UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 33
APRIL 9, 2016

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Suppose that m and n are positive integers whose greatest common divisor is 1. Suppose also that α is a real number such that $\cos m\alpha$, $\cos n\alpha$, and $\cos(m+n)\alpha$ are all rational numbers. Show that $\cos\alpha$ is rational.

Problem 2. Consider a *Pascal array* of numbers whose n^{th} row, for $n \geq 1$, consists of $n+1$ integers , $a_{n,0}$, $a_{n,1}$, …, $a_{n,n}$, subject to the recursive rules $a_{n,0} = 1$ for all $n \geq 0$, $a_{n,n} = 2$ for all $n \geq 1$, and $a_{n+1,i} = a_{n,i-1} + a_{n,i}$, $n \geq 0$, $1 \leq i \leq n$. The array begins:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 2 & & & \\ & & 1 & & 3 & & 2 & & \\ & 1 & & 4 & & 5 & & 2 & \\ 1 & & 5 & & 9 & & 7 & & 2 \end{array}$$

For $n \geq 1$, let $f_n(x) = \sum_{j=0}^{n} a_{nj}x^j$. Determine for which real x the series $\sum_{n=1}^{\infty} f_n(x)$ converges, and find a rational function of x that agrees with it where it converges.

Problem 3. Let p be a polynomial of degree > 1 all of whose roots are real, and put $q(z) = p(z+i) + p(z-i)$ where $i^2 = -1$. Show that all roots of q are real.

Problem 4. An integer $n > 1$ has the property that the binomial coefficient $\binom{n}{i}$ is divisible by n for all integers i with $1 \leq i \leq n-1$. Prove that n is prime.

Problem 5. Let $a_n > 0$ for all n and suppose that

$$\sum_{n=2}^{\infty} a_n^{1-1/\log n} < \infty .$$

Show that

$$\sum_{n=2}^{\infty} a_n < \infty .$$

Problem 6. Determine, with proof, how many integers x, $0 \leq x \leq 2015$, satisfy $x^3 \equiv -1 \bmod 2016$.

Problem 7. Let $\mathcal{F} = \{F_1, F_2, \ldots\}$ be the set of Fibonacci numbers, where $F_1 = 1$, $F_2 = 2$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Let $r(n)$ denote the number of ordered 2016-tuples $(f_1, f_2, \ldots, f_{2016})$ such that

$$f_1 + f_2 + \cdots + f_{2016} = n$$

and $f_i \in \mathcal{F}$ for all i. Determine, with proof, whether the series

$$\sum_{n=1}^{\infty} \frac{r(n)}{n}$$

converges or not.

Problem 8. Let $a_1, a_2, \ldots, a_n$ be complex numbers, and consider the $n \times n$ matrix $B = (b_{ij})$ such that $b_{ii} = 1 + a_i^2$, $1 \leq i \leq n$ and $b_{ij} = a_i a_j$ if $i \neq j$, $1 \leq i, j \leq n$. Show that $\det(B) = 1 + \sum_{i=1}^{n} a_i^2$.

Problem 9. Let ABC be a triangle. Let H_A denote the locus of points X inside the triangle such that the difference between the length of XB and XC is the length of AB minus the length AC, which is a hyperbolic arc connecting A with a point A' interior to the edge BC. Define H_B and H_C similarly. Show that these three hyperbolic arcs have a point P in common, and that there are circles centered at A, B, C and P, respectively, each of which is externally tangent to the other three.

Problem 10. Determine, with proof, whether there exists a function f with the following properties: f is an infinitely differentiable real-valued function of a real variable, $-1 \leq f(x) \leq 1$ for all real x, and $f^{(n)}(0) \to \infty$ as $n \to \infty$.