Algebra 2 QR — August 2023

Problem 1. Let

$$1 \to A \xrightarrow{\alpha} G \xrightarrow{\beta} B \to 1$$

be a short exact sequence of groups, with A and B abelian. Suppose that $\alpha(A)$ is central in G, and let h be an element of G. Show that $g \mapsto hgh^{-1}g^{-1}$ is a group homomorphism from G to G.

Problem 2. Let r, s and t be positive integers, and let G be the group generated by elements a and b modulo the relations $a^r = b^s = 1$, $aba^{-1} = b^t$. Show that G is finite.

Problem 3. Let G be a group of order $4 \cdot 3^n$. Show that G is solvable.

Problem 4. Let Ω/F be a field extension, let E_1 and E_2 be distinct subfields of Ω containing F with $[E_1 : F] = [E_2 : F] = d$, and let K be the subfield of Ω generated by E_1 and E_2. Show that $2d \leq [K : F] \leq d^2$, and give examples where the extreme values $2d$ and d^2 each occur.

Problem 5. Let p be an odd prime. Let K be a subfield of $\mathbb{C}$ that is Galois over $\mathbb{Q}$ of degree p^n. Show that $K \subset \mathbb{R}$.