AIM Qualifying Exam: Advanced Calculus and Complex Variables

January 2020

There are five (5) problems in this examination, each worth 20 points.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

1. Consider the sum

$$\sum_{j=1}^{n-1} \frac{1}{\sqrt{j(n-j)}}.$$

a) (10 points) Find an integral whose Riemann sum is approximated by the sum above.

b) (10 points) Evaluate

$$\lim_{n\to\infty} \sum_{j=1}^{n-1} \frac{1}{\sqrt{j(n-j)}}.$$

2. (20 points) Suppose f_n, $n = 1, 2, \ldots$, and f are continuous functions in the closed interval $[0, 1]$. Assume that

$$f_n \to f \quad \text{as} \quad n \to \infty$$

uniformly in the open interval $(0, 1)$. Prove that

$$f_n(0) \to f(0) \quad \text{and} \quad f_n(1) \to f(1)$$

as $n \to \infty$.

3. Consider the integral

$$\frac{1}{2\pi i}\int_{\gamma_N} \frac{\cot \pi z}{z^2}\, dz,$$

where γ_N is the square contour with vertices at $\pm(N+1/2) \pm i(N+1/2)$ with counter-clockwise sense and $N \in \mathbb{Z}^+$.

a) (5 points) Prove that

$$\lim_{N\to\infty} \frac{1}{2\pi i}\int_{\gamma_N} \frac{\cot \pi z}{z^2}\, dz = 0.$$

You may assume this result for parts (b) and (c).

b) (5 points) Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

c) (10 points) Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

4. Consider the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{\log z}{1+z^2}\,dz.$$

Here $\log z$ has a branch cut on the positive real axis with $\log z = \log r + i\theta$ if $z = re^{i\theta}$ with $0 < \theta < 2\pi$.

a) (10 points) Choose γ to be a suitable contour and evaluate the familiar integral

$$\int_0^\infty \frac{dx}{1+x^2}.$$

b) (10 points) Evaluate

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx.$$

5. (20 points) Evaluate the volume integral

$$\int\int\int_B x^2y^2z^2\,dV$$

over the region B given by $0 \leq x \leq y \leq z \leq 1$.