Algebra 1 QR — August 2023

Problem 1. Let $\mathbb{F}_p$ be the field with p elements and let U be a two dimensional vector space over $\mathbb{F}_p$. How many distinct tensors of the form $u \otimes v$ are there in $U \otimes U$?

Problem 2. Let V be a real vector space of finite dimension n and let $\langle\ ,\ \rangle$ be a non-degenerate symmetric bilinear form on V. Suppose that there is a basis e_1, e_2, $\dots$, e_n of V such that $\langle e_i, e_i \rangle$ is positive for all $1 \leq i \leq n$. What are the possible signatures of $\langle\ ,\ \rangle$?

Problem 3. Let $\mathbb{Z}[i]$ be the subring of the complex numbers generated by i (a square root of -1). Up to isomorphism, how many $\mathbb{Z}[i]$-modules are there with 25 elements? You may use without proof that $\mathbb{Z}[i]$ is a principal ideal domain (PID), and we helpfully point out that $5 = (2+i)(2-i)$ is the prime factorization of 5.

Problem 4. Let A be an $n \times n$ matrix of complex numbers and suppose that the characteristic polynomial of A is $(t-1)^k t^{n-k}$. Show that there is a polynomial $f(x)$ in $\mathbb{C}[x]$ such that $f(A)$ also has characteristic polynomial $(t-1)^k t^{n-k}$, and $f(A)^2 = f(A)$.

Problem 5. Let $\mathbb{Q}(x)$ be the field of rational functions in x with coefficients in $\mathbb{Q}$. Let R be the subring of $\mathbb{Q}(x)$ consisting of functions of the form $\frac{f(x)}{g(x)}$ for f, $g \in \mathbb{Z}[x]$ and $g(0) = 1$. Show that the ideal xR is prime, but not maximal, in R.