39TH UNIVERSITY OF MICHIGAN UNDERGRADUATE MATHEMATICS COMPETITION

1pm-4pm, April 2, 2022

Problem 1. Call a positive integer *almost square* if it is of the form $n^2 - 1$ for an integer $n \geq 2$. Prove that every almost square can be written as the ratio of two almost squares, i.e. $a^2 - 1 = \frac{b^2-1}{c^2-1}$.

Problem 2. For two positive integers a and b, let $a * b$ be the positive integer formed by concatenating their decimal representations in that order. For example, $202 * 2 = 2022$. Write $a \lessdot b$ if $a * b \leq b * a$. For example, $202 \lessdot 2$ because $2022 \leq 2202$. Prove that $\lessdot$ is transitive, i.e. if $a \lessdot b$ and $b \lessdot c$ then $a \lessdot c$.

Problem 3. Determine all nonempty subsets S of the positive integers that satisfy the following two conditions:

- if $a, b \in S$ (possibly equal), then $a + b \in S$;
- if $2a \in S$, then $a \in S$.

Problem 4. Define a sequence of real numbers by

$$a_1 = 1, \quad a_{n+1} = a_n + \frac{a_n^2}{n(n+1)} \text{ for } n \geq 1.$$

Determine whether this sequence converges or diverges as $n \to \infty$.

Problem 5. Prove that any 2×2 real matrix M with determinant 1 can be written as a product of four matrices

$$M = A_1 A_2 A_3 A_4,$$

where each A_i is a 2×2 matrix with ones on the diagonal and at least one zero off the diagonal, i.e.

$$A_i = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ a_i & 1 \end{pmatrix}$$

for real numbers a_i.

Problem 6. Let $P(x)$ be a polynomial with real coefficients satisfying $P(-x) = P(x)$ and $\lim_{x\to+\infty} P(x) = +\infty$. Define a sequence of functions $f_1(x), f_2(x), \ldots$ by

$$f_1(x) = P(x), \quad f_{n+1}(x) = \frac{1}{2}\int_{x-1}^{x+1} f_n(t)\,dt \text{ for } n \geq 1.$$

Prove that this sequence of functions is eventually strictly positive, i.e. for sufficiently large N, $f_N(x) > 0$ for all reals x.

Problem 7. A frog is positioned at an unknown integer on a number line, is facing an unknown direction (left or right), and has an unknown (but constant and integer) hop length. Every second, the frog takes one hop (of constant direction and length). (For example, the frog might start at the integer 2022 and then hop from k to $k - 10$ every second.) You cannot see the frog, but every second you can reach out to an integer of your choice within range in hopes of catching the frog if it is there at that precise time. After n seconds have passed, your range is that you can reach out to any integer k satisfying $|k| \leq n^{1.001}$. Prove that there exists a strategy that will guarantee that you eventually catch the frog.

Problem 8. For integers n, k satisfying $0 \leq k < \frac{n}{2}$, let $g(n,k)$ denote the greatest common divisor of the binomial coefficients $\binom{n}{k}$ and $\binom{n}{k+1}$. Determine all pairs (n,k) such that $g(n,k)$ is a prime number.

Problem 9. Let n be a positive integer. If $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is a permutation of $\{1,\ldots,n\}$, let $f(\sigma)$ be one less than the minimum value attained by $\sigma_i + i$. For example, $f(4213) = \min(4+1, 2+2, 1+3, 3+4) - 1 = 3$. Prove that

$$\sum_{\sigma} f(\sigma) = \sum_{i=1}^{n} i! \cdot i^{n-i},$$

where the left sum runs over all permutations of $\{1,\ldots,n\}$.

Problem 10. For each positive integer n, independently choose a random divisor d_n of n uniformly from the divisors of n. (So $d_1 = 1$, $d_2 = 1$ or 2 with a $1/2$ chance of each, and so on.) Prove that there exists a positive integer N such that the remainder when the sum $d_1 + d_2 + \cdots + d_N$ is divided by 2^{2022} is equally likely to be any of the 2^{2022} different possibilities.

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