UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 26
MARCH 28, 2009

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Let N be a positive integer whose leftmost nonzero digit is a and whose rightmost digit is b. Let $d = |b - a|$. If $N = d^2(d^2 - d - 1)$, find, with proof, all possible values of N.

$d = 0, 1, 2, 3, 4, 5, 6$ yield the respective values $0, -1, 4, 45, 176, 475, 1044$ for N. The values of d for the last five are $0, 1, 5, 1, 3$, respectively, all of which are wrong. If $d = 7$, then $N = 49(49 - 7 - 1) = 49(41) = 2009$ which gives $d = 7$, and so $N = 2009$ is a solution. If $d = 8$, then $N = 64(64 - 8 - 1) = 64(55) = 3520$ which gives $d = 3$, the wrong value. If $d = 9$ then the last digit must be 0, but $N = 81(81 - 9 - 1)$ is not divisible by 10. Hence, $N = 2009$ is the only solution.

Problem 2. Suppose that $p(x)$ is a non-constant polynomial with real coefficients such that there are infinitely many real numbers a such that the equation $p(x) = a$ has more than 1 integer solution for x. Prove that there exists an integer s such that $p(x) = p(s - x)$.

Without loss of generality we may assume that $p(x)$ is monic, so $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0$ for some $c_0, \ldots, c_{n-1} \in \mathbb{R}$. It is clear that $p(x)$ has even degree, so $p(x)$ is bounded from below. There exist $a_1, a_2, a_3, \ldots,\ x_1, x_2, \cdots \in \mathbb{Z}$, and $y_1, y_2, \cdots \in \mathbb{Z}$ such that $\lim_{i\to\infty} a_i \to \infty$, $p(x_i) = p(y_i) = a_i$ and $x_i < y_i$ for all i. Choose $N > 0$ such that $p'(x)$ has no root in the interval $[-N, N]$. Now f is monotone on $[N, \infty)$ and $(-\infty, N]$. For large i, $p(x_i)$ and $p(y_i)$ do not lie in $p([-N, N])$, and they cannot be both in $[N, \infty)$ or both in $(-\infty-, N]$. So $x_i \in (-\infty, -N]$ and$y_i \in [N, \infty)$ for large N. Since $\lim_{i\to\infty} f(x_i) = \lim_{i\to\infty} f(y_i) = \infty$, n must be even.

We have

$$\lim_{i\to\infty} (\frac{x_i}{y_i})^n = \lim_{i\to\infty} \frac{\frac{p(y_i)}{y_i^n}}{\frac{p(x_i)}{x_i^n}} = \frac{1}{1} = 1$$

Since $x_i/y_i < 0$ for large i, we have $\lim_{i\to\infty} x_i/y_i = -1$.

We have

$$0=\lim_{i\to\infty}\frac{p(x_i)-p(y_i)}{(x_i-y_i)y_i^{n-2}}=\lim_{i\to\infty}(x_i+y_i)\frac{(x_i^{n-2}+x_i^{n-4}y_i^2+\cdots+y_i^{n-2})}{y_i^{n-2}}+c_{n-1}\frac{x_i^{n-1}-y_i^{n-1}}{(x_i-y_i)y_i^{n-2}}$$

From

$$\lim_{i\to\infty})\frac{(x_i^{n-2}+x_i^{n-4}y_i^2+\cdots+y_i^{n-2})}{y_i^{n-2}}=\frac{n}{2}$$

and

$$\lim_{i\to\infty}\frac{x_i^{n-1}-y_i^{n-1}}{(x_i-y_i)y_i^{n-2}}=1$$

follows that

$$\lim_{i\to\infty}x_i+y_i=\frac{-2c_{n-1}}{n}.$$

So $s:=-2c_{n-1}/n\in\mathbb{Z}$, and $x_i+y_i=s$ for large i. $p(x)-p(s-x)$ has infinitely many roots, namely $x_1,x_2,\dots$. So $p(x)=p(s-x)$.

Problem 3. An office has 5 employees. In a given 5-day workweek, each employee takes 2 randomly chosen days off. What is the probability that someone is present at the office every day of the week?

For any subset $I\subseteq\{1,2,\dots,5\}$ let $p(I)$ be the probability that no-one is present on day i for all $i\in I$. By the inclusion-exclusion principle, the probability that someone is present every day of the week is

$$\sum_{I\subseteq\{1,2,\dots,5\}}(-1)^{|I|}p(I). \tag{1}$$

If $|I|=k$, then we have

$$p(I)=\Big(\frac{\binom{5-k}{3}}{\binom{5}{3}}\Big)^5,$$

because for each employee, there are $\binom{5}{3}$ ways to choose the three working days among 5 days, and there are $\binom{5-k}{3}$ ways to choose the three working days from $\{1,2,3,4,5\}\setminus I$.

There are $\binom{5}{k}$ ways to choose a subset $I\subseteq\{1,2,3,4,5\}$ with k elements, so (1) is equal to

$$\sum_{k=0}^{2}\sum_{I\subseteq\{1,2,3,4,5\};|I|=k}(-1)^k\Big(\frac{\binom{5-k}{3}}{\binom{5}{3}}\Big)^5=\sum_{k=0}^{2}\binom{5}{k}(-1)^k\Big(\frac{\binom{5-k}{3}}{\binom{5}{3}}\Big)^5=$$

$$=1-5\cdot\Big(\frac{4}{10}\Big)^5+10\cdot\Big(\frac{1}{10}\Big)^5=1-512\cdot10^{-4}+10^{-4}=0.9489.$$

Problem 4. Let $a_n=\prod_{k=n}^{2n}k^{1/k}$. Find specific real constants b, c such that

$$\lim_{n\to\infty}\frac{a_n}{n^b}=c.$$

$\log(a_n) = \sum_{k=n}^{2n} \log k/k$, which, since the sequence of terms is decreasing and approaching 0, differs by an amount $\epsilon_n \to 0$ from $\int_n^{2n} \big(\log x/x\big)\, dx = (1/2)(\log x)^2|_n^{2n} = (1/2)(\log n + \log 2)^2 - (1/2)(\log n)^2 = \log n \log 2 + (1/2)(\log 2)^2$. Exponentiating, we have $a_n = e^{\epsilon_n} e^{\log 2 \log n} e^{(1/2)(\log 2)^2}$, and, consequently, $a_n/n^{\log 2} = e^{\epsilon_n} e^{(1/2)(\log 2)^2}$. It follows that we may choose $b = \log(2)$ and $c = e^{(1/2)(\log 2)^2}$.

Problem 5. There are n contestants, where $n \geq 1$ is an integer. They are told that they will be placed in isolation, and each of them will be randomly assigned an integer between 1 and n, inclusive. Each will then be told what numbers have been assigned to the **other** contestants. Each must guess her or his number. If any one of them is correct, all of them will get a million dollar prize. They can agree on a strategy before they are isolated, but there is no communication allowed after they are isolated. Is there a strategy that enables them to win the prize with certainty? If not, what is the best strategy?

They can win the prize with certainty. In the strategy session, each of them is assigned exactly one of the numbers $0, \ldots, n-1$. The prisoner who is assigned the number i guesses the unique number between 1 and n that makes the sum of all the numbers leave remainder i when divided by n. Exactly one of the contestants will be correct: the sum of the numbers assigned has some remainder j when divided by n, and the contestant assigned the number j will be correct.

Problem 6. For an integer $k \geq 0$ and $x > 0$ define $\log^{(k)}(x)$ recursively by the rules $\log^{(0)}(x) = x$ and $\log^{(k+1)}(x) = \log_e\big(\log^{(k)}(x)\big)$ provided that $\log^{(k)}(x) > 0$. Let $K(x)$ denote the largest integer k such that $\log^{(k)}(x)$ is defined and ≥ 1. For $x \geq 1$, define $p(x) = \prod_{k=0}^{K(x)} \log^{(k)}(x)$. For example, $p(1) = 1$, $p(2) = 2$, $p(3) = 3\log_e(3)$, and $p(21) = 21 \log_e(21) \log_e\big(\log_e(21)\big)$ (since $e < \log_e(21) < e^e$). Determine, with proof, whether the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{p(n)}$$

converges.

Let $e_0 = 1$ and define e_n recursively by the rule $e_{n+1} = e^{e_n}$. When $e_k \leq x < e_{k+1}$ we have that $p(x) = p_k(x)$, where $p_k(x) = \prod_{j=0}^{k} \log^{(j)}(x)$. Note that $\log^h e_k = e_{k-h}$ for $h \leq k$, and $\log^{(k+1)} e_k = 0$. Since x, $\log_e x$, and hence, all of the iterated logs and their products are strictly increasing where defined, the contribution c_k to the sum from integers in $[e_k, e_{k+1})$ is bounded below by $b_k = \displaystyle\int_{\lceil e_k \rceil}^{e_{k+1}} \frac{dx}{p_k(x)}$. If $\lceil e_k \rceil$ is replaced by e_k, the value of the integral changes by an amount $\delta_k \to 0$ with k. By a straightforward induction, $1/p_k$ is the derivative of log^{k+1}, so that the value of the integral with lower limit e_k is $\log^{k+1}(e_{k+1}) - \log^{k+1}(e_k) = 1 - 0 = 1$, and so $c_k \geq 1 - \delta_k$ with $\delta_k \to 0$ as $k \to \infty$. It follows that the series diverges.

Problem 7. Consider the set $S = \{(x, y) \in \mathbb{R}^2 \mid y > |x|\}$ in the xy-coordinate plane. Show that S cannot be expressed as a disjoint union of parabolas of the shape $y = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$.

Problem 8. Call a subset S of the unit interval $[0, 1]$ *special* if whenever it contains x, y it also contains $1-x$ and xy. If $q \in [0, 1]$ let S_q denote the smallest special set that contains $1/2$ and q. Show that for every integer $n > 0$ there exists a rational number $q = q_n$ such that S_q contains all rational numbers in $(0, 1)$ with denominator at most n.

The fact that S_q contains $1/2$ implies that it contains all rationals of the form $a/2^h$ for $a < 2^h$ a positive integer. One may show this by induction on h. To get the new rational numbers needed for $h+1$ multiply each $a/2^h$ for a odd by $1/2$, and then subtract each of the numbers obtained from 1 as well. Let N be a positive integer divisible by all integers from 2 to n, and let 2^h be the greatest power of 2 that is $\leq N$. Then $N < 2^{h+1}$ and $2^h > N/2$. Let $q = 2^h/N$. For $1 \leq b < 2^h$, $b/2^h \in S_q$ and so $(b/2^h)(2^h/N) = b/N \in S_q$, and $1 - (b/N) = (N-b)/N \in S_q$ as well. But either b or $N - b$ is $\leq N/2 < 2^h$.

Problem 9. Let n be a positive integer. Anjolina and Brad play the following game. There is a pile of $2n+1$ cards on the table, numbered $0, 1, 2, \ldots, 2n$. The players take turns and Anjolina begins. In each turn, a player has to take 3 cards from the pile, such that the numbers on the cards sum up to $3n$. If a player cannot find 3 cards with this property, then that player loses. Show that Anjolina has a winning strategy.

In the first turn, Anjolina takes 0, n and $2n$. After that, if Brad chooses x, y and z in round k, then Anjolina takes $2n-x$, $2n-y$ and $2n-z$. Note that $(2n-x)+(2n-y)+(2n-z) = 6n-(x+y+z) = 6n-3n = 3n$. With this strategy, Anjolina ensures that after each of her turns, card a is in the pile if an only if card $2n-a$ is in the pile. If before Brad's turn, x is in the pile, then so is $2n-x$. Now Brad cannot choose both x and $2n-x$, because then the third card would have to be n and n was already taken by Anjolina in her first turn. So Brad chooses x, y and z from the pile, then $2n-x$, $2n-y$ and $2n-z$ are still in the pile. This ensures that Anjolina can continue her strategy. Since Anjolina can always choose 3 cards after Brad's turn, Brad will lose.

Problem 10. Let $n \geq 2$ be an integer. Determine, with proof, all functions f from the real numbers to the real numbers such that for all real numbers x, y, $f(x^n - y^n) = f(x)^n - f(y)^n$.

We show that if n is even then f is identically 0 or the identity function. If n is odd, it is one of these or $f(x) = -x$ for all real x. Letting $x = y$ shows that (1) $f(0) = 0$. Letting $y = 0$ shows that (2) $f(x^n) = f(x)^n$ and letting $x = 0$ shows that (3) $f(-y^n) = -f(y)^n$. Also, when u and v are n the powers, we have that (4) $f(u-v) = f(u) - f(v)$. When n is odd, this shows that (5) f is a homomorphism of additive groups, since every real number is an n th power. If n is even this is still true: (2) and (3) show that $f(-z) = -f(z)$, and this coupled with the fact that (4) holds for u, v nonnegative yields the result. Since $f(1) = f(1^n) = f(1)^n$ we find that $f(1)$ must be 0 or $f(1)^{n-1} = 1$, which implies $f(1) = 1$ if n is even, and $f(1) = \pm 1$ if n is odd. If n is odd and f satisfies the condition so does $-f$. The result follows if we can show that f is identically 0 (respectively, the identity) if $f(1) = 0$ (respectively, 1). Write $f(1) = a$. The desired conclusion holds when x is rational from (5). When n is even, f takes nonnegative numbers to nonnegative numbers. which implies that f is order-preserving, and this gives the desired conclusion easily for all real r. If n is odd, we have that for every real number x and integer m, $f((x+m)^n) = f(x+m)^n = (f(x) + f(m))^n = (f(x) + ma)^n$, and this yields $\sum_{i=1}^{n} \binom{n}{i} m^{n-i} f(x^i) = \sum_{i=1}^{n} \binom{n}{i} f(x)^i m^{n-i} a$ since the terms corresponding to $i = 0$ cancel and $a^t = a$ for $a \geq 1$. This gives $\sum_{i=1}^{n} \binom{n}{i} m^{n-i}(f(x^i) - af(x)^i) = 0$. Choose any n distinct integers m_j. Then the matrix $\left(\binom{n}{i} m_j^{n-i}\right)$ is invertible: after factoring out a nonzero binomial coefficient from every row, we obtain a Vandermonde determinant. It follows that $f(x^i) = af(x)^i$ for all $i \leq n$, and so we may take $i = 2$. But then f is order-preserving, and the result follows.