AIM Qualifying Review Exam: Probability and Discrete Mathematics

August 18, 2023

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Use the pigeonhole principle on the ordinary long division algorithm to show that any rational number $\frac{p}{q}$, where $p < q$ and q does not divide any power of 10, has a decimal expansion that eventually repeats.

Problem 1

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Problem 2

Solve the following recurrence:

$$\begin{cases} h_n &= 4h_{n-1} + 4^n \\ h_0 &= 3. \end{cases}$$

Problem 2

Problem 2

Problem 2

Problem 3

(a) Consider the function

$$f(x) = \begin{cases} C(2x - x^3), & 0 < x < \frac{5}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Could f be a probability density function, possibly changing the value of $f(0)$ and $f(\frac{5}{2})$ if necessary? If so, find C. (Note: we are asking about f, not the cumulative distribution function $F(t) = \int_{-\infty}^{t} f(s)\,ds$.)

(b) Repeat for the following, in which an exponent has been changed from 3 to 2.

$$f(x) = \begin{cases} C(2x - x^2), & 0 < x < \frac{5}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Problem 3

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Problem 3

Problem 4

A player makes a sequence of bets that are independent, and each bet results in the player equally likely to win or to lose \$1. Let W denote the net winnings of a player using the strategy to stop playing immediately after their first win. Find:

(a) $P(W > 0)$

(b) $P(W < 0)$

(c) $E[W]$.

Problem 4

Problem 4

Problem 4

Problem 5

A directed graph with nodes $\{v_1, v_2, \ldots, v_n\}$ is called *ordered* if

- Each arc goes from a node of lower index to higher index. That is, if $v_i \to v_j$ is an arc, then $i < j$.
- Each node except v_n has an out-arc.

The goal in this question is to solve the problem: *Given an ordered graph G, what is the length of the longest path (in number of arcs) from v_1 to v_n?*

(a) Show that the following algorithm will *not* work, by giving a counterexample graph.

```
set w = v_1
set L = 0
while there is an arc out of w
  choose arc w->v_j for minimal j
  set w = v_j
  set L = L + 1
end while
Return L as the length of the longest path
```

(b) Give a algorithm for this problem that runs in time polynomial in n. Hint: Try dynamic programming.

Problem 5

Problem 5

Problem 5