UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 29
MARCH 31, 2012

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. It is March Math Madness. 512 math teams are randomly assigned numbers from 1 to 512. Repeated rounds are played. In the first round, the teams are paired and 256 contests are held. The winner progresses to the next round while the loser is eliminated. Thus, in the second round there are 128 pairs of teams, in the third round 64 pairs, and so forth. Eventually, only two teams are left and they play to determine the champion. Show that no matter how the pairings are done and no matter who wins, at some point in the competition two teams play whose numbers differ by 31 or more.

Problem 2. Determine the number of integers n, $0 \leq n \leq 2012$, such that $\binom{2012}{n}$ is not divisible by 13.

Problem 3. Let W_n be the set of n letter strings formed from the letters a, b and satisfying the additional condition that the maximal substrings consisting entirely of the letter a must have an even number of occurrences of a. E.g., W_4 contains $aabb$, $baab$, and $aaaa$, but not $abba$. Let p_n be the probability that a string in W_n has rightmost character b. Find $\lim_{n\to\infty} p_n$.

Problem 4. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that $\sum_{i=1}^{\infty} a_i$ converges and define $b_n = \sum_{i=n}^{\infty} a_i$ for all n. Show that

$$\sum_{n=1}^{\infty} \frac{a_n}{b_n}$$

diverges.

Problem 5. Let k and n be positive integers with $2k+1 \leq n$. Let f be a bijective function from the set consisting of the first n positive integers, $A_n = \{1, \ldots, n\}$, to itself. Define f to have a k-*maximum* at $h \in A_n$ if f is increasing on the set of integers from $h-k, \ldots, h-1$, h and decreasing on the set h, $h+1$, $\ldots h+k$. For example, there is a 2-maximum at $h = 5$ if $f(3) < f(4) < f(5) > f(6) > f(7)$. Let $M_k(f)$ denote the number of integers that are k-maxima for f, and $m(n,k)$ denote the average value of $M_k(f)$ taken over all possible choices of the bijection $f : A_n \to A_n$. Express $m(n,k)$ in closed form as a rational function of n, k, and $k!$

Problem 6. Suppose that r is a positive integer. Show that for every positive integer n with $n \leq r!$ there exist positive integer m with $m \leq 2r$ and positive integers $k_1 := n > k_2 > \cdots > k_m = 1$ such that

$$\frac{1}{k_{i+1}} - \frac{1}{k_i}$$

is the reciprocal of an integer for $i = 1, 2, \ldots, m-1$. For example if $n = 23 \leq 4!$ then we can take $m = 6 \leq 2 \cdot 4$ and $(k_1, k_2, k_3, k_4, k_5, k_6) = (23, 22, 20, 4, 2, 1)$, because

$$\frac{1}{22} - \frac{1}{23} = \frac{1}{506}, \; \frac{1}{20} - \frac{1}{22} = \frac{1}{220}, \; \frac{1}{4} - \frac{1}{20} = \frac{1}{5}, \; \frac{1}{2} - \frac{1}{4} = \frac{1}{4}, \; \frac{1}{1} - \frac{1}{2} = \frac{1}{2}.$$

Problem 7. Suppose that the positive integers $k_1 < k_2 < \cdots < k_r$ have the property that for every complex solution $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{C}^{2n}$ of the system

$$\begin{aligned} y_1 x_1^{k_1} + \cdots + y_n x_n^{k_1} &= 0 \\ y_1 x_1^{k_2} + \cdots + y_n x_n^{k_2} &= 0 \\ \vdots \qquad & \quad \vdots \\ y_1 x_1^{k_r} + \cdots + y_n x_n^{k_r} &= 0 \end{aligned}$$

we have

$$y_1 x_1^k + \cdots + y_n x_n^k = 0$$

for all positive integers k. Prove that $k_1 k_2 \cdots k_r / n!$ is an integer.

Problem 8. You are playing a game of connect the dots with the devil. The devil tells you how many dots there are, say n, and names them. The devil then decides which pairs of two distinct dots are directly connected: one also says that they form an *edge*, but tells you nothing about this. The array is called a *graph*. Two dots A and B are *connected* if they are the same, or are directly connected, or if there is a *path* between them, i.e., a sequence of dots $A_1, \ldots, A_k$ such that A and A_1, A_i and A_{i+1}, $1 \le i \le k-1$, and A_k and B are edges. The devil's whole array is called *connected* if every dot is connected to every other dot. You get to ask questions. In each question, you are allowed to ask whether two specific dots are directly connected. You are trying to determine whether the whole array is connected or not. You win if you can do this without asking about every pair of dots. Otherwise, the devil wins. In deciding on what to ask, you may take the answers to the previous questions into account. Just to make your life harder, at any point between two questions, the devil can change the whole array, provided that the new array gives the same answers as the old to your previous questions. The devil can do this repeatedly. Determine, for every value of n, who can win the game with best strategy.

Problem 9. Your spaceship is at rest, parked at the origin in $\mathbb{R}$. The control panel of the spaceship displays not only your position x, but also your velocity x', acceleration x'' and all higher derivatives $x^{(k)}$ of your position. All readings are currently at 0.

At time $t = 0$ you enter in new values for n of the derivatives:

$$(x^{(k)}(0), \ldots, x^{(k+n-1)}(0)) := v \in \mathbb{R}^n$$

and hang on for your life until $t = 1$ when you mark down the readouts for n other derivatives:

$$w = (x^{(m_0)}(1), x^{(m_1)}(1), \ldots, x^{(m_{n-1})}(1)).$$

You may assume that $x(t)$ is an analytic function on the interval $[0, 1]$. Show that the determinant of the linear map which sends $v \in \mathbb{R}^n$ to $w \in \mathbb{R}^n$ is either 0 or the reciprocal of an integer.

Problem 10. A room has the shape of a, not necessarily convex, n-gon. For each $n \ge 3$ find the smallest positive integer k such that, regardless of the shape of the room, k lights can be placed in the room such that at every position in the room, at least 1 light is visible.