UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 28
APRIL 7, 2011

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. There are some gas stations on a circular roadway. Together the stations contain just enough gas to make it all the way around. Show that it is possible to start at one of the stations with an empty tank (that is large enough to hold all the gas at all these stations) and make it all the way around.

Problem 2. Four distinct points $\{P_1,\ P_2, P_3,\ P_4\}$ in the plane have the property that the set of six distances between the pairs of them, P_iP_j for $i \neq j$, consists of precisely 2 real numbers. One example is the set of vertices of a square. Find, up to similarity, all configurations of four points with this property.

Problem 3. Find all solutions for $x^3 + y^3 = 3^z$ where x, y and z are integers.

Problem 4. What is the largest number of subsets you can choose from $\{1, \ldots, n\}$ with each subset having an odd number of elements and the intersection of any two distinct subsets having an even number of elements?

Problem 5. Suppose that $a_1, a_2, \ldots$ is a sequence of real numbers such that

$$\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$$

for all positive integers n. For every possible value of a_1, determine $\lim_{n\to\infty} a_n$.

There are more problems on the other side.

Problem 6. Let $r > 0$ be a fixed real number. Suppose that a differentiable function $y = f(x)$ can be defined on the interval $(0, B)$, $B > 0$, so that it is positive and satisfies the differential equation $xy' = y + y^{r+1}$. Suppose that $f(1) = a > 0$. Find, as a function of r and a, the largest possible value of B for which there is such a function.

Problem 7. Suppose that G is a finite group containing elements x, y, z such that $yx = x^2y$, $zy = y^2z$ and $xz = z^2x$. Prove that $x = y = z = 1$.

Problem 8. Show that there is a 4×4 matrix M over the real numbers such that all the entries off the diagonal are nonzero, but all of the entries on the diagonal of M^k are zero when $k \geq 1$ is an integer.

Problem 9. How many of the binomial coefficients $\binom{2011}{r}$, $r = 0, 1, \ldots, 2011$ are even?

Problem 10. One chooses 5 random points in the unit disc. Assume that we have a uniform distribution on the disc, and that the points are chosen independently. What is the probability that the convex hull of the 5 points is a triangle?