UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 30
APRIL 13, 2013

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Let $f(x) = x^{2013} + 2013^x$ on the interval $[0, 1]$. Let g denote the inverse of the function f on the interval $[1, 2014]$. Evaluate

$$\int_1^{2014} g(x)\, dx.$$

Problem 2. The letters **A, E, F, H, J, K, N, O, U** represent distinct digits so that the equations

$$\frac{\mathbf{FUN}}{\mathbf{JOKE}} = 0.\mathbf{HAHAHA}\ldots \qquad \text{and} \qquad \mathbf{JOKE} = 2013$$

are true. Determine, with proof, the values of $\mathbf{A}, \mathbf{F}, \mathbf{H}, \mathbf{N}, \mathbf{U}$.

Problem 3. Let s_n denote the side of a regular n-sided polygon inscribed in a circle of radius n, $n \geq 3$. Determine, with proof, real constants a, b such

$$\lim_{n\to\infty} n^2(a - s_n) = b.$$

Problem 4. A building with n floors has stairs between floor i and floor $i + 1$ for $i = 1, 2, \ldots, n - 1$. The building also has a number of slides. Each slide starts on one of the floors, and ends on some floor below it. For every i with $1 \leq i \leq n - 1$ there exists a slide that starts on a floor above floor i, and ends on floor i or some floor below it. Show that it is possible to start and finish on floor 1, and slide down every slide exactly once without ever having to walk down the stairs.

Problem 5. Suppose you have a real coin and a fake coin. The probability that a flip of the real coin comes up heads is $\frac{1}{2}$. The probability that a flip of the fake coin comes up heads is p, where $0 \leq p \leq 1$ and $p \neq \frac{1}{2}$. The probability that a coin comes up head exactly twice when it is flipped three times is the same for the real and the fake coin. What is p?

Problem 6. Let F_n denote the nth Fibonacci number, defined recursively by the conditions $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Show that the rightmost three digits of a Fibonacci number cannot be 006.

Problem 7. A safe has 4 dials. Each dial has 9 positions. Two of the dials are broken, and the safe can be opened whenever the two dials that are not broken are in the correct position. Show that a safecracker can open the safe in at most 81 tries without knowing the combination or knowing which 2 of the 4 dials are broken. (A *try* consists of changing any number of dials and pulling the door to see if it opens.)

Problem 8. Suppose that $a_1 \geq a_2 \geq \dots$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} n! a_{n!}$ converges. Must $\sum_{n=1}^{\infty} a_n$ necessarily converge?

Problem 9. Four frogs start out on the corners of a square. They play a game in which, at a given turn, one of the frogs, starting at point A, leaps over another, which is at point B, winding up at point C, where B is the midpoint of the line segment AC. The other three frogs stay fixed on that move. After repeated moves of this type can the positions of the four frogs eventually be at the corners of a square of a larger size than the one from which they started? Prove your answer.

Problem 10. Does there exist a 4×4 matrix A with real entries such that

$$A^{1000} = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}?$$

Prove your answer.