see that $sd = a/4$, so it is necessary and sufficient to have $x = s + \frac{a}{4s}$ for some $s \in \mathbb{F}_q^\times$. In other words, we are currently counting the size of the image of the map $x \colon \mathbb{F}_q^\times \to \mathbb{F}_q$ given by

$$x \colon s \mapsto s + \frac{a}{4s}.$$

Now, $x(s_1) = x(s_2)$ if and only if $s_1 + \frac{a}{4s_1} = s_2 + \frac{a}{4s_2}$, which upon clearing fractions and rearranging is equivalent to

$$(4s_1s_2 - a)(s_1 - s_2) = 0.$$

This is now equivalent to $s_1 = s_2$ or $s_1 = \frac{a}{4s_2}$. Thus, for each $s \in \mathbb{F}_q^\times$, we see that $x^{-1}(\{x(s)\}) = \{s, a/(4s)\}$, a set which has size 2 unless a is a square and s is a square root of $a/4$.

To finish, we see that if a is not a square, there are $\frac{q-1}{2}$ values of x. Otherwise, a is a square, and there are two fibers with exactly one element, totaling to $\frac{q-3}{2} + 2 = \frac{q+1}{2}$ values of x. This completes the proof. $\square$

We now have the following cases on b and c.

- Suppose b and c are not squares. Then $y^2 - b$ and $z^2 - c$ are always nonzero, so for $(y^2 - b)(z^2 - c)$ to be a square, either both are squares or neither are squares. Each such pair (y, z) produces two valid values of x, so we have counted

$$2\left(\left(\frac{q-1}{2}\right)^2 + \left(\frac{q+1}{2}\right)^2\right) = q^2 + 1$$

triples (x, y, z) in this case.
- Suppose exactly one of b or c is a square; without loss of generality, say that b is a square. There are two values of y for which $y^2 - b$ vanishes, from which z has any value and $x = 0$, totaling to $2q$ solutions here.

  Continuing, there are $\frac{q-3}{2}$ additional values of y for which $y^2 - b$ is a nonzero square; here, $z^2 - c$ must be a (nonzero) square, giving

$$2\left(\frac{q-3}{2}\right)\left(\frac{q-1}{2}\right) = \frac{q^2 - 4q + 3}{2}$$

additional solutions.

  Lastly, there are $\frac{q-1}{2}$ values of y for which $y^2 - b$ is not a square; here $z^2 - c$ must not be a square, giving

$$2\left(\frac{q-1}{2}\right)\left(\frac{q+1}{2}\right) = \frac{q^2 - 1}{2}$$

more solutions. Summing all three cases gives $2q + \frac{1}{2}(q^2 - 4q + 3) + \frac{1}{2}(q^2 - 1) = q^2 + 1$ solutions.
- Suppose that both b and c are squares. There are two values of y for which $y^2 - b$ from which z has any value and $x = 0$, totaling to $2q$ solutions. There are two values for z for which $z^2 - c$ vanishes again, which adds $2q - 4$ more solutions.

  In the remaining cases, both $y^2 - b$ and $z^2 - c$ must be nonzero. For their product to be a square, either both are squares or neither is a square, so we have counted

$$2\left(\left(\frac{q-3}{2}\right)^2 + \left(\frac{q-1}{2}\right)^2\right) = q^2 - 4q + 5$$

more solutions. In total, there are $2q - 4 + q^2 - 4q + 5 = q^2 + 1$ solutions.

The above casework completes the proof of Lemma 87. □

We are now ready to prove Theorem 83.

Proof of Theorem 83. For given $t \in \mathbb{F}_q$, we will count $A \in \mathrm{Sym}_3^\times(\mathbb{F}_q)$ such that $\operatorname{tr} A = t$. We have two cases.

- Suppose $t = 0$. Then the type of any $A \in \mathrm{Sym}_3^\times(\mathbb{F}_q)$ has one of the following forms.
  - Type $(0, 0, 0)$: there are $(q-1)^3$ matrices here.
  - Permutations of type $(0, a, -a)$ for given $a \in \mathbb{F}_q^\times$: there are $q^3 - q - (q-1)^2$ matrices.
  - Type $(a, b, -a-b)$ for given $a, b, -a-b \in \mathbb{F}_q^\times$: there are $q^3 - (q^2+1)$ matrices.

  Totaling all cases, we have

$$(q-1)^3 + 3(q-1)\left(q^3 - q - (q-1)^2\right) + (q-1)(q-2)\left(q^3 - q^2 - 1\right)$$

matrices. Simplifying, this is $q^5 - q^4$.

- Suppose $t \neq 0$. Then the type of any $A \in \operatorname{Sym}_3^\times(\mathbb{F}_q)$ has one of the following forms.
  - Permutations of type $(t, 0, 0)$: there are $q^3 - q^2 - (q-1)^2$ matrices here.
  - Permutations of type $(a, t-a, 0)$ for given $a, t-a \in \mathbb{F}_q^\times$: there are $q^3 - q - (q-1)^2$ matrices.
  - Type $(a, b, t-a-b)$ for given $a, b, t-a-b \in \mathbb{F}_q^\times$: there are $q^3 - (q^2+1)$ matrices.

  Quickly, note that $a \notin \{0, t\}$ requires $b \notin \{0, t-a\}$ and hence $q-2$ options for b; otherwise $a = t$ requires $b \neq 0$ and hence $q-1$ options for b.

  Totaling all cases, we have

$$3\left(q^3 - q^2 - (q-1)^2\right) + 3(q-2)\left(q^3 - q - (q-1)^2\right) + ((q-2)(q-2) + (q-1))\left(q^3 - \left(q^2+1\right)\right)$$

  matrices. Simplifying, this is $q^5 - q^4 - q^2$.

Combining cases, we see

$$\begin{aligned} \sum_{A \in \operatorname{Sym}_3^\times(\mathbb{F}_q)} \psi(\operatorname{tr} A) &= \sum_{t \in \mathbb{F}_q} \#\left\{A \in \operatorname{Sym}_3^\times(\mathbb{F}_q) : \operatorname{tr} A = t\right\} \psi(t) \\ &= q^2 \psi(0) + \sum_{t \in \mathbb{F}_q} \left(q^5 - q^4 - q^2\right) \psi(t) \\ &= q^2, \end{aligned}$$

which is what we wanted. □

Appendix B. Computation of the Symmetric Gauss Sum

Let $\mathbb{F}_q$ denote the finite field with q elements, where q is an odd prime-power, and let $\operatorname{Sym}_n^\times(\mathbb{F}_q)$ denote the set of invertible symmetric $n \times n$ matrices with entries in $\mathbb{F}_q$. The goal of the present section is to compute the "symmetric" Gauss sum

$$g_n(\omega, \psi, T) \coloneqq \sum_{A \in \operatorname{Sym}_n^\times(\mathbb{F}_q)} \omega(\det A) \psi(\operatorname{tr} AT)$$

where $n \in \mathbb{Z}_{\geq 0}$ is a nonnegative integer, $\omega \colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi \colon \mathbb{F}_q \to \mathbb{C}^\times$ are characters, and $T \in \operatorname{Sym}_n^\times(\mathbb{F}_q)$. Here, $\operatorname{Sym}_0^\times$ is understood to consist of a single empty 0×0 matrix with trace 0 and determinant 1 so that $g_0(\omega, \psi, T) = 1$. In the case where ω is a quadratic character, such sums were considered by [Wal17].

In the following discussion, we will make use of many Gauss sums, so it will be helpful to have the notation

$$g(\omega, \psi) \coloneqq \sum_{a \in \mathbb{F}_q^\times} \omega(a) \psi(a),$$

where ω and ψ are as above. For example, $g_1(\omega, \psi, 1) = g(\omega, \psi)$.

We now state our main result.

Theorem 89. *Let $\omega \colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi \colon \mathbb{F}_q \to \mathbb{C}^\times$ be characters, and let $\chi \colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ denote the nontrivial quadratic character, and fix some $T \in \operatorname{Sym}_n^\times(\mathbb{F}_q)$. Further, assume that ψ is nontrivial.*

- *If $n = 2m$ is an even nonnegative integer, then*

$$g_{2m}(\omega, \psi, T) = \frac{\chi(-1)^m \chi(\det T) q^{m^2}}{\omega(4^m \det T)} \cdot g\left(\omega^2, \psi\right)^m.$$

- *If $n = 2m + 1$ is an odd nonnegative integer, then*

$$g_{2m+1}(\omega, \psi, T) = \frac{q^{m(m+1)}}{\omega(4^m \det T)} \cdot g(\omega, \psi) g\left(\omega^2, \psi\right)^m.$$

Remark 90. The theorem implies that

$$\sum_{A \in \mathrm{Sym}_n^\times(\mathbb{F}_q)} \omega\left(\det AT^{-1}\right) \psi(\operatorname{tr} A) = \sum_{B \in \mathrm{Sym}_n^\times(\mathbb{F}_q)} \omega(\det B) \psi(\operatorname{tr} BT),$$

but this is not obvious: in particular, one cannot apply the variable change $B \coloneqq AT^{-1}$ because AT^{-1} need not be symmetric! We would be interested in a more direct proof of the above equality.

Remark 91. In the "generic" case $\omega^2 \neq 1$, all Gauss sums have magnitude $\sqrt{q}$ (see Proposition 93), so Theorem 89 implies

$$|g_n(\omega, \psi, T)| = q^{n(n+1)/4} = q^{\frac{1}{2}\binom{n+1}{2}}.$$

This is roughly what we expect to be true from "square-root cancellation": $|\mathrm{Sym}_n(\mathbb{F}_q)| = q^{\binom{n+1}{2}}$.

B.1. **Quadratic Twists of Gauss Sums.** The goal of this subsection is to prove the following result.

Proposition 92. *Let $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ be characters, and let $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ denote the nontrivial quadratic character. Then*

$$\omega(4) g(\omega, \psi) g(\omega\chi, \psi) = g\left(\omega^2, \psi\right) g(\chi, \psi).$$

Proof. Expanding out the Gauss sums, we are trying to show that

$$\sum_{a,b \in \mathbb{F}_q^\times} \omega(4ab) \chi(b) \psi(a+b) \stackrel{?}{=} \sum_{a,b \in \mathbb{F}_q^\times} \omega\left(a^2\right) \chi(b) \psi(a+b).$$

Fixing some $d \in \mathbb{F}_q^\times$ and $t \in \mathbb{F}_q$, it is enough to show that

$$\sum_{\substack{a+b=t \\ 4ab=d}} \chi(b) \stackrel{?}{=} \sum_{\substack{a+b=t \\ a^2=d}} \chi(b) \tag{B.1.1}$$

and then sum over all possible values of d and t. At this point, the proof has become combinatorial number theory. For convenience, extend χ to $\mathbb{F}_q$ by $\chi(0) \coloneqq 0$, and allow $a, b \in \mathbb{F}_q$ in the right-hand sum above; this will not change its value.

For example, suppose that d is not a square. Then the right-hand side of (B.1.1) is empty and hence zero. On the other hand, we claim that the left-hand side is zero. Let $(a_1, b_1), \ldots, (a_m, b_m)$ denote the solutions to the system of equations $a + b = t$ and $4ab = d$. Because d is not a square, $a_k \neq b_k$ for each k—in fact, if a_k is a square, then b_k is not a square (and vice versa). Thus, if (a, b) is a solution, then (b, a) is a distinct solution with

$\{\chi(a),\chi(b)\}=\{1,-1\}$, so the two pairs (a,b) and (b,a) contribute $1-1=0$ to the left-hand side of (B.1.1). It follows that the left-hand side vanishes.

Thus, in the rest of the proof, we may assume that $d=x^2$ where $x\in\mathbb{F}_q^\times$, so the right-hand side of (B.1.1) reads

$$\chi(t+x)+\chi(t-x).$$

To continue, observe that solving the system of equations $a+b=t$ and $4ab=d$ is equivalent to having $a=t-b$ and

$$(2b-t)^2=t^2-d.$$

As such, for our next case, suppose that t^2-d fails to be a square. Then the left-hand side of (B.1.1) is empty and hence vanishes, so we want to show that the right-hand side also vanishes. Well, $t^2-d=(t+x)(t-x)$ is then not a square, so both are nonzero, and one is a square while the other is not a square. Thus, $\chi(t+x)+\chi(t-x)=0$, as needed.

Thus, in the rest of the proof, we may assume that $t^2-d=y^2$ for some $y\in\mathbb{F}_q$. Quickly, we deal with the case where $y=0$. On one hand, we have $t^2=d$, so $t=\pm x$, so the right-hand side of (B.1.1) is $\chi(2t)$. On the other hand, we see the left-hand side of (B.1.1) is $\chi(t/2)$, so we finish by noting $\chi(2t)=\chi(t/2)$.

At the current point, we can now say that $t^2=x^2+y^2$ where $x,y\in\mathbb{F}_q^\times$, and the left-hand side of (B.1.1) is $\chi\left(\frac{t+y}{2}\right)+\chi\left(\frac{t-y}{2}\right)$, so we are trying to show that

$$\chi\left(\frac{t+y}{2}\right)+\chi\left(\frac{t-y}{2}\right)\stackrel{?}{=}\chi(t+x)+\chi(t-x). \tag{B.1.2}$$

Because $(t-x)(t+x)=y^2$ and $\left(\frac{t+y}{2}\right)\left(\frac{t-y}{2}\right)=\frac{1}{4}x^2$, we see that all values above are nonzero, and $\chi\left(\frac{t+y}{2}\right)=\chi\left(\frac{t-y}{2}\right)$ and $\chi(t+x)=\chi(t-x)$. Because, these values are in $\{\pm1\}$, we see that it is enough to show that $\chi(t+x)=1$ if and only if $\chi\left(\frac{t+y}{2}\right)=1$.

The main claim, now, is that $\chi(t+x)=1$ implies that $\chi\left(\frac{t+y}{2}\right)=1$. This approximately boils down to the enumeration of Pythagorean triples. The above logic grants that $\chi(t+x)=\chi(t-x)=1$, so both $t+x$ and $t-x$ are squares; write $t+x=x_1^2$ and $t-x=x_2^2$ for $x_1,x_2\in\mathbb{F}_q^\times$. Adjusting signs, we may assume that $y=x_1x_2$. Thus,

$$\frac{t+y}{2}=\frac{1}{2}\left(\frac{x_1^2+x_2^2}{2}+x_1x_2\right)=\left(\frac{x_1+x_2}{2}\right)^2$$

is a square, and we know $\frac{t+y}{2}$ is nonzero from the above logic, so $\chi\left(\frac{t+y}{2}\right)=1$, as desired.

To finish the proof, we must show the reverse implication: we claim that $\chi\left(\frac{t+y}{2}\right)=1$ implies $\chi(t+x)=1$. Well, we see that $\left(\frac{x}{2}\right)^2+\left(\frac{y}{2}\right)^2=\left(\frac{t}{2}\right)^2$, so the argument of the previous paragraph tells us that $\chi\left(\frac{t}{2}+\frac{y}{2}\right)=1$ implies

$$\chi(t+x)=\chi\left(\frac{t+x}{4}\right)=\chi\left(\frac{\frac{t}{2}+\frac{x}{2}}{2}\right)=1,$$

as desired. $\square$

Before continuing, it will be helpful to have the following well-known fact about the quadratic Gauss sum. Because the proof is so quick, we include the proof.

Proposition 93. *Let $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ denote nontrivial characters. Then*

$$g(\omega,\psi)g\left(\omega^{-1},\psi^{-1}\right) = q.$$

Thus, if $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ denotes the nontrivial quadratic character, then $g(\chi,\psi)^2 = \chi(-1)q$.

Proof. For the first claim, we want to show

$$\sum_{a,b\in\mathbb{F}_q^\times} \omega(a/b)\psi(a-b) \stackrel{?}{=} q.$$

Well, set $c \coloneqq a/b$ so that the sum is

$$\sum_{c\in\mathbb{F}_q^\times}\left(\omega(c)\sum_{a\in\mathbb{F}_q^\times}\psi(a-ac)\right).$$

If $c \neq 1$, then the inner sum is $-\psi(0) + \sum_{a\in\mathbb{F}_q}\psi(a-ac) = -1$. Otherwise, if $c = 1$, then the inner sum is $q-1$. In total, we are left with

$$(q-1) + \sum_{c\in\mathbb{F}_q^\times\setminus\{1\}} -\omega(c) = q - \sum_{c\in\mathbb{F}_q^\times}\omega(c) = q,$$

which is what we wanted.

For the second claim, we see

$$g\left(\chi^{-1},\psi^{-1}\right) = \sum_{a\in\mathbb{F}_q^\times}\chi(a)\psi(-a) = \chi(-1)\sum_{a\in\mathbb{F}_q^\times}\chi(a)\psi(a) = \chi(-1)g(\chi,\psi),$$

so the second claim follows from the first. □

B.2. **The Main Computation.** In this subsection, we prove Theorem 89. The key idea is to use Gaussian elimination of symmetric matrices in order to be able to use induction; note that this idea was used to count the number of invertible symmetric matrices over $\mathbb{Z}/n\mathbb{Z}$ in [BM87]. As such, the hard work is done in the following lemma.

Lemma 94. *Let $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ be characters, and let $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ denote the nontrivial quadratic character. Further, assume that ψ is nontrivial. For any positive integer n and $d_1,\ldots,d_{n+1}\in\mathbb{F}_q^\times$,*

$$g_{n+1}(\omega,\psi,\operatorname{diag}(d_1,\ldots,d_{n+1})) = g_n(\omega,\psi,\operatorname{diag}(d_1,\ldots,d_n))\cdot\frac{\chi(d_1\cdots d_n)\chi(d_{n+1})^n}{\omega(d_{n+1})}\cdot g\left(\omega\chi^n,\psi\right)g(\chi,\psi)^n.$$

Proof. For brevity, set $T \coloneqq \operatorname{diag}(d_1,\ldots,d_{n+1})$. For a matrix square $A\in M_m(\mathbb{F}_q)$, we use the notation $A_{k\ell}$ denote the entry of A in the kth row and ℓth column. Now, for some $A\in\operatorname{Sym}_{n+1}^\times$, there are two cases.

- Suppose $A_{n+1,n+1}\neq 0$; here, set $T' \coloneqq \operatorname{diag}(d_1,\ldots,d_n)$ for brevity. For our Gaussian elimination, we note that the map

$$\begin{aligned}\operatorname{Sym}_n^\times(\mathbb{F}_q)\times\mathbb{F}_q^n\times\mathbb{F}_q^\times &\to \{A\in\operatorname{Sym}_{n+1}^\times(\mathbb{F}_q) : A_{n+1,n+1}\neq 0\}\\ (A'\ ,\ v\ \ ,\ c) &\mapsto \begin{bmatrix}1 & v\\ & 1\end{bmatrix}\begin{bmatrix}A' & \\ & c\end{bmatrix}\begin{bmatrix}1 & \\ v^\intercal & 1\end{bmatrix} = \begin{bmatrix}A'+cvv^\intercal & cv\\ cv^\intercal & c\end{bmatrix}\end{aligned}$$

is a bijection. Indeed, $A_{n+1,n+1}$ uniquely determines c, the values $A_{k,n+1}$ for $1 \leq k \leq n$ uniquely determine v, and then the rest of the matrix uniquely determines A'. Using this bijection, we see that

$$\begin{aligned}
S_{\neq 0} &\coloneqq \sum_{\substack{A \in \operatorname{Sym}_{n+1}^\times(\mathbb{F}_q) \\ A_{n+1,n+1} \neq 0}} \omega(\det A)\psi(\operatorname{tr} AT) \\
&= \sum_{\substack{A \in \operatorname{Sym}_n^\times(\mathbb{F}_q) \\ v \in \mathbb{F}_q^n, c \in \mathbb{F}_q^\times}} \omega(c \det A')\psi\left(\operatorname{tr} A'T' + c \operatorname{tr} vv^\intercal T' + cd_{n+1}\right) \\
&= g_n(\omega, \psi, T') \sum_{c \in \mathbb{F}_q^\times} \omega(c)\psi(cd_{n+1}) \sum_{v \in \mathbb{F}_q^n} \psi\left(c \operatorname{tr} vv^\intercal T'\right) \\
&= g_n(\omega, \psi, T') \sum_{c \in \mathbb{F}_q^\times} \omega(c)\psi(cd_{n+1}) \prod_{k=1}^n \left(\sum_{a \in \mathbb{F}_q} \psi\left(cd_k a^2\right) \right).
\end{aligned}$$

Quickly, we claim that

$$\sum_{a \in \mathbb{F}_q} \psi\left(cd_k a^2\right) \stackrel{?}{=} \sum_{a \in \mathbb{F}_q} (1 + \chi(cd_k a))\psi(a),$$

where we have extended χ to $\mathbb{F}_q$ by $\chi(0) \coloneqq 0$. Indeed, for any $b \in \mathbb{F}_q$, we see that $\psi(b)$ appears on the left-hand side 0 times if b does not have the form $cd_k a^2$, appears 1 time if $b = 0$, and appears 2 times if b is nonzero and has the form $cd_k a^2$; these values are exactly $1 + \chi(cd_k a)$ in all cases. As such, the claim follows, and because ψ is nontrivial, we actually have

$$\sum_{a \in \mathbb{F}_q} \psi\left(cd_k a^2\right) = \sum_{a \in \mathbb{F}_q} \chi(cd_k a)\psi(a) = \chi(cd_k)g(\chi, \psi).$$

Plugging this in, we see that

$$\begin{aligned}
S_{\neq 0} &= g_n(\omega, \psi, T') \sum_{c \in \mathbb{F}_q^\times} \omega(c)\chi(c)^n\psi(cd_{n+1})\chi(d_1 \cdots d_n)g(\chi, \psi)^n \\
&= g_n(\omega, \psi, T') \cdot \frac{\chi(d_1 \cdots d_n)\chi(d_{n+1})^n}{\omega(d_{n+1})} \sum_{c \in \mathbb{F}_q^\times} \omega(c)\chi(c)^n\psi(c)g(\chi, \psi)^n \\
&= g_n(\omega, \psi, T') \cdot \frac{\chi(d_1 \cdots d_n)\chi(d_{n+1})^n}{\omega(d_{n+1})} \cdot g\left(\omega\chi^n, \psi\right) g(\chi, \psi)^n.
\end{aligned}$$

- Suppose $A_{n+1,n+1} = 0$; here, set $T' \coloneqq \operatorname{diag}(d_1, \ldots, d_{n-1})$ for brevity. The computation in the previous case implies that we would like to show

$$\sum_{\substack{A \in \operatorname{Sym}_{n+1}^\times(\mathbb{F}_q) \\ A'_{n+1,n+1} = 0}} \omega(\det A)\psi(\operatorname{tr} AT) \stackrel{?}{=} 0.$$

In fact, let e_{n+1} denote the nth basis vector, and for any $v \in k^{n-1}$ and $c \in k$, we claim

$$S(v,c) := \sum_{\substack{A \in \mathrm{Sym}_{n+1}^{\times}(\mathbb{F}_q) \\ Ae_{n+1}=(v,c,0)}} \omega(\det A)\psi(\operatorname{tr} AT) \stackrel{?}{=} 0.$$

To do Gaussian elimination, we would like to assume $c \neq 0$. Well, because A is invertible, we know that $A_{k,n+1} \neq 0$ for some $1 \leq k \leq n$ (recall $A_{n+1,n+1} = 0$ already), so if the sum is to be nonempty, we may assume that $c \neq 0$ or $v_k \neq 0$ for some k. If $v_k \neq 0$, then note swapping the kth row and column with the nth row and column (of both A and T) will not affect the trace or detetrminant but does switch v_k with c, which grants $c \neq 0$.

We now do Gaussian elimination: note that there is a bijection

$$\begin{aligned} \mathrm{Sym}_{n-1}^{\times}(\mathbb{F}_q) \times \mathbb{F}_q^{n-1} \times \mathbb{F}_q &\to \{A \in \mathrm{Sym}_{n+1}^{\times}(\mathbb{F}_q) : Ae_{n+1} = (v,c,0)\} \\ (A' \;,\; w \quad ,\; d) &\mapsto \begin{bmatrix} 1 & \frac{1}{c}v & w \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} A' & & \\ & d & c \\ & c & \end{bmatrix} \begin{bmatrix} 1 & & \\ \frac{1}{c}v^\intercal & 1 & \\ w^\intercal & & 1 \end{bmatrix} \end{aligned}$$

(Here, $\mathrm{Sym}_0^{\times}$ is understood to consist of only the "empty" 0×0 matrix.) To see that this is a bijection, we expand out the matrix product as

$$\begin{bmatrix} A' + \frac{d}{c^2}vv^\intercal + (vw^\intercal + wv^\intercal) & dv + cw & v \\ dv^\intercal + cw^\intercal & d & c \\ v^\intercal & c & 0 \end{bmatrix},$$

so we see that $A_{n,n}$ forces d, which then forces w from $A_{k,n}$ as $1 \leq k \leq n$; the rest of the data then forces A'. Thus,

$$\begin{aligned} S(v,c) &= \sum_{\substack{A' \in \mathrm{Sym}_{n-1}^{\times}(\mathbb{F}_q) \\ w \in \mathbb{F}_q^{n-1}, d \in \mathbb{F}_q}} \omega\left(-c^2 \det A'\right) \psi\left(\operatorname{tr} A'T' + \frac{d}{c^2} \operatorname{tr} vv^\intercal T' + 2 \operatorname{tr} vw^\intercal T' + dd_n\right) \\ &= \sum_{A' \in \mathrm{Sym}_{n-1}^{\times}(\mathbb{F}_q)} \omega\left(-c^2 \det A'\right) \psi(\operatorname{tr} AT') \left(\sum_{d \in \mathbb{F}_q} \psi\left(dd_n + \frac{d}{c^2} \operatorname{tr} vv^\intercal T'\right) \sum_{w \in \mathbb{F}_q^{n-1}} \psi(2 \operatorname{tr} vw^\intercal T') \right). \end{aligned}$$

Beginning with the innermost sum, we see $\operatorname{tr} vw^\intercal T' = d_1v_1w_1 + \cdots + d_{n-1}v_{n-1}w_{n-1}$, so this sum is

$$\sum_{w \in \mathbb{F}_q^{n-1}} \psi(2 \operatorname{tr} vw^\intercal T') = \prod_{k=1}^{n-1} \left(\sum_{w_k \in \mathbb{F}_q} \psi(2d_kv_kw_k) \right).$$

In order for these inner sums to be nonzero, we note that we must have $v_k = 0$ for each k because ψ is a nontrivial character. Thus, we may assume $v = 0$, from which we see

$$S(0,c) = \left(\sum_{A' \in \mathrm{Sym}_{n-1}^{\times}(\mathbb{F}_q)} \omega(\det A')\psi(\operatorname{tr} A'T') \right) \left(\sum_{d \in \mathbb{F}_q} \psi(d_nd) \right) = 0,$$

so we conclude in this case as well.

Summing the above two cases finishes the proof of Lemma 94. $\square$

We are now ready to prove Theorem 89.

Proof of Theorem 89. Quickly, we reduce to the case where T is diagonal. Indeed, by choosing an orthogonal basis for the symmetric bilinear form given by T, we receive some $g \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $D \coloneqq gTg^\intercal$ is diagonal. As such, we compute

$$\begin{aligned} g_n(\omega, \psi, T) &= \sum_{A \in \mathrm{Sym}_n^\times(\mathbb{F}_q)} \omega(\det A)\psi(\operatorname{tr} AT) \\ &= \sum_{A \in \mathrm{Sym}_n^\times(\mathbb{F}_q)} \omega(\det A)\psi\left(\operatorname{tr} g^{-\intercal}Ag^{-1}D\right) \\ &= \sum_{A \in \mathrm{Sym}_n^\times(\mathbb{F}_q)} \omega\left(\det g^\intercal Ag\right)\psi\left(\operatorname{tr} AD\right) \\ &= \omega(\det g)^2 g_n(\omega, \psi, D). \end{aligned}$$

Now, suppose we have proven the theorem for diagonal matrices. In this case, we see $g_n(\omega, \psi, D) = (\det D)^{-1} g_n(\omega, \psi, 1)$, so $\det D = (\det g)^2(\det T)$ implies that

$$g_n(\omega, \psi, T) = (\det T)^{-1} g_n(\omega, \psi, 1),$$

which is the theorem for T, as desired.

Thus, we may assume that $T \coloneqq \operatorname{diag}(d_1, \ldots, d_n)$. At this point, we induct on n. For $n = 0$ and $n = 1$, there is nothing to say. For the induction, assume $n \geq 2$, and we use Lemma 94; for brevity, set $T' \coloneqq \operatorname{diag}(d_1, \ldots, d_{n-1})$. There are two cases.

- Suppose that $n = 2m$ is an even positive integer. In this case, Lemma 94 and induction yields

  $$\begin{aligned} g_{2m}(\omega, \psi) &= g_{2m-1}(\omega, \psi) \cdot \frac{\chi(\det T)}{\omega(d_{n+1})} \cdot g(\omega\chi, \psi) g(\chi, \psi)^{2m-1} \\ &= \frac{\chi(\det T) q^{(m-1)m}}{\omega(4^{m-1} \det T)} \cdot g(\omega, \psi) g\left(\omega^2, \psi\right)^{m-1} g(\omega\chi, \psi) g(\chi, \psi)^{2m-1}. \end{aligned}$$

  By Proposition 92, this is

  $$g_{2m}(\omega, \psi) = \frac{\chi(\det T) q^{m^2-m}}{\omega(4^m \det T)} \cdot g\left(\omega^2, \psi\right)^m g(\chi, \psi)^{2m}.$$

  Lastly, Proposition 93 yields

  $$g_{2m}(\omega, \psi) = \frac{\chi(-1)^m \chi(\det T) q^{m^2}}{\omega(4^m \det T)} \cdot g\left(\omega^2, \psi\right)^m.$$

- Suppose $n = 2m + 1$ is an odd positive integer with $m \geq 1$. In this case, Lemma 94 and induction yields

  $$\begin{aligned} g_{2m+1}(\omega, \psi) &= g_{2m}(\omega, \psi) g(\omega, \psi) \cdot \frac{\chi(\det T')}{\omega(d_{n+1})} \cdot g(\chi, \psi)^{2m} \\ &= \frac{\chi(-1)^m q^{m^2}}{\omega(4^m \det T)} \cdot g\left(\omega^2, \psi\right)^m g(\omega, \psi) g(\chi, \psi)^{2m}. \end{aligned}$$

From here, Proposition 93 implies

$$g_{2m+1}(\omega,\psi)=\frac{q^{m^2+m}}{\omega(4)^m}\cdot g(\omega,\psi)g\left(\omega^2,\psi\right)^m.$$

The above cases complete the induction. $\square$

B.3. **A Gamma Matrix Computation.** In this subsection, we use Theorem 89 to compute the finite-field analogue of a γ-matrix attachaed to the prehomogeneous space $\operatorname{Sym}_n(\mathbb{F}_q)$. For context, the p-adic analogue of Theorem 89 is intimately related to zeta functions attached to prehomogeneous spaces; see [KS97, Section 3] or [Ike17, Section 2]. We refer to [Sat89] for the general theory of prehomogeneous spaces.

In our case, we note that $(\mathrm{GL}_n, \operatorname{Sym}_n)$ is a prehomogeneous space, where the action is given by $g\cdot A := gAg^\intercal$. In other words, there is a proper algebraic subset $S\subsetneq \operatorname{Sym}_n(\overline{k})$ such that $\operatorname{Sym}_n(\overline{k})\setminus S$ is a single $\mathrm{GL}_n(\overline{k})$-oribt. To see this, for any field k, we note that two invertible symmetric matrices $A,B\in\operatorname{Sym}_n(k)$ have some $g\in\mathrm{GL}_n(k)$ such that $g\cdot A=B$ if and only if $\det A$ and $\det B$ are the same element in $k^\times/k^{\times 2}$; thus, when passing to the algebraic closure, $\operatorname{Sym}_n^\times(\overline{k})$ is a Zariski-open $\mathrm{GL}_n(\overline{k})$-orbit in $\operatorname{Sym}_n(\overline{k})$.

We now define our zeta function. Let the $\mathrm{GL}_n(\mathbb{F}_q)$-orbits of $\operatorname{Sym}_n^\times(\mathbb{F}_q)$ be denoted by $Y_1\sqcup Y_{-1}$, corresponding to if $A\in\operatorname{Sym}_n^\times(\mathbb{F}_q)$ has square or non-square determinant, respectively. Now, because the proper algebraic subset $S\subseteq\operatorname{Sym}_n(\mathbb{F}_q)$ is cut out by det, our attached zeta functions are

$$Z_k(\omega,\varphi):=\sum_{A\in Y_k}\omega(\det A)\varphi(A),$$

where $\omega\colon\mathbb{F}_q^\times\to\mathbb{C}^\times$ is a character and $\varphi\colon\operatorname{Sym}_n(\mathbb{F}_q)\to\mathbb{C}$ is some test function; let $S(\operatorname{Sym}_n(\mathbb{F}_q))$ denote this space of test functions. Now, fix once and for all a nontrivial additive character $\psi\colon\mathbb{F}_q\to\mathbb{C}^\times$, so we may define the Fourier transform

$$\mathcal{F}_\psi\varphi(A):=\sum_{B\in\operatorname{Sym}_n(\mathbb{F}_q)}\varphi(B)\psi(\operatorname{tr}AB).$$

Remark 95. To view $\mathcal{F}_\psi$ as a Fourier transform, we claim $\mathcal{F}_{\psi^{-1}}\circ\mathcal{F}_\psi=q^{\binom{n+1}{2}}$. It suffices to check this result on indicators 1_C where $C\in\operatorname{Sym}_n(\mathbb{F}_q)$. Then we see $\mathcal{F}_\psi 1_C(B)=\psi(\operatorname{tr}BC)$ for any $B\in\operatorname{Sym}_n(\mathbb{F}_q)$, so

$$\left(\mathcal{F}_{\psi^{-1}}\mathcal{F}_\psi 1_C\right)(A)=\sum_{B\in\operatorname{Sym}_n(\mathbb{F}_q)}\psi(\operatorname{tr}(C-A)B).$$

If $A=C$, then the sum is $q^{\binom{n+1}{2}}$. Otherwise, $A':=C-A\neq 0$, and we need the sum to vanish. Well, if $A'_{k'\ell'}\neq 0$ for some indices k' and ℓ', then consider the matrix $B(k',\ell')$ by $B(k',\ell')_{k\ell}=1_{\{k,\ell\}=\{k',\ell'\}}$, which gives

$$\sum_{b\in\mathbb{F}_q}\psi(\operatorname{tr}A'bB(k',\ell'))=\sum_{b\in\mathbb{F}_q}\sum_{k,\ell=1}^n\psi(bA'_{k\ell}B(k',\ell')_{\ell k})=\sum_{b\in\mathbb{F}_q}\psi(2bA'_{k'\ell'})=0.$$

Grouping the rest of the sum by $\operatorname{Sym}_n(\mathbb{F}_q)/\mathbb{F}_qB(k',\ell')$ shows that $\sum_{B\in\operatorname{Sym}_n(\mathbb{F}_q)}\psi(\operatorname{tr}A'B)=0$, as needed.

A functional equation of zeta functions attached to prehomogeneous spaces is typically a result relating $Z_\bullet(\omega, \varphi)$ to a dual version $Z_\bullet(\omega^{-1}, \mathcal{F}_\psi\varphi)$; some such results exist in the literature [DG98], but we will prove an analogue here for completeness. To prove our analogue, we begin with the following multiplicity-two result.

Proposition 96. *Fix notation as above, and let* $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ *be a character.*

(a) For any $k \in \{\pm 1\}$ *and* $g \in \mathrm{GL}_n(\mathbb{F}_q)$ *and* $\varphi \in S(\mathrm{Sym}_n(\mathbb{F}_q))$*, we have*

$$Z_k(\omega, g \cdot \varphi) = \omega(\det g)^2 Z_k(\omega, \varphi).$$

(b) The functionals $Z_1(\omega)$ *and* $Z_2(\omega)$ *are a basis of the space*

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{F}_q)}\left(S(\mathrm{Sym}_n(\mathbb{F}_q))^\circ, \omega^2 \circ \det\right),$$

where $S(\mathrm{Sym}_n(\mathbb{F}_q))^\circ$ *denotes the functionals on* $\mathrm{Sym}_n(\mathbb{F}_q)$ *supported on* $\mathrm{Sym}_n^\times(\mathbb{F}_q)$*.*

Proof. Quickly, we recall that the $\mathrm{GL}_n(\mathbb{F}_q)$-action on $\mathrm{Sym}_n(\mathbb{F}_q)$ is given by $(g \cdot \varphi)(A) = \varphi\left(g^{-1} \cdot A\right) = \varphi\left(g^{-1} A g^{-\intercal}\right)$. From this one can see that $S\left(\mathrm{Sym}_n^\times(\mathbb{F}_q)\right)^\circ$ is in fact a $\mathrm{GL}_n(\mathbb{F}_q)$-subrepresentation of $\mathrm{Sym}_n(\mathbb{F}_q)$.

To see (a), we directly compute

$$\begin{aligned} Z_k(\omega, g \cdot \varphi) &= \sum_{A \in Y_k} \omega(\det A)(g \cdot \varphi)(A) \\ &= \sum_{A \in Y_k} \omega(\det A)\varphi\left(g^{-1} \cdot A\right) \\ &= \sum_{A \in Y_k} \omega(\det g \cdot A)\varphi(A) \\ &= \omega(\det g)^2 \sum_{A \in Y_k} \omega(\det A)\varphi(A), \end{aligned}$$

which is what we wanted.

Thus, we spend most of our time on (b). Fix representatives $A_1 \in Y_1$ and $A_{-1} \in Y_{-1}$. Then we see that $Z_1(\omega)$ and $Z_2(\omega)$ are at least linearly independent as functionals on $S\left(\mathrm{Sym}_n(\mathbb{F}_q)\right)^\circ$ because $Z_k(\omega, 1_{A_\ell}) = 1_{k=\ell}\omega(\det A_\ell)$.

It remains to show that Z_1 and Z_2 span this eigenspace. The main point is that $\mathrm{Sym}_n^\times(\mathbb{F}_q)$ has only two orbits, so any eigenvector Z is essentially determined by two values. Rigorously, without loss of generality, we replace Z with

$$Z - \frac{Z(1_{A_1})}{\omega(\det A_1)} \cdot Z_1(\omega) - \frac{Z(1_{A_{-1}})}{\omega(\det A_{-1})} \cdot Z_{-1}(\omega)$$

so that $Z(1_{A_1}) = Z(1_{A_{-1}}) = 0$. We now claim that $Z = 0$, which will complete the proof. It is enough to show that $Z(1_A) = 0$ for any $A \in \mathrm{Sym}_n^\times(\mathbb{F}_q)$.

Well, $\mathrm{Sym}_n^\times(\mathbb{F}_q) = Y_1 \sqcup Y_{-1}$, so without loss of generality, take $A \in Y_1$. Then we may find $g \in \mathrm{GL}_n(\mathbb{F}_q)$ so that $A = g \cdot A_1$, so

$$1_A(B) = 1_{g \cdot A_1}(B) = 1_{A_1}\left(g^{-1} \cdot B\right) = (g \cdot 1_{A_1})(B)$$

for any $B \in \mathrm{Sym}_n(\mathbb{F}_q)$. Thus, because Z is an eigenvector,

$$Z(1_A) = Z(g \cdot 1_{A_1}) = \omega(\det g)^2 Z(1_{A_1}) = 0,$$

as desired. ∎

Remark 97. In fact, for any eigenvector Z, the proof of Proposition 96 shows that

$$Z(\varphi) = \frac{Z(1_{A_1})}{\omega(\det A_1)} \cdot Z_1(\omega, \varphi) + \frac{Z(1_{A_{-1}})}{\omega(\det A_{-1})} \cdot Z_{-1}(\omega, \varphi)$$

for any $\varphi \in S(\operatorname{Sym}_n(\mathbb{F}_q))^\circ$. Here, we recall $A_1 \in Y_1$ and $A_{-1} \in Y_{-1}$ are any representatives.

To use Proposition 96, we thus want to show that $\varphi \mapsto Z\left(\omega^{-1}, \mathcal{F}_\psi \varphi\right)$ is an eigenvector. This follows formally from the following lemma.

Lemma 98. *For any $g \in \mathrm{GL}_n(\mathbb{F}_q)$, the following diagram commutes.*

$$\begin{tikzcd} S(\operatorname{Sym}_n(\mathbb{F}_q)) \arrow[r, "\mathcal{F}_\psi"] \arrow[d, "g"'] & S(\operatorname{Sym}_n(\mathbb{F}_q)) \arrow[d, "g^{-\intercal}"] \\ S(\operatorname{Sym}_n(\mathbb{F}_q)) \arrow[r, "\mathcal{F}_\psi"] & S(\operatorname{Sym}_n(\mathbb{F}_q)) \end{tikzcd}$$

Proof. This is a direct computation. For any $g \in \mathrm{GL}_n(\mathbb{F}_q)$ and $\varphi \in S(\operatorname{Sym}_n(\mathbb{F}_q))$ and $A \in \operatorname{Sym}_n(\mathbb{F}_q)$, we compute

$$\begin{aligned} (\mathcal{F}_\psi g\varphi)(A) &= \sum_{B \in \operatorname{Sym}_n(\mathbb{F}_q)} (g\varphi)(B)\psi(\operatorname{tr} AB) \\ &= \sum_{B \in \operatorname{Sym}_n(\mathbb{F}_q)} \varphi\left(g^{-1}Bg^{-\intercal}\right)\psi(\operatorname{tr} AB) \\ &= \sum_{B \in \operatorname{Sym}_n(\mathbb{F}_q)} \varphi(B)\psi\left(\operatorname{tr} AgBg^{\intercal}\right) \\ &= \sum_{B \in \operatorname{Sym}_n(\mathbb{F}_q)} \varphi(B)\psi\left(\operatorname{tr} g^{\intercal}AgB\right) \\ &= \mathcal{F}_\psi\varphi\left(g^{\intercal} \cdot A\right) \\ &= \left(g^{-\intercal}\mathcal{F}_\psi\varphi\right)(A), \end{aligned}$$

which is what we wanted. ∎

Theorem 99. *Fix notation as above. Let $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a character. For any $k \in \{\pm 1\}$, there exist unique constants $\gamma_{k,1}(\omega)$ and $\gamma_{k,-1}(\omega)$ such that*

$$Z_k\left(\omega^{-1}, \mathcal{F}_\psi\varphi\right) = \gamma_{k,1}(\omega)Z_1(\omega,\varphi) + \gamma_{k,-1}(\omega)Z_{-1}(\omega,\varphi)$$

for any $\varphi \in S(\operatorname{Sym}_n(\mathbb{F}_q))$ supported on $\operatorname{Sym}_n^\times(\mathbb{F}_q)$.

Proof. This follows formally from Proposition 96 and Lemma 98. Indeed, it is enough to show that the functional $\varphi \mapsto Z_k\left(\omega^{-1}, \mathcal{F}_\psi\varphi\right)$ on $S(\operatorname{Sym}_n(\mathbb{F}_q))$ is a $\mathrm{GL}_n(\mathbb{F}_q)$-eigenvector with eigenvalue $\omega^2 \circ \det$. Well, for any $\varphi \in S(\operatorname{Sym}_n(\mathbb{F}_q))$ and $g \in \mathrm{GL}_n(\mathbb{F}_q)$, we use Lemma 98 to compute

$$\begin{aligned} Z_k\left(\omega^{-1}, \mathcal{F}_\psi(g\varphi)\right) &= Z_k\left(\omega^{-1}, g^{-\intercal}\mathcal{F}_\psi\varphi\right) \\ &= \left(\omega^{-1}\left(\det g^{-\intercal}\right)\right)^2 Z_k\left(\omega^{-1}, \mathcal{F}_\psi\varphi\right) \\ &= \omega(\det g)^2 Z_k\left(\omega^{-1}, \mathcal{F}_\psi\varphi\right), \end{aligned}$$

as desired. ∎

The main point of this subsection is to explicitly compute the constants $\gamma_{k,\ell}(\omega)$, which make up the "change-of-basis" γ-matrix. To this end, we have the following result.

Theorem 100. *Fix notation as above. Let $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a character, and let $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ be the nontrivial quadratic character. For any $k, \ell \in \{\pm 1\}$, we have*

$$c_{k,\ell}(\omega) = \frac{g_n\left(\omega^{-1}, \psi, 1\right) + k\ell g_n\left(\omega^{-1}\chi, \psi, 1\right)}{2}. \tag{B.3.1}$$

In particular, we have the following.

- *If $n = 2m$ is an even nonnegative integer, then*
$$c_{k,\ell}(\omega) = \chi(-1)^m \omega(4)^m q^{m^2} g\left(\omega^{-2}, \psi\right)^m 1_{k=\ell}.$$
- *If $n = 2m+1$ is an odd nonnegative integer, then*
$$c_{k,\ell}(\omega) = \omega(4)^m q^{m(m+1)} g\left(\omega^{-2}, \psi\right)^m \cdot \frac{g\left(\omega^{-1}, \psi\right) + k\ell g\left(\omega^{-1}\chi, \psi\right)}{2}.$$

Proof. The last computations follow from directly from plugging (B.3.1) into Theorem 89, so we will spend our time proving (B.3.1). Using Remark 97, we see

$$\gamma_{k,\ell}(\omega) = \frac{Z_k\left(\omega^{-1}\mathcal{F}_\psi 1_{A_\ell}\right)}{\omega(\det A_\ell)} = \frac{1}{\omega(\det A_\ell)} \sum_{A \in Y_k} \omega^{-1}(\det A)\mathcal{F}_\psi 1_{A_\ell}(A),$$

where $A_\ell \in Y_\ell$ is some representative. A direct computation shows $\mathcal{F}_\psi 1_{A_\ell}(A) = \psi(\operatorname{tr} AA_\ell)$, so

$$\gamma_{k,\ell}(\omega) = \frac{1}{\omega(\det A_\ell)} \sum_{A \in Y_k} \omega^{-1}(\det A)\psi(\operatorname{tr} AA_\ell).$$

To express this in terms of g_ns, we need to change the sum from over $A \in Y_k$ to over $A \in \operatorname{Sym}_n^\times(\mathbb{F}_q)$. To this end, we note that $A \in Y_k$ if and only if $\chi(\det A) = k$ and is $-k$ otherwise, so a direct computation shows that $1_{Y_k} = \frac{1}{2}(1 + k\chi \circ \det)$. Thus,

$$\begin{aligned}
\gamma_{k,\ell}(\omega) &= \frac{1}{\omega(\det A_\ell)} \sum_{A \in \operatorname{Sym}_n^\times} \omega^{-1}(\det A)\psi(\operatorname{tr} AA_\ell)\left(\frac{1 + k\chi(\det A)}{2}\right) \\
&= \frac{g_n\left(\omega^{-1}, \psi, A_\ell\right) + k g_n\left(\omega^{-1}\chi, \psi, A_\ell\right)}{2\omega(\det A_\ell)}.
\end{aligned}$$

To finish up, we note that Theorem 89 implies that $g_n\left(\omega^{-1}, \psi, A_\ell\right) = \omega(\det A_\ell) g_n\left(\omega^{-1}, \psi, 1\right)$ and

$$g_n\left(\omega^{-1}\chi, \psi, A_\ell\right) = \omega(\det A_\ell)\chi(\det A_\ell) g_n\left(\omega^{-1}\chi, \psi, 1\right) = \omega(\det A_\ell)\ell g_n\left(\omega^{-1}\chi, \psi, 1\right),$$

from which substitution completes the proof. ∎

Corollary 101. *Fix notation as above. Let $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a character. The functions $\varphi \mapsto Z_\bullet\left(\omega^{-1}, \mathcal{F}_\psi\varphi\right)$ form a basis of the space*

$$\operatorname{Hom}_{\mathrm{GL}_n(\mathbb{F}_q)}\left(S(\operatorname{Sym}_n(\mathbb{F}_q))^\circ, \omega^2 \circ \det\right),$$

where $S(\operatorname{Sym}_n(\mathbb{F}_q))^\circ$ denotes the functionals on $\operatorname{Sym}_n(\mathbb{F}_q)$ supported on $\operatorname{Sym}_n^\times(\mathbb{F}_q)$.

Proof. For brevity, define $Z'_\bullet(\varphi) := Z_\bullet(\omega^{-1}, \mathcal{F}_\psi\varphi)$. Note that $Z'_\bullet$ is in fact an eigenvector by the proof of Theorem 99, and this space has basis given by $Z_1(\omega)$ and $Z_2(\omega)$ by Proposition 96. Now, the constants $(\gamma_{k,\ell})_{k,\ell\in\{\pm1\}}$ make a change-of-basis matrix from $\{Z_1(\omega), Z_2(\omega)\}$ to $\{Z'_1, Z'_2\}$, so it suffices to show that

$$\det\begin{bmatrix}\gamma_{1,1} & \gamma_{1,-1}\\ \gamma_{-1,1} & \gamma_{-1,-1}\end{bmatrix} \overset{?}{\neq} 0.$$

To use Theorem 100, we set $g_+ := g_n(\omega^{-1}, \psi, 1)$ and $g_- := g_n(\omega^{-1}\chi, \psi, 1)$, from which we compute

$$\det\begin{bmatrix}\gamma_{1,1} & \gamma_{1,-1}\\ \gamma_{-1,1} & \gamma_{-1,-1}\end{bmatrix} = \det\frac{1}{2}\begin{bmatrix}g_+ + g_- & g_+ - g_-\\ g_+ - g_- & g_+ + g_-\end{bmatrix} = g_+g_-.$$

Now, g_+ and g_- are nonzero by Theorem 89 (and Proposition 93), so we are done. $\square$

Remark 102. Combining the above computation with Remark 91, in the "generic" case $\omega^2 \neq 1$, we have

$$\left|\det\begin{bmatrix}\gamma_{1,1} & \gamma_{1,-1}\\ \gamma_{-1,1} & \gamma_{-1,-1}\end{bmatrix}\right| = q^{\binom{n+1}{2}}.$$

If we were to normalize $\mathcal{F}_\psi$ to $\mathcal{F}^*_\psi := q^{-\frac{1}{2}\binom{n+1}{2}}\mathcal{F}_\psi$ and redefine everything with the normalized Fourier transform, then this determinant would have absolute value 1. This normalization factor is desirable because Remark 95 implies $\mathcal{F}^*_{\psi^{-1}} \circ \mathcal{F}^*_\psi = 1$.

B.4. **Combinatorics.** In this subsection, we use Theorem 89 to compute the number of symmetric invertible matrices over $\mathbb{F}_q$ with specified trace and determinant. This requires a more complete understanding of the sums $g_n(\omega, \psi, T)$ than Theorem 89 provides; in particular, we need to understand the case when ψ is trivial. Nonetheless, the method of proof Theorem 89 still applies.

Proposition 103. *Fix a nonnegative integer n and some $T \in \operatorname{Sym}^\times_n(\mathbb{F}_q)$.*

(a) Let $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a nontrivial character. If n is odd or $\omega^2 \neq 1$, then $g_n(\omega, 1, T) = 0$.
(b) Let $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ be the nontrivial quadratic character. If $n = 2m$ is even, then

$$g_{2m}(\chi, 1, T) = \chi(-1)^m q^{m^2}\prod_{k=0}^{m-1}\left(q^{2k+1} - 1\right).$$

Proof. For the proof of (a), we have two cases.

- Suppose $\omega^2 \neq 1$. Then for any $g \in \mathrm{GL}_n(\mathbb{F}_q)$, we see that $A \in \operatorname{Sym}^\times_n(\mathbb{F}_q)$ if and only if $gAg^\intercal \in \operatorname{Sym}^\times_n(\mathbb{F}_q)$, so
$$g_n(\omega, 1, T) = \sum_{A\in\operatorname{Sym}^\times_n(\mathbb{F}_q)}\omega(\det A) = \sum_{A\in\operatorname{Sym}^\times_n(\mathbb{F}_q)}\omega(\det gAg^\intercal) = \omega(\det g)^2 g_n(\omega, 1, T).$$
Thus, to conclude $g_n(\omega, 1, T) = 0$, it suffices to find $g \in \mathrm{GL}_n(\mathbb{F}_q)$ with $\omega(\det g)^2 \neq 1$. Well, $\omega^2 \neq 1$, so find $c \in \mathbb{F}_q^\times$ such that $\omega(c)^2 \neq 1$ and then set $g := \operatorname{diag}(c, 1, \ldots, 1)$.
- Suppose n is odd. By the previous case, we may assume that $\omega^2 = 1$. Now, for any $c \in \mathbb{F}_q^\times$, we see that $A \in \operatorname{Sym}^\times_n(\mathbb{F}_q)$ if and only if $cA \in \operatorname{Sym}^\times_n(\mathbb{F}_q)$, so
$$g_n(\omega, 1, T) = \sum_{A\in\operatorname{Sym}^\times_n(\mathbb{F}_q)}\omega(\det A) = \sum_{A\in\operatorname{Sym}^\times_n(\mathbb{F}_q)}\omega(c\det A) = \omega(c)^n g_n(\omega, 1, T).$$

Now, if we did have $g_n(\omega, 1, T) \neq 0$, then we would have $\omega(c)^n = 1$ for all $c \in \mathbb{F}_q^\times$ and hence $\omega^n = 1$; however, n is odd and $\omega^2 = 1$ already, so it would follow $\omega = 1$. However, $\omega \neq 1$ by hypothesis.

For the proof of (b), we imitate the proof of Theorem 89. As an analogue of Lemma 94, we claim that

$$g_{2m+2}(\chi, 1, T) \stackrel{?}{=} g_{2m}(\chi, 1, T) \cdot \chi(-1) q^{2m+1} \left(q^{2m+1} - 1\right) \tag{B.4.1}$$

for any nonnegative integer m. Note that (B.4.1) will complete the proof of (b) by an induction.

Now, the proof of (B.4.1) is analogous to Lemma 94; there are two cases. Set $n \coloneqq 2m$ for brevity.

- We sum over $A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q)$ with $A_{n+2,n+2} \neq 0$. As in Lemma 94, we have the following bijection.

$$\begin{aligned} \operatorname{Sym}_{n+1}^\times(\mathbb{F}_q) \times \mathbb{F}_q^{n+1} \times \mathbb{F}_q^\times &\to \{A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q) : A_{n+2,n+2} \neq 0\} \\ (A' \;,\; v \qquad ,\; c) &\mapsto \begin{bmatrix} 1 & v \\ & 1 \end{bmatrix} \begin{bmatrix} A' & \\ & c \end{bmatrix} \begin{bmatrix} 1 & \\ v^\intercal & 1 \end{bmatrix} \end{aligned}$$

  It follows that

$$\sum_{\substack{A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q) \\ A_{n+2,n+2} \neq 0}} \chi(\det A) = \left(\sum_{A' \in \operatorname{Sym}_{n+1}^\times(\mathbb{F}_q)} \chi(\det A) \right) \left(\sum_{c \in \mathbb{F}_q^\times, v \in \mathbb{F}_q^{n+1}} \omega(c) \right),$$

  but the left sum vanishes by (a) because it is $g_{2m+1}(\chi, 1, T) = 0$. Thus, there is no contribution in this case.
- We sum over $A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q)$ with $A_{n+2,n+2} = 0$. In light of the previous case, we expect all contribution from this case. Let e_{n+2} denote the $(n+2)$nd basis vector. For any $v \in \mathbb{F}_q^n$ and $c \in \mathbb{F}_q$, we claim that

$$\sum_{\substack{A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q) \\ Ae_{n+2} = (v,c,0)}} \chi(\det A) \stackrel{?}{=} g_{2m}(\chi, 1, T) \cdot \chi(-1) q^{2m+1},$$

  from which the claim will follow upon summing over all vectors $(v, c) \in \mathbb{F}_q^{n+1}$ with at least one nonzero entry. Quickly, because some entry in $(v, c) \in \mathbb{F}_q^{n+1}$, we note that we can rearrange the rows and columns of A to allow us to assume that $c \neq 0$.

  Thus, as in Lemma 94, we have the following bijection.

$$\begin{aligned} \operatorname{Sym}_n^\times(\mathbb{F}_q) \times \mathbb{F}_q^n \times \mathbb{F}_q &\to \{A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q) : Ae_{n+2} = (v, c, 0)\} \\ (A' \;,\; w \;\;,\; d) &\mapsto \begin{bmatrix} 1 & \frac{1}{c}v & w \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} A' & & \\ & d & c \\ & c & \end{bmatrix} \begin{bmatrix} 1 & & \\ \frac{1}{c}v^\intercal & 1 & \\ w^\intercal & & 1 \end{bmatrix} \end{aligned}$$

  It follows that

$$\sum_{\substack{A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q) \\ Ae_{n+2} = (v,c,0)}} \chi(\det A) = \sum_{w \in \mathbb{F}_q^n, d \in \mathbb{F}_q} \left(\sum_{A' \in \operatorname{Sym}_n^\times(\mathbb{F}_q)} \chi\left(-c^2 \det A\right) \right).$$

  which is what we wanted upon noting $\chi\left(-c^2\right) = \chi(-1)$ and collecting sums.

Combining the above casework completes the proof of (b). $\square$

Remark 104. One can prove (b) by a combinatorial argument, directly counting the number of invertible symmetric matrices with square determinant; this is done in [Mac69, Theorem 4]. We have included the above proof to emphasize the strength of the Gaussian elimination technique to compute these Gauss sums.

The last sum $g_n(\omega, \psi, T)$ to consider is the case where ω and ψ are both trivial. Equivalently, we are counting the number of invertible symmetric $n \times n$ matrices with entries in $\mathbb{F}_q$. This result is well-known; for example, see [Mac69, Theorem 2]. However, to emphasize the strength of our method (and for completeness), we will present a proof using Gaussian elimination, as done in [BM87] in the case of $\mathbb{F}_p$.

Proposition 105. *Fix a nonnegative integer n and some $T \in \operatorname{Sym}_n^\times(\mathbb{F}_q)$.*

(a) If $n = 2m$ is even, then

$$g_{2m}(1, 1, T) = q^{m^2+m} \prod_{k=0}^{m-1} \left(q^{2k+1} - 1\right).$$

(b) If $n = 2m + 1$ is odd, then

$$g_{2m+1}(1, 1, T) = q^{m^2+m} \prod_{k=0}^{m} \left(q^{2k+1} - 1\right).$$

Proof. The proof will be by induction on n. In analogy to Lemma 94, the main claim is that

$$g_{n+2}(1, 1, T) \stackrel{?}{=} q^{n+1}(q - 1)g_{n+1}(1, 1, T) + q^{n+1}\left(q^{n+1} - 1\right) g_n(1, 1, T) \tag{B.4.2}$$

for any nonnegative integer n. The proof of Equation (B.4.2) uses the typical Gaussian elimination technique.

- We sum over $A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q)$ where $A_{n+2,n+2} \neq 0$. As in Lemma 94, we have the following bijection.

$$\begin{aligned} \operatorname{Sym}_{n+1}^\times(\mathbb{F}_q) \times \mathbb{F}_q^{n+1} \times \mathbb{F}_q^\times &\to \{A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q) : A_{n+2,n+2} \neq 0\} \\ (A' \ , \ v \qquad , \ c) &\mapsto \begin{bmatrix} 1 & v \\ & 1 \end{bmatrix} \begin{bmatrix} A' & \\ & c \end{bmatrix} \begin{bmatrix} 1 & \\ v^\intercal & 1 \end{bmatrix} \end{aligned}$$

  It follows that the number of matrices in this case is $q^{n+1}(q - 1)g_{n+1}(1, 1, T)$.
- We sum over $A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q)$ where $A_{n+2,n+2} = 0$. Let e_{n+2} denote the $(n + 2)$nd basis vector. For any $v \in \mathbb{F}_q^n$ and $c \in \mathbb{F}_q$, we claim that

$$\#\left\{A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q) : Ae_{n+2} = (v, c, 0)\right\} \stackrel{?}{=} q^{n+1} g_n(1, 1, T),$$

  from which (B.4.2) will follow by summing over all $(v, c) \in \mathbb{F}_q^{n+1}$ with at least one nonzero entry. Quickly, because some entry in $(v, c) \in \mathbb{F}_q^{n+1}$, we note that we can rearrange the rows and columns of A to allow us to assume that $c \neq 0$.

  Now, as in Lemma 94, we have the following bijection.

$$\begin{aligned} \operatorname{Sym}_n^\times(\mathbb{F}_q) \times \mathbb{F}_q^n \times \mathbb{F}_q &\to \{A \in \operatorname{Sym}_{n+2}^\times(\mathbb{F}_q) : Ae_{n+2} = (v, c, 0)\} \\ (A' \ , \ w \ \ , \ d) &\mapsto \begin{bmatrix} 1 & \frac{1}{c}v & w \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} A' & & \\ & d & c \\ & c & \end{bmatrix} \begin{bmatrix} 1 & & \\ \frac{1}{c}v^\intercal & 1 & \\ w^\intercal & & 1 \end{bmatrix} \end{aligned}$$

The desired equality follows.

Summing the above cases completes the proof of (B.4.2).

We now complete the proof by an induction on n. For $n = 0$ and $n = 1$, there is nothing to say. Now, to synthesize cases, we note that

$$q^{m^2+m}\prod_{k=0}^{m}\left(q^{2k+1}-1\right) = q^{\frac{1}{2}(2m+1)(2m+2)}\prod_{\substack{1\le k\le 2m+1\\ k\text{ odd}}}\left(1-\frac{1}{q^k}\right)$$

and analogously for the even case. Thus, for our induction, we take $n \geq 0$ and use (B.4.2) to see $g_{n+2}(1,1,T)$ is

$$\begin{aligned}&q^{n+1}(q-1)g_{n+1}(1,1,T) + q^{n+1}\left(q^{n+1}-1\right)g_n(1,1,T)\\ &= q^{\frac{1}{2}(n+2)(n+1)}\left(q^{n+1}(q-1)\prod_{\substack{n<k\le n+1\\ k\text{ odd}}}\left(1-\frac{1}{q^k}\right) + \left(q^{n+1}-1\right)\right)\prod_{\substack{1\le k\le n\\ k\text{ odd}}}\left(1-\frac{1}{q^k}\right).\end{aligned}$$

If n is odd, we have

$$q^{\frac{1}{2}(n+2)(n+1)}\left(q^{n+2}-1\right)\prod_{\substack{1\le k\le n\\ k\text{ odd}}}\left(1-\frac{1}{q^k}\right),$$

which simplifies correctly. If n is even, we have

$$q^{\frac{1}{2}(n+2)(n+1)}\underbrace{\left(q^{n+1}(q-1)\left(1-\frac{1}{q^{n+1}}\right)+\left(q^{n+1}-1\right)\right)}_{q(q^{n+1}-1)}\prod_{\substack{1\le k\le n\\ k\text{ odd}}}\left(1-\frac{1}{q^k}\right),$$

which still simplifies correctly. This completes the induction. $\square$

We are now ready for our combinatorics.

Theorem 106. *Let n be a nonnegative integer, and fix some $T \in \operatorname{Sym}_n^\times(\mathbb{F}_q)$. Further, fix $d \in \mathbb{F}_q^\times$ and $t \in \mathbb{F}_q$.*

(a) Suppose that $n = 2m+1$ is odd. Then the number of $A \in \operatorname{Sym}_{2m+1}^\times(\mathbb{F}_q)$ such that $\det A = d$ and $\operatorname{tr} AT = t$ is

$$\frac{q^{m^2+m}}{q(q-1)}\left(\prod_{k=0}^{m}\left(q^{2k+1}-1\right)-(q-1)^{m+1}\right)$$
$$+\,q^{m^2+m}\cdot\#\left\{(x,y_1,\ldots,y_m): x+(y_1+\cdots+y_m)=t,\ \frac{x(y_1\cdots y_m)^2}{4^m\det T}=d\right\}.$$

(b) Suppose that $n = 2m$ is even. Let $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ denote the nontrivial quadratic character. Then the number of $A \in \operatorname{Sym}_{2m}^\times(\mathbb{F}_q)$ such that $\det A = d$ and $\operatorname{tr} AT = t$ is

$$\frac{q^{m^2}}{q(q-1)}\left((q^m+\chi(-1)^m\chi(d))\prod_{k=0}^{m-1}\left(q^{2k+1}-1\right)-\chi(-1)^m\left(\chi(d)+\chi(\det T)\right)(q-1)^m\right)$$
$$+\,\chi(-1)^m\chi(\det T)q^{m^2}\cdot\#\left\{(y_1,\ldots,y_m): y_1+\cdots+y_m=t,\ \frac{(y_1\cdots y_m)^2}{4^m\det T}=d\right\}.$$

Proof. We prove these separately.

(a) For any characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$, we claim that

$$\begin{aligned} g_n(\omega,\psi,T) &\stackrel{?}{=} \frac{q^{m(m+1)}}{\omega\left(4^m \det T\right)} \cdot g(\omega,\psi) g\left(\omega^2,\psi\right)^m \\ &\quad + \frac{g_n(1,1,T) - q^{m(m+1)}(q-1)^{m+1}}{q(q+1)} \sum_{a\in\mathbb{F}_q^\times, b\in\mathbb{F}_q} \omega(a)\psi(b). \end{aligned}$$

This is by casework. If ψ is nontrivial, the second sum on the right-hand side vanishes, so the claim follows from Theorem 89. If ψ is trivial and ω is nontrivial, then the right-hand side vanishes, and left-hand side vanishes by Proposition 103. Lastly, if both ψ and ω are trivial, then both sides are $g_n(1,1,T)$.

Now, we notice that full expansion gives

$$\frac{1}{\omega\left(4^m \det T\right)} \cdot g(\omega,\psi) g\left(\omega^2,\psi\right) = \sum_{x,y_1,\ldots,y_m\in\mathbb{F}_q^\times} \omega\left(\frac{x(y_1\cdots y_m)^2}{4\det T}\right)\psi(x+(y_1+\cdots+y_m)),$$

so by summing appropriately over all ω and ψ, we see that the number of $A \in \operatorname{Sym}_n^\times(\mathbb{F}_q)$ such that $\det A = d$ and $\operatorname{tr} AT = t$ is

$$\begin{aligned} &\frac{g_n(1,1,T) - q^{m(m+1)}(q-1)^{m+1}}{q(q+1)} \\ &+ q^{m^2+m} \cdot \#\left\{(x,y_1,\ldots,y_m) : x + (y_1+\cdots+y_m) = t, \frac{x(y_1\cdots y_m)^2}{4^m \det T} = d\right\}. \end{aligned}$$

To finish, we note that we can simplify the first term with from Proposition 105.

(b) For any characters $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$, we claim that

$$\begin{aligned} g_n(\omega,\psi,T) &\stackrel{?}{=} \frac{\chi(-1)^m \chi(\det T) q^{m^2}}{\omega\left(4^m \det T\right)} \cdot g\left(\omega^2,\psi\right)^m \\ &\quad + \frac{g_n(\chi,1,T) - \chi(-1)^m q^{m^2}(q-1)^m}{q(q-1)} \sum_{a\in\mathbb{F}_q^\times, b\in\mathbb{F}_q} \chi(a)\omega(a)\psi(b) \\ &\quad + \frac{g_n(1,1,T) - \chi(-1)^m \chi(\det T) q^{m^2}(q-1)^m}{q(q-1)} \sum_{a\in\mathbb{F}_q^\times, b\in\mathbb{F}_q} \omega(a)\psi(b). \end{aligned}$$

Again, this is by casework. If ψ is trivial, this is Theorem 89; otherwise, ψ is trivial. Then if $\omega^2 \neq 1$ (i.e., $\omega \notin \{1,\chi\}$) the right-hand side vanishes, and the left-hand side vanishes by Proposition 103. Lastly, if $\omega \in \{1,\chi\}$, then both sides are equal by construction.

Now, as in (a), by full expansion of $\omega\left(4^m \det T\right)^{-1} g\left(\omega^2,\psi\right)^m$ and summing the claim over all ω and ψ appropriately, we see that the number of $A \in \operatorname{Sym}_n^\times(\mathbb{F}_q)$ such that $\det A = d$ and $\operatorname{tr} AT = t$ is

$$\frac{g_n(\chi,1,T)-\chi(-1)^m q^{m^2}(q-1)^m}{q(q-1)}\cdot\chi(d)+\frac{g_n(1,1,T)-\chi(-1)^m\chi(\det T)q^{m^2}(q-1)^m}{q(q-1)}$$
$$+\chi(-1)^m\chi(\det T)q^{m^2}\cdot\#\left\{(y_1,\dots,y_m):y_1+\cdots+y_m=t,\frac{(y_1\cdots y_m)^2}{4^m\det T}=d\right\}.$$

It remains to simplify the first two terms. On one hand, we note Proposition 103 gives

$$\frac{g_n(\chi,1,T)-\chi(-1)^m q^{m^2}(q-1)^m}{q(q-1)}\cdot\chi(d)=\frac{\chi(-1)^m q^{m^2}}{q(q-1)}\left(\prod_{k=0}^{m-1}\left(q^{2k+1}-1\right)-(q-1)^m\right)\chi(d).$$

On the other hand, Proposition 105 gives

$$\frac{g_n(1,1,T)-\chi(-1)^m\chi(\det T)q^{m^2}(q-1)^m}{q(q-1)}=\frac{q^{m^2}}{q(q-1)}\left(q^m\prod_{k=0}^{m-1}\left(q^{2k+1}-1\right)-\chi(-1)^m\chi(\det T)(q-1)^m\right).$$

Summing the above two equalities completes the simplification. □

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