Thirty-Sixth
University of Michigan
Undergraduate Mathematics Competition
April 13, 2019

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Evaluate the series

$$\frac{1}{2\sqrt{3}+3\sqrt{2}}+\frac{1}{3\sqrt{4}+4\sqrt{3}}+\frac{1}{4\sqrt{5}+5\sqrt{4}}+\cdots$$

Problem 2. Show that 3^n, with $n \geq 3$ an integer, cannot have only odd digits in its decimal representation.

Problem 3. Suppose that a, b, c are positive real numbers with

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \leq 5.$$

What is the largest possible value of $\frac{a}{b}$?

Problem 4. In a magical isosceles triangle $\triangle ABC$ we have $|AC| = |BC|$. Let D be the midpoint between A and B. The inscribed circle of ABC intersects the line segment CD in a point E that is in the interior of the triangle. Suppose that $|AB| = 15$ and $|CE| = 8$. Determine $|AC|$.

Problem 5. We have a set $\mathcal{D}$ of N Associate Deans, and we form m different committees whose members are in $\mathcal{D}$, with the cardinalities of the committees being $k_1, k_2, \ldots, k_m$, and no committee is a subcommittee of any other. Show that

$$\sum_{i=1}^{m} k_i!(N-k_i)! \ \leq \ N!\,.$$

Problem 6. Suppose that $a_1, a_2, \ldots, a_{2n+1}$ are real numbers such that when any one of the is removed, the remaining $2n$ of them can be partitioned into two collections of n terms with equal sums. Show that the a_i must all be equal.

Problem 7. For a given positive real number r and real x, y let $N_r(x, y)$ be the number of pairs of integers (m, n) satisfying $(x-m)^2 + (y-n)^2 \leq r^2$. Evaluate

$$\int_0^1 \int_0^1 N_r(x, y)\, dx\, dy$$

as a function of r.

Problem 8. Let a and b be relatively prime positive integers, and let $\mathcal{S}$ be the set of those nonnegative integers n that can be written $n = ua + vb$ where u and v are nonnegative integers. Show that

$$\sum_{n \in \mathcal{S}} z^n = \frac{1 - z^{ab}}{(1 - z^a)(1 - z^b)}$$

for $|z| < 1$.

Problem 9. Let f be a continuously differentiable map from the unit interval $I = [0, 1]$ to the unit square $J = [0, 1] \times [0, 1]$, and suppose that the boundary ∂J is in the image of f. Prove that there exist $0 \leq s < t \leq 1$ such that $f(s) = f(t)$, and the arc length of $f|_{[s,t]}$ is greater or equal to 2.

Problem 10. Two prisoners must play the following game to save their lives. They learn the rule of the game, can work on strategy, but once the game starts, they are unable to communicate. The first prisoner is taken into a room, which contains a chessboard. Each of the 64 fields of the chessboard has a coin, showing either heads or tails. The warden points to one of the coins. The first prisoner turns over exactly one coin (which may or may not be the one that the warden pointed to). The first prisoner is taken to his/her solitary cell. The second prisoner is led in. Upon examining the coins on the board, the second prisoner must identify the coin that the warden pointed to. Describe a strategy by which this can be done.

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