April, 2018

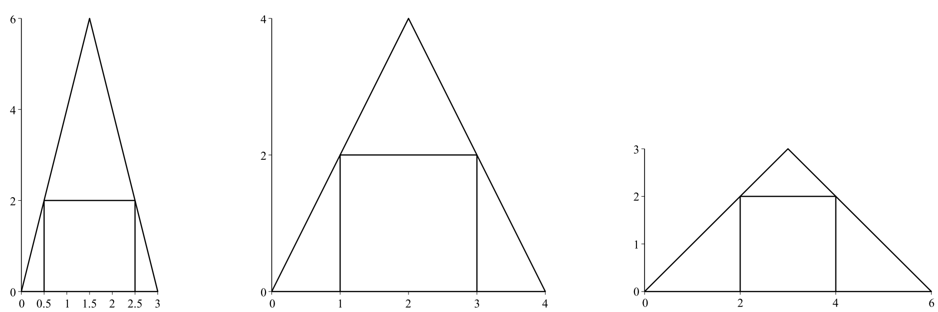
Thirty-Fifth
University of Michigan
Undergraduate Mathematics Competition
SOLUTIONS

1. Let n be the desired number. Then $n + 2$ is divisible by $\operatorname{lcm}[2, 3, 4, 5, 6, 7, 8, 9, 10] = 9 \cdot 8 \cdot 7 \cdot 5 = 2520$. Thus $n = 2518$.

2. We compute the area of the triangle in two ways. On one hand, the area of the triangle is $bh/2$. On the other hand, the trapezoid that encloses the square has parallel sides of lengths 2 and b, and height 2, so its area is $\frac{1}{2}(b + 2) \cdot 2 = b + 2$, and the small triangle that sits on the square has height $h - 2$ and base 2, so its area is $\frac{1}{2}(h - 2) \cdot 2 = h - 2$. On summing these two quantities we see that the area of the large triangle is $b + h$. Thus

$$b + h = \frac{bh}{2}\,.$$

This is symmetric in b and h. If $b \le h$, then the left hand side above is $\le 2h$, which gives $b \le 4$. We try possible values of b. If $b = 1$, then $h = -2$, which is nonsense. If $b = 2$, then $b = 0$, which again is nonsense. If $b = 3$, then $h = 6$. By symmetry we also have $b = 6$, $h = 3$. If $b = 4$, then $h = 4$. To complete the solution, we show that these pairs arise in geometry, not just in the equation displayed above. The configurations are depicted below.



3. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, which is a cube root of 1. Then we have the desired identity when $P_1 = z - \omega$ and $P_2 = z - \overline{\omega}$. To see this, note that

$$(\sqrt{z - \omega} \pm \sqrt{z - \overline{\omega}})^2 = z - \omega \pm 2\sqrt{(z - \omega)(z - \overline{\omega})} + z - \overline{\omega} = 2z + 1 \pm \sqrt{z^2 + z + 1}\,.$$

4. Suppose that all numbers up to and including k are in $\mathcal{A}$, and that $k > 3$. By property (2) we know that $2k \in \mathcal{A}$. Since k, $2k$, and 3 are distinct, it follow that $(k+2k+3)/3 = k+1 \in \mathcal{A}$. By induction it follows that all positive integers are in $\mathcal{A}$. To complete the proof we need to show that $3 \in \mathcal{A}$. We apply property (3) four times: $(1 + 4 + 16)/3 = 7 \in \mathcal{A}$, $(4 + 7 + 16)/3 = 9 \in \mathcal{A}$, $(2 + 7 + 9)/3 = 6 \in \mathcal{A}$, and finally $(1 + 2 + 6)/3 = 3 \in \mathcal{A}$.

5. *First Solution* Once the slot that Michigan is in is known, there is precisely one other slot (of 63) which would result in Ohio State playing Michigan in the first round. The probability of this is 1/63. Similarly, there are 2 places (out of 63) that lead to meeting in the second round, provided that both teams win in the first round. The probability of this is $(2/63) \times 1/2^2 = 1/(2 \times 63)$. In order that the times meet in the third round there are 4 slots that Ohio State must lie in, and then both teams must also win their first 2 games. This has probability $(4/63) \times 1/2^4 = 1/(2^2 \cdot 63)$. Continuing in this way, we see that Ohio State must be in one of 32 slots in order to meet in the sixth (and final) round, and both teams must win their first 5 games. The probability of this is $(2^5/63) \times (1/2^{10}) = 1/(2^5 \cdot 63)$. These are disjoint events, and their union is the event we are concerned with, so the desired probability is the sum $1/63 + 1/(2 \cdot 63) + \cdots + 1/(2^5 \cdot 63) = 1/32$.

   Second Solution Consider a single-elimination tournament involving n teams. For $1 \le i < j \le n$ let $X_{ij} = 1$ if team i plays a gain against team j, $X_{ij} = 0$ otherwise. The teams are randomly permuted, so X_{ij} is a Bernoulli random variable with parameter equal to some number p that is the same for all choices of i, j with $i \ne j$. Let $X = \sum_{1 \le i < j \le n} X_{ij}$ be the total number of games played. Then X is a random variable, but in fact X is constant: $X = n - 1$ because $n - 1$ teams need to be eliminated to determine a winner, and each game eliminates one team. Thus

$$n - 1 = E[X] = E\Big[\sum_{1 \le i < j \le n} X_{ij}\Big] = \sum_{1 \le i < j \le n} E[X_{ij}] = \sum_{1 \le i < j \le n} p = \binom{n}{2} p\,.$$

   Hence $p = 2/n$. When $n = 64$, $p = 1/32$.

6. Let $F(x) = f(x)\sin x + a\cos x$. Then $F(0) = F(2\pi) = a$, so by Rolle's theorem there is a $b \in (0, 2\pi)$ such that $F'(b) = 0$. That is, $f'(b)\cos b + (f(b) - a)\sin b = 0$, so the two vectors in question are orthogonal.

7. *First solution.* Suppose that r is rational, say $r = a/q$. Put $\rho = \cos 2\pi a/q + i \sin 2\pi a/q = e^{2\pi i a/q}$. Then $\rho^q = 1$ so ρ is an algebraic integer. Also, $(1/\rho)^q = 1$, so $1/\rho$ is an algebraic integer. The sum of two algebraic integers is an algebraic integer:

$$\rho + 1/\rho = 2\cos 2\pi r = 6/5\,.$$

   A rational number is an algebraic integer if and only if it is a rational integer, and $6/5 \notin \mathbb{Z}$, so $r \notin \mathbb{Q}$.

   Second solution. If r is rational, then $3/5 + i4/5$ is a root of unity in the Gaussian field $\mathbb{Q}(\sqrt{-1})$. It is known that the roots of unity in this field are precisely $1, i, -1, -i$. Thus r is irrational.

   Third solution. Suppose that $r = a/q$. Then by de Moivre's formula $(3/5 + i4/5)^q = (\cos 2\pi r + i \sin 2\pi r)^q = 1$, which gives

$$(3 + 4i)^q = 5^q\,. \tag{1}$$

   There are (at least) two ways to see that this cannot hold:

(a) The ring of integers in $\mathbb{Q}(\sqrt{-1})$ has unique factorization, the integers are the numbers $a+ib$ with $a, b \in mathbbZ$, $2+i$, $2-i$ are distinct primes, and $3+4i = (2+i)^2$, $5 = (2+i)(2-i)$. The identity (1) reads $(2+i)^{2q} = (2+i)^q(2-i)^q$, which gives $(2+i)^q = (2-i)^q$, a violation of unique factorization.

(b) Suppose that $(a_1+ib_1)(a_2+ib_2) = a_3+ib_3$. Suppose also that $a_1 \equiv a_2 \equiv 3 \pmod 5$ and that $b_1 \equiv b_2 \equiv 4 \pmod 5$. Then $a_3 = a_1a_2-b_1b_2 \equiv 3 \pmod 5$ and $b_3 = a_1b_2+a_2b_1 \equiv 4 \pmod 5$. Thus if $(3+i4)^q = a+ib$, then $a \equiv 3 \pmod 5$ and $b \equiv 4 \pmod 5$. The claim in (1) is that $a \equiv b \equiv 0 \pmod 5$. Contradiction.

8. *First solution.* Let x_m be a point in $[0,1]$ where $|f(x)|$ is minimal, and let x_M be a point in $[0,1]$ where $|f(x)|$ is maximal. Then

$$\big|f(x_M) - f(x_m)\big| = \Big|\int_{x_m}^{x_M} f'(x)\,dx\Big| \le \int_0^1 |f'(x)|\,dx\,.$$

By the triangle inequality we see that $|f(x_M)|-|f(x_m)| \le |f(x_M)-f(x_m)|$. On combining this with the above and rearranging, we find that

$$|f(x_M)| \le |f(x_m)| + \int_0^1 |f'(x)|\,dx\,.$$

But clearly

$$|f(x_m)| \le \int_0^1 |f(x)|\,dx,$$

so we have the stated bound.

Second solution. Suppose that $0 \le x \le 1$. By integration by parts we see that

$$\int_0^x f(u)\,du = xf(x) - \int_0^x f'(u)u\,du,$$

and that

$$\int_x^1 f(u)\,du = (1-x)f(x) - \int_x^1 f'(u)(u-1)\,du\,.$$

We add these identities and rearrange to see that

$$f(x) = \int_0^1 f(u)\,du + \int_0^x f'(u)u\,du + \int_x^1 f'(u)(u-1)\,du\,.$$

The stated inequality now follows by the triangle inequality.

9. Choose v_1, v_2 to be a basis for the kernel of T and v_4, v_5 to be a basis for the kernel of S. Since the kernels meet only in 0, these extend to a basis v_1, v_2, v_3, v_4, v_5 for V, and since the span of v_1, v_2, v_3 does not meet the kernel of S, its image is 3 dimensional, and we may choose a basis w_1, w_2, w_3, w_4, w_5 for W such that $w_i = S(v_i)$ for $i = 1, 2, 3$. Then the matrix A of S is the direct sum of a 3×3 identity matrix and and a 2×2 0 matrix, and the first two columns of the matrix B of T are 0. This does not change of we replace w_4 and w_5 by two other vectors with the same span. If $a \ne 0$, the fact that $aS+T$ has rank 3 implies that the 3×3 submatrix C in the lower right corner B has rank 1 for all a. The 2×2 matrix in the lower right hand corner of C must be 0: if c_{ij} in this corner is nonzero,

the 2 by 2 minor involving c_{ij} and $a+c_{11}$ cannot vanish identically as a function of a. This shows that the fourth and fifth rows of B have nonzero entries only in the third column. Hence, for a suitable new choice of w_4 and w_5, the fifth rows of both of the matrices of S and T are is 0, and both images are therefore contained in the span of w_1, w_2, w_3, w_4.

10. The field $\mathbb{Q}$ is countably infinite. Let the sequence $a_1, a_2, \ldots$ be a list of all rational numbers, each one occurring once, and set $P_n(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$. Now put

$$g(x,y) = \sum_{n=1}^{\infty} P_n(x)P_n(y)\,.$$

Then

$$g(a_{N+},y) = \sum_{n=1}^{N} P_n(a_{N+1})P_n(y) \in \mathbb{Q}[y], \qquad g(x,a_{N+1}) = \sum_{n=1}^{N} P_n(x)P_n(a_{N+1}) \in \mathbb{Q}[x]\,.$$

Moreover, since $P_n(y)$ has exact degree n and $P_N(a_{N+1}) \neq 0$, it follows that $g(a_{N+1}, y)$ has exact degree N. Since this N can be arbitrarily large, it follows that $g(x,y)$ is not a polynomial.