1. The cone $CX$ on a topological space $X$ is the quotient of $X \times [0,1]$ by identifying all the points $(x, 1)$ to a single point $\ast$. Suppose that there exists an open neighborhood $U$ of the point $\ast$ in $CX$ which is homeomorphic to $\mathbb{R}^n$. What are the possible sequences of groups $H_0(X), H_1(X), H_2(X), \ldots$?

**Solution:** For $n = 0$, clearly, $X$ has to be empty. For $n > 0$, we have

$$H_k(U, U \setminus \{\ast\}) \cong H_k(CX, CX \setminus \{\ast\}) \cong \tilde{H}_{k-1}(X)$$

(since $CX \setminus \{\ast\} \simeq X$). The left-hand side is $\mathbb{Z}$ for $k = n$ and 0 else. Thus, $X$ has to have the homology of the $(n-1)$-sphere. So the only possible sequence for $n = 0$ is

$$0, 0, 0, \ldots,$$

for $n = 1$, it is

$$\mathbb{Z} \oplus \mathbb{Z}, 0, 0 \ldots,$$

for $n = 2$, it i

$$\mathbb{Z}, \mathbb{Z}, 0, \ldots,$$

etc.

2. Let $F_n$ be the free group on $n$ generators. For which $n = 1, 2, \ldots$ does there exist an injective homomorphism $h : F_n \to F_2$? For which $n = 1, 2, \ldots$ can $h$ be chosen so the image has finite index in $F_2$?

**Solution:** Yes on both counts for $n \geq 2$. We need to find a covering space with fundamental group $F_n$ of the graph with a single vertex and two edges. Any $(n-1)$-fold connected covering would do. For $n = 1$, the subgroup clearly exists, but not of finite index (since a covering has to have degree $> 0$).

3. Let $X$ be a path-connected space with basepoint $\ast$. Let $x \neq y \in X$ be points, and let $Y = X/x \sim y$.

(a) Is the homomorphism $p_* : \pi_1(X, \ast) \to \pi_1(Y, \ast)$ induced by the projection $p : X \to Y$ necessarily injective?
(b) Can $p_*$ be an isomorphism?

**Solution:** One proves that $\pi_1$ remains the same if we replace $Y$ by the space $Y'$ obtained by attaching $[0, 1]$ by identifying $0, 1$ with $x, y$, respectively. Then $\pi_1(Y, \ast)$ is the free product of $\pi_1(X, \ast)$ with $\mathbb{Z}$ where $p_*$ is the injection to the first factor. Thus, $p_*$ is always injective, and never an isomorphism.

4. Describe all the homotopy equivalence classes of CW-complexes which have exactly three cells in dimension $0, 1, n$, respectively, for a given $n > 1$.

**Solution:** The 1-skeleton is $S^1$. For $n = 2$, the possibly different homotopy types arise by different choices of the homotopy class of the 2-cell attaching map, which is classified by degree $\in \mathbb{Z}$. However, degree $n \in \mathbb{Z}$ is identified with $-n$ by reversing the orientation of the cell. Thus, we get different homotopy equivalence classes for attaching maps of degrees $0, 1, 2, \ldots$ (which are proved different using the first homology group).

For $n > 2$, the attaching map is $S^{n-1} \to S^1$, which is homotopic to a constant map (since it lifts to the universal cover of $S^1$, which is contractible). Thus, there is only one class.

5. Compute the homology of the space $\mathbb{R}P^3/\mathbb{R}P^1$ (where $\mathbb{R}P^1$ is embedded in the standard way).

**Solution:** By using the long exact sequence in homology, we get homology groups $\mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}$. (One can also see directly that the quotient space is homotopy equivalent to $S^2 \vee S^3$.)