AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 2020

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

(a) Show that for any matrix $\mathbf{A}$, the sum $\sum_{k=1}^{\infty} \frac{\mathbf{A}^k}{k!}$ converges (to a matrix with finite entries).

(b) A certain matrix $\mathbf{B}$ has an eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue 2 and another eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with eigenvalue 4.

Find the entries of $e^{\mathbf{B}}$ explicitly. Here $e^{\mathbf{B}} = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{B}^k}{k!}$, and $\mathbf{I}$ is the 2-by-2 identity matrix.

Problem 1

Problem 1

Problem 1

Problem 2

Consider the system of ODEs

$$dx/dt = y$$
$$dy/dt = z$$
$$dz/dt = x$$

(a) For which initial conditions do solutions tend to the origin as $t \to +\infty$?

(b) Solve the equations with the initial condition $x(0) = 1, y(0) = 0, z(0) = 0$.

(c) Draw a sketch of all the different types of trajectories (include at least three trajectories) and describe them in words.

Problem 2

Problem 2

Problem 2

Problem 3

A 3-by-3 matrix $\mathbf{M}$ satisfies $\mathbf{Ma} = \mathbf{b}$, $\mathbf{Mb} = \mathbf{c}$, and $\mathbf{Mc} = \mathbf{a}$

where $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(a) Find the entries of $\mathbf{M}$ in the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

(b) Find the entries of $\mathbf{M}$ in the standard basis.

Now consider a 3-by-3 matrix $\mathbf{N}$ that satisfies $\mathbf{Na} = \mathbf{b}$ and $\mathbf{Nb} = \mathbf{c}$ (only two of the relations satisfied by $\mathbf{M}$), and with the additional requirement that the determinant of $\mathbf{N} = 0$.

(c) What are the dimensions of the range and nullspace of $\mathbf{N}$? Explain.

(d) Is $\mathbf{N}$ unique? I.e. are there multiple $\mathbf{N}$ with the above properties? Explain.

Problem 3

Problem 3

Problem 3

Problem 4

Consider the system

$$dx/dt = -y + |x|^{3/2}$$
$$dy/dt = -y + x^2 y$$

(a) Determine the stability of all critical points.

(b) For which initial conditions do solutions exist (at least for a short time) and for which initial conditions are such solutions unique? Explain your reasoning.

Problem 4

Problem 4

Problem 4

Problem 5

A solid slab occupies $-1 \leq x \leq 1$ and is initially at temperature 1. At $t = 0$ the temperatures at the ends of the slab are suddenly changed to 0 and held there for $t > 0$. The temperature $u(x,t)$ is described by the following equation:

$$\partial_t u - \partial_{xx} u = 0 \ , \ -1 < x < 1, \ t > 0$$

with initial and boundary conditions:

$$\begin{aligned} u(x,0) &= 1 \ , \ -1 \leq x \leq 1 \\ u(\pm 1, t) &= 0 \ \ , \ t > 0. \end{aligned}$$

Solve for $u(x,t)$ throughout $-1 < x < 1$, and for all $t > 0$, in the form of a series of known functions.

Problem 5

Problem 5

Problem 5