Thirty-Sixth
University of Michigan
Undergraduate Mathematics Competition
Problems with Solutions
April, 2019

Problem 1. Evaluate the series

$$\frac{1}{2\sqrt{3}+3\sqrt{2}}+\frac{1}{3\sqrt{4}+4\sqrt{3}}+\frac{1}{4\sqrt{5}+5\sqrt{4}}+\cdots$$

Solution 1. We observe that

$$\begin{aligned}\frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}&=\frac{n\sqrt{n+1}-(n+1)\sqrt{n}}{(n\sqrt{n+1}+(n+1)\sqrt{n})(n\sqrt{n+1}-(n+1)\sqrt{n})}\\&=\frac{n\sqrt{n+1}-(n+1)\sqrt{n}}{n^2(n+1)-n(n+1)^2}=\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\,.\end{aligned}$$

This is a telescoping series with the term $n^{-1/2}$ tending to zero, so the value of the series is $2^{-1/2}$.

Problem 2. Show that 3^n, with $n \geq 3$ an integer, cannot have only odd digits in its decimal representation.

Solution 2. Since $3^4 = 81 \equiv 1 \pmod{20}$, it follows that the powers of 3 modulo 20 are $1, 3, 9, 7$. Write 3^k in its decimal expansion, $3^k = \sum d_j 10^j$. Since $20|10^2$, it follows that $3^k \equiv d_1 10 + d_0 \pmod{20}$. In order that this should be one of the numbers $1, 3, 9, 7$ modulo 20, it is necessary that d_1 is even.

Problem 3. Suppose that a, b, c are positive real numbers with

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq 5.$$

What is the largest possible value of $\frac{a}{b}$?

Solution 3. By the arithmetic-geometric mean inequality we see that

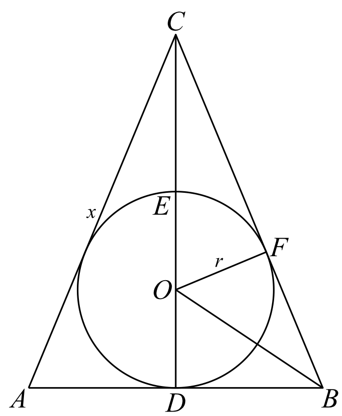
$$2\sqrt{\frac{b}{a}}\leq\frac{b}{c}+\frac{c}{a}\,.$$

Thus

$$\frac{a}{b}+2\sqrt{\frac{b}{a}}\leq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq 5\,.$$

Put $x = \sqrt{a/b}$. The inequality $x^2 + 2/x \leq 5$ is equivalent to $x^3 - 5x + 2 \leq 0$. This polynomial has roots $-1-\sqrt{2}, -1+\sqrt{2}, 2$ in increasing order, so we deduce that $x \leq 2$, which is to say that $a/b \leq 4$. On the other hand, the triple $(a,b,c) = (4,1,2)$ is admissible, so the largest possible value of a/b is 4.

Problem 4. In a magical isosceles triangle $\triangle ABC$ we have $|AC| = |BC|$. Let D be the midpoint between A and B. The inscribed circle of ABC intersects the line segment CD in a point E that is in the interior of the triangle. Suppose that $|AB| = 15$ and $|CE| = 8$. Determine $|AC|$.



Solution 4. We let $x = |AC|$, and r the radius of the inscribed circle. We further let O be the center of the inscribed circle and F the intersection of the inscribed circle with the segment BC. We note that the right triangle ODB is congruent to the right triangle OFB. Since $|DB| = 15/2$, it follows that $|FB| = 15/2$, and hence that $|CF| = x - 15/2$. The smaller right triangle OFC is similar to the larger right triangle BDC. Hence

$$\frac{|OF|}{|DB|} = \frac{|CF|}{|CD|} = \frac{|CO|}{|BC|}\,.$$

That is,

$$\frac{r}{15/2} = \frac{x - 15/2}{2r+8} = \frac{r+8}{x}\,. \tag{1}$$

On cross-multiplying the first and second fractions above, we find that

$$\frac{15}{2}\left(x - \frac{15}{2}\right) = r(2r+8)\,. \tag{2}$$

On cross-multiplying the first and third fractions in (1), we find that $rx = \frac{15}{2}(r+8)$, which is to say that

$$r\left(x - \frac{15}{2}\right) = 60\,. \tag{3}$$

To eliminate x, we multiply both sides of (2) by r, and both sides of (3) by $15/2$, and subtract the one equation from the other. This gives $r^3+4r^2-225=0$. This cubic polynomial factors as $(r-5)(r^2+9r+45)$. Here the quadratic factor has no real root, so we deduce that $r=5$. On substituting this into (3) we deduce that $|AC|=x=39/2$.

Problem 5. We have a set $\mathcal{D}$ of N Associate Deans, and we form m different committees whose members are in $\mathcal{D}$, with the cardinalities of the committees being $k_1, k_2, \ldots, k_m$, and no committee is a subcommittee of any other. Show that

$$\sum_{i=1}^{m} k_i!(N-k_i)! \;\le\; N!\,.$$

Solution 5. Let $\mathcal{S}_i$ denote the set of Associate Deans in the i^{th} committee. For each i, arrange the Deans linearly, with members of $\mathcal{S}_i$ in the first k_i places, and the other Deans in the remaining $N-k_i$ places. The Deans in the committee can be ordered in $k_i!$ different ways, and the Deans not in the committee can be ordering in $(N-k_i)!$ different ways, making $k_i!(N-k_i)!$ orderings altogether. To see that these collections of permutations are pairwise disjoint, suppose that $i\ne j$ and that $k_i\le k_j$. In a listing corresponding to $\mathcal{S}_i$, the first k_i entries are the members of $\mathcal{S}_i$. However, in a listing for $\mathcal{S}_j$, the first j entries are all members of $\mathcal{S}_j$, and if the first i of these were all members of $\mathcal{S}_i$, then we would have $\mathcal{S}_i\subseteq\mathcal{S}_j$, which we know is not the case. Since these various collections of permutations are pairwise disjoint, the sum of their cardinalities is at most the total number of permutations, $N!$.

One may carry the reasoning further: We note that $k_i!(N-k_i)!\ge\lfloor N/2\rfloor!(N-\lfloor N/2\rfloor)!$. Hence the left hand side of the identity is $\ge D\lfloor N/2\rfloor!(N-\lfloor N/2\rfloor)!$, and so

$$D\le\binom{N}{\lfloor N/2\rfloor}.$$

This is a result in combinatorics known as *Sperner's Lemma.*

Problem 6. Suppose that $a_1, a_2, \ldots, a_{2n+1}$ are real numbers such that when any one of the is removed, the remaining $2n$ of them can be partitioned into two collections of n terms with equal sums. Show that the a_i must all be equal.

Solution 6. We rephrase the hypothesis by saying that for each i there is a linear form L_i in the $a's$ that vanishes. The coefficient of a_i in this linear form is 0, while for $j\ne i$, n of the a_j have coefficient 1, and n of them have coefficient -1. Let C be the $(2n+1)\times(2n+1$ matrix whose i^{th} row is a listing of the coefficients of L_i. We know that $C\boldsymbol{a}=\mathbf{0}$. Let $\boldsymbol{x}$ be the vector all of whose coordinates are 1. We note that $A\boldsymbol{x}=\mathbf{0}$. In order to show that $\boldsymbol{a}$ lies in the subspace generated by $\boldsymbol{x}$, it suffices to show that C has rank at least $2n$, since rank + nullity $=2n+1$. Let $D=[d_{ij}]$ be the $(2n)\times(2n)$ matrix formed from the first $2n$ rows and columns of C. We show that D is nonsingular by showing that its determinant is odd. Let $E=[e_{ij}]$ be the matrix with $e_{ij}=|d_{ij}|$. Thus all off-diagonal elements of E are 1, and all diagonal elements are 0. The determinant of a matrix is a polynomial in its entries with integer coefficients, and $e_{ij} \equiv d_{ij} \pmod 2$, so $\det D \equiv \det E \pmod 2$. To complete the solution it suffices to show that $\det E$ is odd. This may be done in several ways. For example, with suitable elementary row and column operations we may reduce E to triangular form, and thus show that $\det E = -(2n-1)$. Alternatively, it suffices to observe that $E^2 \equiv I \pmod 2$.

Problem 7. For a given positive real number r and real x, y let $N_r(x,y)$ be the number of pairs of integers (m,n) satisfying $(x-m)^2 + (y-n)^2 \le r^2$. Evaluate

$$\int_0^1 \int_0^1 N_r(x,y)\,dx\,dy$$

as a function of r.

Solution 7. Let $I(x,y) = 1$ if $x^2 + y^2 \le r^2$, and $I(x,y) = 0$ otherwise. Thus

$$N_r(x,y) = \sum_{(m,n)\in\mathbb{Z}^2} I(x-m, y-n),$$

and hence

$$\begin{aligned}\int_0^1 \int_0^1 N_r(x,y)\,dx\,dy &= \sum_{(m,n)\in\mathbb{Z}^2} \int_0^1 \int_0^1 I(x-m, y-n)\,dx\,dy \\ &= \sum_{(m,n)\in\mathbb{Z}^2} \int_{-m}^{-m+1} \int_{-n}^{-n+1} I(x,y)\,dx\,dy\,.\end{aligned}$$

The squares $[-m, -m+1] \times [-n, -n+1]$ tile the plane $\mathbb{R}^2$, so the above is

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x,y)\,dx\,dy = \pi r^2.$$

Problem 8. Let a and b be relatively prime positive integers, and let $\mathcal{S}$ be the set of those nonnegative integers n that can be written $n = ua + vb$ where u and v are nonnegative integers. Show that

$$\sum_{n\in\mathcal{S}} z^n \;=\; \frac{1 - z^{ab}}{(1-z^a)(1-z^b)}$$

for $|z| < 1$.

Solution 8. Suppose that n, u, v are integers such that $n = ua + vb$. For a given n there will be infinitely many pairs (u,v) for which this equation holds, but in all such pairs, the number u satisfies the congruence $ua \equiv n \pmod b$. Since $(a,b) = 1$, this congruence has exactly one solution u_0 with $0 \le u_0 < b$. If $n = ua + vb$ is a representation of n with nonnegative

u, v, and $u \geq b$, then $u = u_0 + kb$ for some positive integer k. Thus $n = u_0 a + (v + ka)b$ and we see that any integer n that can be represented with nonnegative u, v has a unique representation $n = u_0 a + v_0 b$ with $0 \leq u_0 < b$. Thus

$$\sum_{n \in \mathcal{S}} z^n = \sum_{j=0}^{b-1} \sum_{k=0}^{\infty} z^{ja+kb} = \Big(\sum_{j=0}^{b-1} z^{ja} \Big) \Big(\sum_{k=0}^{\infty} z^{kb} \Big) = \frac{1 - z^{ab}}{1 - z^a} \cdot \frac{1}{1 - z^b}$$

for $|z| < 1$.

Alternatively, let $r(n)$ denote the number of representations of n in the form $n = ua + vb$ with nonnegative u, v. Thus $\sum_n r(n) z^n = (1 - z^a)^{-1}(1 - z^b)^{-1}$. One can show that $r(n) = 0$ or 1 if $0 \leq n < ab$. Also, if $n \geq ab$, then $r(n) = r(n - ab) + 1$. Hence $\sum_{n \in \mathcal{S}} z^n = \sum_n (r(n) - r(n - ab)) z^n$, which gives a second proof of the identity.

Problem 9. Let f be a continuously differentiable map from the unit interval $I = [0, 1]$ to the unit square $J = [0, 1] \times [0, 1]$, and suppose that the boundary ∂J is in the image of f. Prove that there exist $0 \leq s < t \leq 1$ such that $f(s) = f(t)$, and the arc length of $f|_{[s,t]}$ is greater or equal to 2.

Solution 9. Assume the conclusion is false. Then if $f(s), f(t)$ belong to one side J of the square, there does not exist a $s < u < t$ such that $f(u)$ belongs to the opposite side J'. (Otherwise, the set $\{f(x) \in J | x < u\}$ is both open and closed.) Thus, thee must exist two sides, J, K such that $0 \in f^{-1}(U \cup K)$ is connected (and thus equal to some $[0, q]$, as is1 $\in f^{-1}(J' \cup K')$ (where J', K' denote the opposite sides), and thus equal to $[q, 1]$. Thus $f(q) \in ((J \cup K) \cap (J' \cup K'))$. Thus, the other point in $(J \cup K) \cap (J' \cup K')$ must be of the form $f(s) = f(t)$ for some $s < q < t$. These points satisfy the requirements of the problem (which also is a contradiction).

Problem 10. Two prisoners must play the following game to save their lives. They learn the rule of the game, can work on strategy, but once the game starts, they are unable to communicate. The first prisoner is taken into a room, which contains a chessboard. Each of the 64 fields of the chessboard has a coin, showing either heads or tails. The warden points to one of the coins. The first prisoner turns over exactly one coin (which may or may not be the one that the warden pointed to). The first prisoner is taken to his/her solitary cell. The second prisoner is led in. Upon examining the coins on the board, the second prisoner must identify the coin that the warden pointed to. Describe a strategy by which this can be done.

Solution 10. Let V be the vector space $V = F_2^6$. For $0 \leq n < 64$ we write $n = \sum_{i=0}^{5} b_i 2^i$, and put $b(n) = (b_0, b_1, \ldots, b_5) \in V$. Row by row, starting at the top left and ending at the bottom right, we number the squares of the board $0, 1, \ldots, 63$. Let $\mathcal{H}$ be the set of those h, $0 \leq h < 64$ such that the coin in square h shows heads. Let w be the number of the square indicated by warden. Form the sum

$$b(w) + \sum_{h \in \mathcal{H}} b(h)\,.$$

This is a member of V, and there is a unique integer f, $0 \le f < 64$, such that $b(f)$ is equal to the above. The first prisoner should turn over ('flip') the coin in square numbered f. Let $\mathcal{H}'$ denote the set of numbers of the squares now showing heads. If $f \notin \mathcal{H}$, then

$$\sum_{h \in \mathcal{H}'} b(h) = b(f) + \sum_{h \in \mathcal{H}} b(h).$$

If $f \in \mathcal{H}$, then

$$\sum_{h \in \mathcal{H}'} b(h) = -b(f) + \sum_{h \in \mathcal{H}} b(h) = b(f) + \sum_{h \in \mathcal{H}} b(h)$$

since $-b(f) = b(f)$. Thus in either case,

$$\sum_{h \in \mathcal{H}'} b(h) = b(f) + \sum_{h \in \mathcal{H}} b(h) = b(w) + 2 \sum_{h \in \mathcal{H}} b(h) = b(w).$$

The second prisoner should calculate the sum on the above left; this provides the binary expansion of the number of the square that the warden identified.