AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2023

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Consider the set of real 2-by-2 matrices $\mathbf{A}$ such that $\mathbf{A} = \mathbf{A}^T$.

(a) Let $\mathbf{x}$ and $\mathbf{y}$ be independent eigenvectors of $\mathbf{A}$, i.e. $\mathbf{x} \neq \alpha\mathbf{y}$ for any scalar α. For which $\mathbf{A}$ in the set mentioned above is $\mathbf{x}^T\mathbf{y}$ always zero, and for which $\mathbf{A}$ in the set could $\mathbf{x}^T\mathbf{y}$ be nonzero? Justify your answer and give an example of a matrix for each of the two cases.

(b) Show that any $\mathbf{A}$ in the set can be written $\mathbf{x}_1\mathbf{x}_1^T + \mathbf{x}_2\mathbf{x}_2^T$ for some $\mathbf{x}_1$ and $\mathbf{x}_2 \in \mathbb{C}^2$.

Problem 1

Problem 1

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Problem 2

Find a basis for each of these subspaces of $\mathbb{R}^4$ (or $\mathbb{R}^2$ and $\mathbb{R}^5$ in part d). Justify your answers.

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1,1,0,0)^T$ and $(1,0,1,1)^T$.

(d) The column space (in $\mathbb{R}^2$) and the null space (in $\mathbb{R}^5$) of $\mathbf{U} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

Problem 2

Problem 2

Problem 2

Problem 3

Consider the system of ODEs

$$dx/dt = (1+x)\sin y \;\; ; \;\; dy/dt = 1 - x - \cos y. \tag{1}$$

(a) Determine all critical points.

(b) Find the corresponding linear system near each critical point.

(c) Find the eigenvalues of each linear system. What conclusions can you then draw about the stability of the nonlinear system near each critical point?

Problem 3

Problem 3

Problem 3

Problem 4

Consider the initial value problem

$$y^{(4)} + 2y'' + y = g(t),\ y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 0.$$

(a) What is the most general class of functions $g(t)$ that guarantees the solution exists for all real t?

(b) Solve the initial value problem in the special case $g(t) = 3t + 4$.

Problem 4

Problem 4

Problem 4

Problem 5

Solve the PDE

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

for $u(x,t)$ in the domain $\{t > 0\ ;\ 0 < x < 1\}$ with the boundary conditions and initial conditions:

$$u(0,t) = 0\ ,\ u(1,t) = 1$$
$$u(x,0) = x + \sin(\pi x)\ ,\ \frac{\partial u}{\partial t}(x,0) = \sin(\pi x).$$

Problem 5

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