UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 30
APRIL 13, 2013

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Let $f(x) = x^{2013} + 2013^x$ on the interval $[0, 1]$. Let g denote the inverse of the function f on the interval $[1, 2014]$. Evaluate

$$\int_1^{2014} g(x)\, dx.$$

Problem 2. The letters **A, E, F, H, J, K, N, O, U** represent distinct digits so that the equations

$$\frac{\mathbf{FUN}}{\mathbf{JOKE}} = 0.\mathbf{HAHAHA}\ldots \qquad \text{and} \qquad \mathbf{JOKE} = 2013$$

are true. Determine, with proof, the values of $\mathbf{A}, \mathbf{F}, \mathbf{H}, \mathbf{N}, \mathbf{U}$.

Problem 3. Let s_n denote the side of a regular n-sided polygon inscribed in a circle of radius n, $n \geq 3$. Determine, with proof, real constants a, b such

$$\lim_{n\to\infty} n^2(a - s_n) = b.$$

Problem 4. A building with n floors has stairs between floor i and floor $i+1$ for $i = 1, 2, \ldots, n-1$. The building also has a number of slides. Each slide starts on one of the floors, and ends on some floor below it. For every i with $1 \leq i \leq n-1$ there exists a slide that starts on a floor above floor i, and ends on floor i or some floor below it. Show that it is possible to start and finish on floor 1, and slide down every slide exactly once without ever having to walk down the stairs.

Problem 5. Suppose you have a real coin and a fake coin. The probability that a flip of the real coin comes up heads is $\frac{1}{2}$. The probability that a flip of the fake coin comes up heads is p, where $0 \leq p \leq 1$ and $p \neq \frac{1}{2}$. The probability that a coin comes up head exactly twice when it is flipped three times is the same for the real and the fake coin. What is p?

Problem 6. Let F_n denote the nth Fibonacci number, defined recursively by the conditions $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Show that the rightmost three digits of a Fibonacci number cannot be 006.

Problem 7. A safe has 4 dials. Each dial has 9 positions. Two of the dials are broken, and the safe can be opened whenever the two dials that are not broken are in the correct position. Show that a safecracker can open the safe in at most 81 tries without knowing the combination or knowing which 2 of the 4 dials are broken. (A *try* consists of changing any number of dials and pulling the door to see if it opens.)

Problem 8. Suppose that $a_1 \geq a_2 \geq \dots$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} n! a_{n!}$ converges. Must $\sum_{n=1}^{\infty} a_n$ necessarily converge?

Problem 9. Four frogs start out on the corners of a square. They play a game in which, at a given turn, one of the frogs, starting at point A, leaps over another, which is at point B, winding up at point C, where B is the midpoint of the line segment AC. The other three frogs stay fixed on that move. After repeated moves of this type can the positions of the four frogs eventually be at the corners of a square of a larger size than the one from which they started? Prove your answer.

Problem 10. Does there exist a 4×4 matrix A with real entries such that

$$A^{1000} = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}?$$

Prove your answer.

Solution 1. Let $r = 2013$. Thought of as $\int_1^{r+1} g(y)\,dy$, this is the same as the area between bounded by the y-axis, the line $y = r+1$, and the curve $x = g(y)$, which is the same as $y = f(x)$. Hence its sum with the area between $y = f(x)$ and the x-axis on $[0,1]$ is the area of the rectangle bounded by the x-axis, $x = 1$, the y-axis, and $y = r+1$, which is $r+1$. Hence, we obtain

$$r + 1 - \int_0^1 (x^r + r^x)dx.$$

Since $r^x = e^{cx}$ where $c = \log_e r$, we obtain

$$r + 1 - \left(\frac{1}{r+1} + \frac{e^{cx}}{c}\Big|_0^1 \right) = r + 1 - \frac{1}{r+1} - \left(\frac{e^c - 1}{c} \right) = r + 1 - \frac{1}{r+1} - \frac{r-1}{\log_e r},$$

which, in this case, is

$$2014 - \frac{1}{2014} - \frac{2012}{\log_e 2013}.$$

Solution 2. We have

$$\frac{\mathbf{FUN}}{2013} = \frac{\mathbf{FUN}}{\mathbf{JOKE}} = \frac{\mathbf{HA}}{99},$$

so

$$99 \cdot \mathbf{FUN} = 2013 \cdot \mathbf{HA}.$$

Dividing both sides by 33 gives

$$3 \cdot \mathbf{FUN} = 61 \cdot \mathbf{HA}.$$

HA is at most

$$3 \cdot \frac{999}{61} < \frac{3 \cdot 1000}{60} = 50,$$

at least 45 (because the digits $0, 1, 2, 3$ do not appear in **HA**) and it is divisible by 3. So we have **HA** $= 45$ or **HA** $= 48$. If **HA** $= 45$, then **FUN** $= 45 \cdot 61/3 = 549$. But then **H** $=$ **U** $= 4$ leads to a contradiction. So **HA** $= 48$ gives **FUN** $= 48 \cdot 61/3 = 976$.

Solution 3. Drop a perpendicular from the center of the circle P to the midpoint M of a side, and let V be a vertex for that side. Then one has a right triangle $\triangle PMV$. The angle at P is $(1/2)(2\pi/n) = \pi/n$, and so $\sin(\pi/n) = (s_n/2)/n$ and $s_n = 2n\sin(\pi/n)$. It follows from the power series for sin that

$$s_n = 2n\left(\frac{\pi}{n} - \frac{1}{6}\left(\frac{\pi}{n}\right)^3 + f\left(\frac{\pi}{n}\right)\left(\frac{\pi}{n}\right)^5 \right) = 2\pi - \frac{\pi^3}{3n^2} + f\left(\frac{\pi}{n}\right)\frac{2\pi^5}{n^4}$$

where f is an analytic function, and

$$n^2(2\pi - s_n) = \frac{\pi^3}{3} - f\left(\frac{\pi}{n}\right)\frac{2\pi^5}{n^2}.$$

Hence, one may take $a = 2\pi$ and $b = \pi^3/3$.

Solution 4. Construct a sequence of slides $S_1, \dots, S_r$ as follows: S_1 is a slide that starts on floor n. Suppose that slide S_i has been defined and it starts on s_i and ends on e_i. If $e_i > 1$, then there exists a slide that starts on floor e_i or above, and ends on a floor below level e_i. Call this slide S_{i+1}. This way we construct slides $S_1, \dots, S_r$ with $n > e_1 > e_2 > \cdots > e_r = 1$, $s_1 = n$ and $s_{i+1} \geq e_i$ for $i = 1, 2, \dots, r-1$. Note that $S_1, \dots, S_r$ are distinct, and that, starting on floor n we can take the slides $S_1, \dots, S_r$ (in that order) without ever having to walk down stairs. Let $T_1, \dots, T_p$ be the remaining slides. Starting on floor 1, we can take the slides $T_1, \dots, T_p$ according to increasing start level. Then we walk up to the n-floor and take the slides $S_1, \dots, S_r$.

Solution 5. The probability for 2 heads among 3 tosses is

$$\binom{4}{2}\left(\frac{1}{2}\right)^3$$

for the fair coin, and

$$\binom{4}{2}p^2(1-p)$$

for the fake coin. So we have

$$p^2 - p^3 = \frac{1}{8}$$

and

$$(p - \tfrac{1}{2})(p^2 - \tfrac{1}{2}p - \tfrac{1}{4}) = p^3 - p^2 + \tfrac{1}{8} = 0.$$

It follows that

$$p^2 - \tfrac{1}{2}p - \tfrac{1}{4} = 0$$

and

$$p = \frac{\frac{1}{2} \pm \sqrt{\frac{5}{4}}}{2} = \frac{1}{4} \pm \frac{\sqrt{5}}{4}.$$

Since

$$\frac{1}{4} - \frac{\sqrt{5}}{4} < 0,$$

we get

$$p = \frac{1+\sqrt{5}}{4}.$$

Solution 6. Working modulo 8, the Fibonacci numbers are 0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, ..., after which the sequence is clearly periodic. Note that 6 does not occur. Since $8 \mid 1000$, a Fibonacci number cannot have the form $1000m + 6$.

Solution 7. Let the 9 positions be $0, 1, 2, \dots, 8$. Try all the combinations

$$(x, y, x+y, x+2y)$$

where $x, y \in \{0, 1, \dots, 8\}$ and we calculate modulo 9. Each of the pairs (x, y), $(x, x+y)$, $(x+2y)$, $(y, x+y)$, $(y, x+2y)$, $(x+y, x+2y)$ runs through all combinations (i, j) with $0 \leq i, j \leq 8$.

Solution 8. Let
$$a_n = \frac{1}{n\log(n)\log\log(n)}$$
for $n \geq 3$ and define $a_1 = a_2 = a_3$. Then the sequence $a_1, a_2, \ldots$ is weakly decreasing and positive. The sequence
$$\sum_{n=3}^{\infty} a_n \geq \int_3^{\infty} \frac{1}{x\log(x)\log\log(x)}\,dx = \infty.$$
diverges by comparison. Since $i(n+1-i) \geq n$, we get $n!^2 \geq n^n$ by taking the product over $i = 1, 2, \ldots, n$. and $\log(n!) \geq \frac{n}{2}\log(n) \geq n$ for $n \geq 9 > e^2$.

We have
$$n!a_{n!} = \frac{1}{\log(n!)\log\log(n!)} \leq \frac{2}{n\log^2(n)}.$$
Since
$$\sum_{n=3}^{\infty} \frac{1}{n\log(n)^2} \leq \int_2^{\infty} \frac{1}{x\log^2(x)}\,dx = \frac{1}{\log(2)}$$
converges by comparison, the series $\sum_{n=9}^{\infty} n!a_{n!}$ converges by comparison as well.

Solution 9. No. Set up coordinates so that the four frogs are initially at lattice points forming a unit square. Since the positions A, B, C, D are replaced by $2B - A, B, C, D$, they are always lattice points. This means the side of the square they form can never be smaller than 1. Hence, they cannot reach a smaller square. But since these moves are reversible they cannot reach a larger square either.

Solution 10. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the eigenvalues of A. Then the eigenvalues of A are $\lambda_1^{1000}, \lambda_2^{1000}, \lambda_3^{1000}, \lambda_4^{1000}$. Since the eigenvalues of the matrix
$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$
are $-1, -1, -2, -2$, we may assume that $\lambda_1^{1000} = \lambda_2^{1000} = -1$ and $\lambda_3^{1000} = \lambda_4^{1000} = -2$. Clearly, $\lambda_1, \ldots, \lambda_4$ cannot be real. For the complex conjugate $\overline{\lambda}_1$ of λ_1 we have $\overline{\lambda}_1^{1000} = \overline{-1} = -1$. So $\overline{\lambda}_1 = \lambda_2$. Similarly $\overline{\lambda}_3 = \lambda_4$. In particular, $\lambda_1 \neq \lambda_2$ and $\lambda_3 \neq \lambda_4$. So $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are distinct, and A is diagonalizable. But then A^{1000} is also diagonalizable. Contradiction.