1. If $f[a,b] \to \mathbb{R}$ is a continuous function, prove that

$$6\int_a^b \int_a^z \int_a^y f(x)f(y)f(z)\,dx\,dy\,dz = \left(\int_a^b f(x)\,dx\right)^3$$

in the following steps:

a) First, observe that the domain of integration of the iterated integral is $a \leq x \leq y \leq z \leq b$.

b) Second, argue that the value of the iterated integral is the same for every domain $a \leq x_1 \leq x_2 \leq x_3 \leq b$ where x_1, x_2, x_3 is some permutation of x, y, z.

c) Deduce the formula given by the right hand side.

$x \in [a, y]$
$y \in [a, z]$
$z \in [a, b]$

combine to give $a \leq x \leq y \leq z \leq b$.

The value of the integral is the same over the domain

$$a \leq x_1 \leq x_2 \leq x_3 \leq b$$

with x_1, x_2, x_3 being any permutation of x, y, z because $f(x)f(y)f(z)$ is unchanged when x, y, z are permuted.

There are 6 domains $a \leq x_1 \leq x_2 \leq x_3 \leq b$ corrs to 6 permutations. Their union is

$$a \leq x, y, z \leq b.$$

Therefore $6\int_a^b \int_a^z \int_a^y f(x)f(y)f(z)\,dx\,dy\,dz$

$$= \left(\int_a^b f(x)\,dx\right)^3.$$

2. Let $f_n : [0,1] \to \mathbb{R}$, $n = 1, 2, \ldots$, be continuous functions such that for every $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ with

$$|f_m(x) - f_n(x)| < \epsilon$$

for all $m, n \geq N$. You may assume the well-known result that there exists a continuous function f which is the uniform limit of f_n as $n \to \infty$.
Suppose in addition that f_n are continuously differentiable and that for every $\epsilon > 0$ there exists an N such that

$$|f'_m(x) - f'_n(x)| < \epsilon$$

for all $m, n \geq N$.

a) Prove that f is continuously differentiable.

b) Prove that f' is the uniform limit of f'_n as $n \to \infty$.

f is the uniform limit of $f_1, f_2, \ldots$

Similarly, let g be the uniform limit of $f'_1, f'_2, \ldots$

$$\frac{f(x+h) - f(x)}{h} = \lim_{n \to \infty} \frac{f_n(x+h) - f_n(x)}{h}$$

with $h > 0$ if $x = 0$
$h < 0$ if $x = 1$
$h \neq 0$ otherwise

$$= \lim_{n \to \infty} f'_n(x + \theta h) \text{ for some } \theta \in [0,1]$$

by the mean value thm
θ may depend on n, x, h.

For sufficiently large n, $|f'_n(x + \theta_n h) - g(x + \theta_n h)| < \epsilon$.

Thus $\lim_{n \to \infty} f'_n(x + \theta_n h) = g(x + \theta h)$ for some θ.
if not there will be a contra.
(see next page)

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} g(x + \theta h) = g(x) = \frac{df}{dx}.$$

why is $\lim_{n\to\infty} f_n'(x+\theta_n h) = g(x+\theta h)$ for some $\theta \in [0,1]$?

$$\lim_{n\to\infty} f_n'(x+\theta_n h) = \frac{f(x+h)-f(x)}{h}.$$

Thus the limit exists.

$\{g(x+\theta h) \mid 0 \le \theta \le 1\}$ is a compact set.

If the limit is not in this compact set,

dist $f_n'(x+\theta_n h)$ from compact set

$> \epsilon$ for some $\epsilon > 0$ and n sufficiently large.

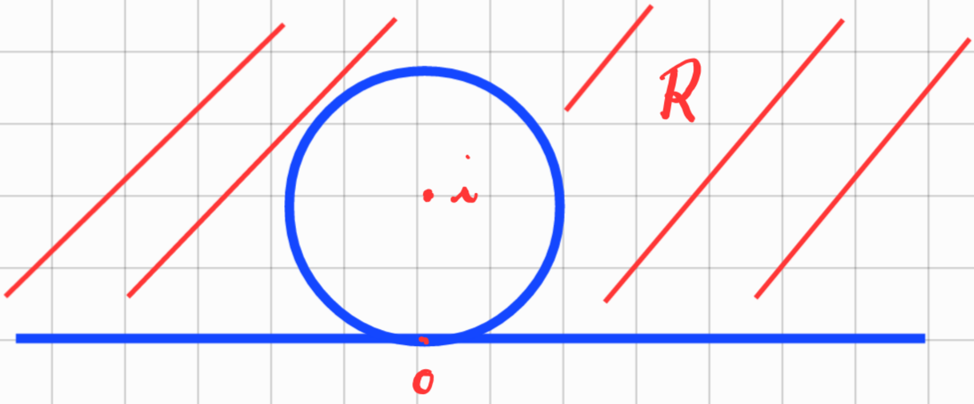
But that would contradict the unif of cvg of f_n' to g.

3. Let R be the region in $\mathbb{C}$ given by

$$\{z|\operatorname{Im} z > 0\} \cap \{z||z - i| > 1\}.$$

a) Sketch the image of R under the map $w = \frac{1}{z}$.

b) Find a conformal map from R to the region $-1 < \operatorname{Im} z < 1$.

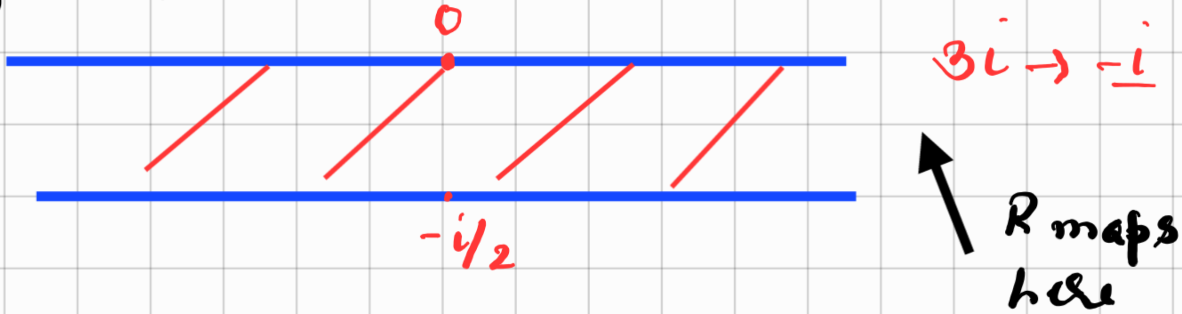


$0 \to \infty$ under $w = 1/z$

Thus both $|z - i| = 1$ and $\operatorname{Im} z = 0$ map to lines.

The lines must be parallel because ∞ is their only point of intersection.

The real line maps to the real line, obviously. $2i \to -\frac{i}{2}$. Thus $|z-i| = 1$ must map to a horiz line through $-i/2$.



$w = \frac{4}{z} + i$ maps to $-1 < \operatorname{Im} z < 1$.

4. Let $p(z) = 1 - 4z^2 + z^n$ with $n > 2$ being an integer. How many roots of $p(z) = 0$ lie in the region

$$1 - \epsilon < |z| < 1 + \epsilon$$

for $\epsilon = \frac{1}{100}$ (or some other small number) and in the limit $n \to \infty$? Explain your answer.

On the circle $|z| = 1 + \epsilon$, $|z^n| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Thus $|z^n| > |1 - 4z^2|$ for n large.

By Rouché's thm, all n roots of $p(z) = 0$ lie inside $|z| < 1 + \epsilon$.

$|z^n| \to 0$ as $n \to \infty$ on the circle $|z| = 1 - \epsilon$.

$|1 - 4z^2| \geq |4z^2| - 1 = 4(1-\epsilon)^2 - 1 > 2$ for ϵ small.

Again by Rouché's thm two roots of $p(z) = 0$ lie in $|z| < 1 - \epsilon$.

Thus $n-2$ roots will lie in

$$1 - \epsilon < |z| < 1 + \epsilon$$

for n large.

5. The Bernoulli polynomials $\phi_n(x)$ are defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{\phi_n(x)}{n!} z^n.$$

a) Verify that $\phi_0(x) = 1$ and $\phi_1(x) = x - \frac{1}{2}$.

b) If n is a positive integer and x is a real number, find the residue of

$$\frac{e^{2\pi i z x}}{z^{2n}} \times \frac{1}{\sin \pi z}$$

$$\frac{z e^{xz}}{e^z - 1} = \frac{z\left(1 + xz + \cdots\right)}{\left(1 + z + \frac{z^2}{2} + \cdots - 1\right)}$$

$$= \frac{1 + xz + \cdots}{1 + \frac{z}{2} + \cdots}$$

$$= \left(1 - \frac{z}{2} + \cdots\right)\left(1 + xz + \cdots\right)$$

$$= 1 + \left(x - \frac{1}{2}\right) z + \cdots$$

Therefore $\varphi_0(x) = 1$, $\varphi_1(x) = x - \frac{1}{2}$.

$$\operatorname{Res}\left(\frac{e^{2\pi i z x}}{z^{2n}} \cdot \frac{1}{\sin \pi z} ; z = 0\right)$$

$$= \operatorname{Res}\left(\frac{1}{z^{2n}} \cdot \frac{e^{2\pi i z x} \cdot 2i}{e^{i\pi z} - e^{-i\pi z}} ; z = 0\right)$$

$$= 2i \cdot \operatorname{Res}\left(\frac{e^{i\pi z(2x+1)}}{z^{2n}\left(e^{2\pi i z} - 1\right)} ; z = 0\right)$$

$$= 2i \cdot \operatorname{Res}\left(\frac{z e^{i\pi z(2x+1)}}{z^{2n+1}\left(e^{2\pi i z}-1\right)} ; z=0 \right)$$

$$= 2i \cdot \text{Coeff of } z^{2n} \text{ in } \frac{z e^{i\pi z(2x+1)}}{e^{2\pi i z}-1}$$

$$= 2i \cdot \text{coeff of } z^{2n} \text{ in } \frac{z e^{2\pi i z\left(x+\frac{1}{2}\right)}}{e^{2\pi i z}-1}$$

$$= 2i \cdot \frac{1}{(2\pi i)} \cdot (2\pi i)^{2n} \cdot \text{Coeff of } w^{2n} \text{ in } \frac{w e^{w(x+1/2)}}{e^{w}-1}$$

after sub $w = 2\pi i z$

$$= \frac{1}{\pi} \cdot (-1)^{n} \cdot (2\pi)^{2n} \cdot \varphi_{2n}\left(x+\frac{1}{2}\right) \cdot \frac{1}{(2n)!} .$$