Department of Mathematics, University of Michigan
Complex Analysis Qualifying Exam
August 15, 2023; Morning Session

Problem 1: Let f be an analytic function in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $f(0) = 0$ and $|f(z)| < 2023$ for all $z \in \mathbb{D}$. Assume also that f satisfies the property $f(iz) = f(z)$ for all $z \in \mathbb{D}$. Prove that $|f(\frac{1}{7})| < 1$.

Solution: Consider the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of the function f; note that $a_0 = 0$ as $f(0) = 0$. The property $f(iz) = f(z)$ implies $i^n f^{(n)}(0) = f^{(n)}(0)$ and hence $a_n = (n!)^{-1} f^{(n)}(0) = 0$ unless n is a multiple of 4. Therefore, one can write $f(z) = g(z^4)$, where $g(z) = \sum_{n=1}^{\infty} a_{4n} z^n$. The function g is analytic in the unit disc $\mathbb{D}$ and satisfies $|g(z)| < 2023$ for all $z \in \mathbb{D}$, as well as $g(0) = 0$. Thus, Schwarz–Pick's lemma gives the desired estimate

$$\left| f\left(\frac{1}{7}\right)\right| = \left| g\left(\frac{1}{7^4}\right)\right| \le \frac{2023}{7^4} < \frac{2023}{45^2} < 1\,.$$

Problem 2: Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ be the upper half-plane. Find a conformal mapping from the domain

$$\mathbb{H} \smallsetminus \{\, z \in \mathbb{H} : \; z = e^{i\theta}, \; \theta \in (0, \tfrac{\pi}{2}]\,\}$$

(i.e., $\mathbb{H}$ slit along a circular arc) back onto $\mathbb{H}$. You may write your solution as a composition of simpler maps.

Solution: First, consider the linear-fractional transform $f_1(z) = (z-1)/(z+1)$, which maps $\mathbb{R}$ to $\mathbb{R}$, the point $+1$ to 0, the point -1 to ∞, and the point $i = e^{i\frac{\pi}{2}}$ to i. The image of the domain in question under f_1 is $\mathbb{H} \smallsetminus \{z \in \mathbb{H} : \Re z = 0, \Im z \le 1\}$; the upper half plane with a straight vertical cut. Next, consider the mapping $f_2(z) = z^2$, which conformally maps this half-plane with a vertical cut onto $\mathbb{C} \smallsetminus [-1, +\infty)$; the full plane cut along a ray. Finally, set $f_3(z) = \sqrt{z+1}$ and consider $f_3 \circ f_2 \circ f_1$.

Problem 3: Use contour integration to evaluate the integral

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \cdot \frac{dx}{1+x^2}\,.$$

[*Simplification:* If you experience difficulties, you can first change the variable of integration to $t = (1+x)/(1-x)$ and use contour integration for the new integral.]

Solution: We use the calculus of residues for the function $f(z) = \sqrt{\dfrac{1+z}{1-z}} \cdot \dfrac{1}{(1+z^2)}$ defined for $z \in \mathbb{C} \smallsetminus [-1, 1]$, where the branch of the square root is chosen so that $\lim_{y \to 0+} f(x+iy) > 0$ for $x \in (-1, 1)$. (Note that f is single-valued in the domain $\mathbb{C} \smallsetminus [-1, 1]$.) Given (small) $\varepsilon > 0$, consider the contour γ_ε that consists of the segment $S_\varepsilon^+ := [-1 + i\varepsilon, 1 + i\varepsilon]$ (oriented from left to right), a half-circle of radius ε around the point $+1$ (oriented clockwise), the segment $S_\varepsilon^- := [1 - i\varepsilon, -1 - i\varepsilon]$ (oriented from right to left) and a half-circle around the point -1 (oriented clockwise). Also, let $R > 2$ be big enough and Γ_R denote the circle of radius R centered at the origin, oriented counterclockwise. Cauchy's residue theorem gives

$$\int_{\gamma_\varepsilon} f(z)dz + \int_{\Gamma_R} f(z)dz \;=\; 2\pi i \cdot (\operatorname{res}(f, i) + \operatorname{res}(f, -i)).$$

It is clear that $|f(z)| = O(R^{-2})$ for $z \in \Gamma_R$. Hence, $\int_{\Gamma_R} f(z)dz = O(R^{-1})$ and one has (e.g., by considering the limit $R \to +\infty$) the equality

$$\int_{\gamma_\varepsilon} f(z)dz \;=\; 2\pi i \cdot (\mathrm{res}(f,i) + \mathrm{res}(f,-i)).$$

Also, $|f(z)| = O(|z-1|^{-1/2})$ near the point $+1$ and $|f(z)| = O(|z+1|^{1/2})$ near the point -1, which means that $\int_{z:|z\pm 1|=\varepsilon} |f(z)||dz| = O(\varepsilon^{1/2})$ as $\varepsilon \to 0$. Further,

$$\lim_{\varepsilon\downarrow 0} \int_{S_\varepsilon^+} f(z)dz \;=\; \lim_{\varepsilon\downarrow 0} \int_{S_\varepsilon^-} f(z)dz \;=\; \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \cdot \frac{dx}{1+x^2}$$

due to the uniform integrability in neighborhoods of the points ± 1. (The same sign for S_ε^- is the result of the compensation of two minuses: the first comes from the opposite value of the square root, the second from the right-to-left orientation of the segment.) Therefore,

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \cdot \frac{dx}{1+x^2} \;=\; \pi i \cdot (\mathrm{res}(f,i) + \mathrm{res}(f,-i))$$

and it remains to compute the residues of the function f at its (simple) poles $\pm i$. Since $(1\pm i)/(1\mp i) = \pm i$, we have

$$\mathrm{res}(f,i) = \pm e^{i\frac{\pi}{4}}/(2i) \quad \text{and} \quad \mathrm{res}(f,-i) = \pm e^{-i\frac{\pi}{4}}/(-2i)\,.$$

In order to avoid a careful consideration of the signs of the square roots one can use the fact that the answer must be purely real and positive, namely

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \cdot \frac{dx}{1+x^2} \;=\; \frac{\pi}{2} \cdot (e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}) = \frac{\pi}{\sqrt{2}}\,.$$

Problem 4: Let $\alpha \in \mathbb{C}$ satisfy $|\alpha| = 1$. Consider the equation $\sin z = \dfrac{\alpha}{z^2}$ for $z \in \mathbb{C}$.

(a) Prove that for each $k \in \mathbb{Z} \smallsetminus \{0\}$ this equation has exactly one solution inside the vertical strip $|\Re z - \pi k| < \frac{\pi}{2}$.

(b) How many solutions (counted with multiplicities) does this equation have inside the vertical strip $|\Re z| < \frac{\pi}{2}$?

Solution: (a) It is easy to see that $|\sin z| = \frac{1}{2}(e^{\Im z} + e^{-\Im z}) \geq 1$ if $\Re z = \frac{\pi}{2} + \pi k$. Moreover, $|\sin z| \geq \frac{1}{2}(e^{|\Im z|} - e^{-|\Im z|}) \geq 1$ if $|\Im z|$ is large enough. Therefore, Rouche's theorem applied in each rectangle $[-\frac{\pi}{2}+\pi k; \frac{\pi}{2}+\pi k] \times [-C, C]$ with $k \neq 0$ implies that the function $\sin z - \alpha z^{-2}$ has the same number of roots (counted with multiplicities) inside this rectangle as the function $\sin z$. As all roots of the function $\sin z$ are simple and located at points πk, $k \in \mathbb{Z}$, this proves the claim by sending $C \to +\infty$.

(b) A similar reasoning can be applied in rectangles $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-C, C]$ to the functions $z^2 \sin z - \alpha$ and $z^2 \sin z$. Since $|z^2 \sin z| \geq |z^2| > 1 = |\alpha|$ on the boundary of this rectangle (provided that C is chosen large enough), the entire function $z^2 \sin z - \alpha$ has exactly *three* roots (counted with multiplicity) inside such a rectangle (for all C large enough) and hence in the full vertical strip $|\Re z| < \frac{\pi}{2}$. As $z = 0$ cannot be a root, the same is true for the equation $\sin z = \alpha z^{-2}$.

Problem 5: Let $a_k \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for all $k \in \mathbb{N}$. Consider functions

$$B_n(z) := \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z}, \quad z \in \mathbb{D}.$$

(a) Prove that the sequence $\{B_n\}_{n=1}^{\infty}$ contains a subsequence that converges uniformly on compact subsets of the unit disc $\mathbb{D}$.

(b) Assume that $\limsup_{n\to\infty}(1 - |a_n|) > 0$. Prove that each subsequential limit of the functions B_n is identically zero in $\mathbb{D}$.

(c) Prove that the same holds if $\sum_{n=1}^{\infty}(1 - |a_n|) = +\infty$.

Solution: (a) Each factor is a linear-fractional transform of the unit disc $\mathbb{D}$ onto itself. In particular, $|B_n(z)| = 1$ if $|z| = 1$ and the functions B_n are uniformly bounded inside $\mathbb{D}$. Montel's theorem says that families of uniformly bounded holomorphic functions are normal and thus the sequence $(B_n)_{n=1}^{\infty}$ has a locally uniformly convergent subsequence.

(b) Assume that $\limsup_{n\to\infty}(1-|a_n|) = \varepsilon > 0$. Then, the number of zeros (counted with multiplicity) of B_n inside the disc $(1-\frac{1}{2}\varepsilon)\mathbb{D}$ grows to infinity as $n \to \infty$. On the other hand, if there existed a non-trivial subsequential limit B of B_n, then B would have only finitely many isolated zeros of finite order in $(1 - \frac{1}{2}\varepsilon)\mathbb{D}$, a contradiction. (Note that each point a that appears at least m times in the sequence $(a_n)_{n=1}^{\infty}$ must be a zero of B of order at least m since a is a zero of order m of each B_n with n large enough and the convergence of B_n to B holds for all derivatives.)

(c) First, note that $|B_n(0)| = \prod_{k=1}^{n} |a_k| \to 0$ as $n \to \infty$ since $\sum_{k=1}^{\infty}(1-|a_k|) = +\infty$. Our goal is to prove a similar claim for other z's in $\mathbb{D}$. It is not hard to see that

$$\left|\frac{z - a_n}{1 - \bar{a}_n z} + a_n\right| \;=\; \frac{|z| \cdot (1 - |a_n|^2)}{|1 - \bar{a}_n z|} \;\le\; \frac{1 - |a_n|}{2} \quad \text{if} \quad |z| \le \frac{1}{5}.$$

Therefore,

$$|B_n(z)| \;=\; \prod_{k=1}^{n}\left|\frac{z - a_n}{1 - \bar{a}_n z}\right| \;\le\; \prod_{k=1}^{n} \frac{1 + |a_n|}{2} \underset{n\to\infty}{\to} 0 \quad \text{if} \;\; |z| \le \frac{1}{5}$$

(since $\sum_{k=1}^{\infty}(1 - \frac{1}{2}(1 + |a_k|)) = \frac{1}{2}\sum_{k=1}^{\infty}(1 - |a_k|) = +\infty$). It follows from this estimate that each subsequential limit B of B_n must identically vanish at least in the disc $\frac{1}{5}\mathbb{D}$. Since B is an analytic function in $\mathbb{D}$, it then necessarily vanishes everywhere in the unit disc.