AIM Qualifying Exam: Advanced Calculus and Complex Variables

August 2023

There are five (5) problems in this test, each worth 20 points.

For the most part, there is sufficient room in this booklet for all your work. However, if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

1. If $f[a,b] \to \mathbb{R}$ is a continuous function, prove that

$$6\int_a^b \int_a^z \int_a^y f(x)f(y)f(z)\,dx\,dy\,dz = \left(\int_a^b f(x)\,dx\right)^3$$

in the following steps:

a) First, observe that the domain of integration of the iterated integral is $a \leq x \leq y \leq z \leq b$.

b) Second, argue that the value of the iterated integral is the same for every domain $a \leq x_1 \leq x_2 \leq x_3 \leq b$ where x_1, x_2, x_3 is some permutation of x, y, z.

c) Deduce the formula given by the right hand side.

2. Let $f_n : [0,1] \to \mathbb{R}$, $n = 1, 2, \ldots$, be continuous functions such that for every $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ with

$$|f_m(x) - f_n(x)| < \epsilon$$

for all $m, n \geq N$. You may assume the well-known result that there exists a continuous function f which is the uniform limit of f_n as $n \to \infty$.
Suppose in addition that f_n are continuously differentiable and that for every $\epsilon > 0$ there exists an N such that

$$|\frac{df_m}{dx}(x) - \frac{df_n}{dx}(x)| < \epsilon$$

for all $m, n \geq N$.

a) Prove that f is continuously differentiable.

b) Prove that $\frac{df}{dx}$ is the uniform limit of $\frac{df_n}{dx}$ as $n \to \infty$.

3. Let R be the region in $\mathbb{C}$ given by

$$\{z|\text{Im}z > 0\} \cap \{z||z-i| > 1\}.$$

a) Sketch the image of R under the map $w = \frac{1}{z}$.

b) Find a conformal map from R to the region $-1 < \text{Im}z < 1$..

4. Let $p(z) = 1 - 4z^2 + z^n$ with $n > 2$ being an integer. How many roots of $p(z) = 0$ lie in the region

$$1 - \epsilon < |z| < 1 + \epsilon$$

for $\epsilon = \frac{1}{100}$ (or some other small number) and in the limit $n \to \infty$? Explain your answer.

5. The Bernoulli polynomials $\phi_n(x)$ are defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{\phi_n(x)}{n!} z^n.$$

a) Verify that $\phi_0(x) = 1$ and $\phi_1(x) = x - \frac{1}{2}$.

b) If n is a positive integer and x is a real number, find the residue of

$$\frac{e^{2\pi izx}}{z^{2n}} \times \frac{1}{\sin \pi z}$$

at $z = 0$ in terms of a Bernoulli polynomial.