38TH UNIVERSITY OF MICHIGAN UNDERGRADUATE MATHEMATICS COMPETITION

April 3, 2021

Instructions: You have three hours to work on the problems and write up your solutions. During this time you must work on the problems alone and cannot consult other people, books, the internet or any other sources. You are also not allowed to use calculators, computer algebra systems, and so on. At the end of the three-hour period (4pm on April 3) you must stop working, and you should upload legible scans or photos of your solutions to the Canvas page where this exam was posted by 4:30pm at latest. If you have any technical difficulties, you can also e-mail your solutions to `pixton@umich.edu`.

Problem 1. Define a sequence recursively by $a_1 = 1$ and for all $n \geq 1$, $a_{n+1} = a_n^n + 1$. Determine, with proof, the rightmost (i.e. units) digit of a_{2021}.

Problem 2. Suppose $0 < a_1 < a_2 < \cdots < a_{2021}$ is an increasing sequence of positive integers such that $a_1 + \cdots + a_k$ is a multiple of k for each $k = 1, 2, \ldots, 2021$. Determine, with proof, the minimum possible value of a_{2021}.

Problem 3. Let n be a positive integer. Prove that

$$\int_0^{n-1} (x)_n \, dx \;\leq\; 0,$$

where $(x)_n = x(x-1)\cdots(x-n+1)$.

Problem 4. Let n be a positive integer. Determine all ways of coloring some of the integers $1, 2, \ldots, n$ red and coloring the rest blue such that there are no solutions to the equation

$$r_1 + r_2 = b_1 + b_2,$$

where r_1, r_2 (possibly equal) are red and b_1, b_2 (possibly equal) are blue.

Problem 5. Suppose $f, g : [0,1] \to \mathbb{R}$ are twice continuously differentiable functions, i.e. f'', g'' exist and are continuous. Suppose that $f(0) = f(1)$ and $g(0) = g(1)$. Prove that there exists t in the open interval $(0,1)$ such that

$$f'(t)f''(t) = g'(t)g''(t).$$

Problem 6. For every integer $n \geq 0$, let $a_n := \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n-2i}{i}$. Prove that $\lambda := \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$ exists, and find a polynomial equation in one variable with integer coefficients that λ satisfies.

Problem 7. Find all pairs of positive integers (m, n) such that $m^3 + 1$ is a multiple of n and $n + 1$ is a multiple of m.

Problem 8. Let n be a positive integer that is not divisible by 3 or 4. Let G be a subgroup of S_n, the group of permutations of n objects. Prove that if G contains both a 3-cycle and an n-cycle, it also contains a 5-cycle.

Problem 9. Let n be a positive integer, and suppose that $x_1, \ldots, x_n$ are distinct nonzero real numbers. Prove that

$$\sum_{i=1}^{n} \frac{1}{\prod\limits_{\substack{1 \leq j \leq n \\ j \neq i}} \left(1 - \frac{x_i}{x_j}\right)} = \sum_{i=1}^{n} \frac{1}{\prod\limits_{\substack{1 \leq j \leq n \\ j \neq i}} \left(1 - \frac{x_j}{x_i}\right)}.$$

Problem 10. Let a be an integer. Prove that there is a positive integer n such that

$$\sum_{k=1}^{n} k^k \equiv a \pmod{2021}.$$

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