Problem 1: Let $f$ be an analytic function in the unit disc $D = \{ z \in \mathbb{C} : |z| < 1 \}$ such that $f(0) = 0$ and $|f(z)| < 2023$ for all $z \in D$. Assume also that $f$ satisfies the property $f(iz) = f(z)$ for all $z \in D$. Prove that $|f(\frac{i}{2})| < 1$.

Problem 2: Let $\mathbb{H} = \{ z \in \mathbb{C} : \Im z > 0 \}$ be the upper half-plane. Find a conformal mapping from the domain $\mathbb{H} \setminus \{ z \in \mathbb{H} : z = e^{i\theta}, \theta \in (0, \frac{\pi}{2}) \}$ (i.e., $\mathbb{H}$ slit along a circular arc) back onto $\mathbb{H}$. You may write your solution as a composition of simpler maps.

Problem 3: Use contour integration to evaluate the integral
$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \cdot \frac{dx}{1+x^2}.$$  
[Simplification: If you experience difficulties, you can first change the variable of integration to $t = (1+x)/(1-x)$ and use contour integration for the new integral.]

Problem 4: Let $\alpha \in \mathbb{C}$ satisfy $|\alpha| = 1$. Consider the equation $\sin z = \alpha z^2$ for $z \in \mathbb{C}$.
(a) Prove that for each $k \in \mathbb{Z} \setminus \{0\}$ this equation has exactly one solution inside the vertical strip $|\Re z - \pi k| < \frac{\pi}{2}$.
(b) How many solutions (counted with multiplicities) does this equation have inside the vertical strip $|\Re z| < \frac{\pi}{2}$?

Problem 5: Let $a_k \in D = \{ z \in \mathbb{C} : |z| < 1 \}$ for all $k \in \mathbb{N}$. Consider functions
$$B_n(z) := \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z}, \quad z \in D.$$  
(a) Prove that the sequence $\{B_n\}_{n=1}^{\infty}$ contains a subsequence that converges uniformly on compact subsets of the unit disc $D$.
(b) Assume that $\limsup_{n \to \infty} (1 - |a_n|) > 0$. Prove that each subsequential limit of the functions $B_n$ is identically zero in $D$.
(c) Prove that the same holds if $\sum_{n=1}^{\infty} (1 - |a_n|) = +\infty$. 