UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 21
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SOLUTIONS

Problem 1. Show that $n! + 2004$ is not a perfect square for any positive integer n.

Solution: The number 2004 is divisible by 3 but not by 9. For $n \geq 6$, $n!$ is divisible by 9, so $n! + 2004$ is divisible by 3 but not by 9. It follows that $n! + 2004$ cannot be a square for $n \geq 6$. One easily checks that $n! + 2004$ is not a square for $n = 1, 2, 3, 4, 5$ either. ($1! + 2004 = 2005$, $2! + 2004 = 2006$, $3! + 2004 = 2010$, $4! + 2004 = 2028$, $5! + 2004 = 2124$ and $44^2 = 1936$, $45^2 = 2025$, $46^2 = 2116$, $47^2 = 2209$)

Problem 2. Twenty-four delegates sit around a round table. Two delegates can speak to each other if at most 4 people sit between them. After a break each person sits down again, not necessarily in the same seat as before. Show that there exist two delegates who are able to speak to each other before and after the break.

Solutions: In fact, there are two delegates such there are at most 3 people between them, before and after the break. Consider 5 delegates who sit next to each other before the break. Each pair of these 5 can talk to each other. If there are at least 4 delegates between each two of them after the break, then there will be at least $5 \cdot 4 + 5 = 25$ delegates. Contradiction. One can find 2 of these 5 delegates with at most 3 people in between them after the break.

Problem 3. For positive integers n, let $S(n) = \lfloor\sqrt{n}\rfloor$ denote the integer part of the positive square root of n. Call a non-empty set of positive integers X *rooted* if whenever $n, m \in X$, then $S(n) + S(m) \in X$ (including the case when $n = m$). Find, with proof, all rooted sets of positive integers that do not contain 4.

Solution: A rooted set that does not contain 4 cannot contain an integer $n > 4$, for if n were a smallest such integer, we have $S(n) \geq 2$ and so $S(n) + S(n) \geq 4$, but $2S(n) < \sqrt{n}\sqrt{n} = n$ since $2 < \sqrt{n}$ and $S(n) \leq \sqrt{n}$. Thus $X \subseteq \{1, 2, \}$. But $S(n) = 1$ for $n \in \{1, 2, 3\}$, and so $S(n)+S(m) = 2$ for all $n, m \in X$. Thus $X \subseteq \{1, 2, 3\}$ is rooted if and only if $2 \in X$ and the four possibilities are $\{2\}$, $\{1, 2\}$, $\{2, 3\}$ and $\{1, 2, 3\}$.

Problem 4. A tetrahedron has as base equilateral triangle ABC and a fourth vertex V not in the plane of $\triangle ABC$ such that $|VA| = |VB| = |VC|$. Thus, $\triangle VAB$, $\triangle VBC$ and $\triangle VCA$ are congruent isosceles triangles. Let α be the angle at vertex V in each of these triangles, and let β be the interior angle of the tetrahedron between the planes of any two of these triangles. Express $y = \cos(\beta)$ in terms of $x = \cos(\alpha)$.

Solution: Define the unit vectors $u = \vec{AV}/|\vec{AV}|$, $v = \vec{BV}/|\vec{BV}|$ and $w = \vec{CV}/|\vec{CV}|$. Then $u \cdot v = v \cdot w = w \cdot u = x = \cos(\alpha)$. Now $u - xv$ is a vector in $\triangle VAB$ perpendicular to BV and $w - xv$ is a vector in triangle VBC perpendicular to BV. We have $|u-xv|^2 = (u-xv)\cdot(u-xv) = u\cdot u - 2x(u\cdot v) + x^2(v\cdot v) = 1 - 2x^2 + x^2 = 1 - x^2$, so $|w - xv| = |u - xv| = \sqrt{1-x^2}$. We have

$$y = \frac{(u-xv)\cdot(w-xv)}{|u-xv||w-xv|} = \frac{u\cdot w - x(u\cdot v) - x(v\cdot w) + x^2(v\cdot v)}{1-x^2} =$$

$$= \frac{x - x^2 - x^2 + x^2}{1-x^2} = \frac{x-x^2}{1-x^2} = \frac{x}{1+x}.$$

Problem 5. Let a be a fixed real number strictly between -2 and 2 and let A_n be the $n \times n$ matrix which has a along the diagonal, 1's along the super- and sub-diagonals and 0's everywhere else. For example, we have

$$A_2 = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}, \quad A_3 = \begin{pmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{pmatrix}, \quad A_4 = \begin{pmatrix} a & 1 & 0 & 0 \\ 1 & a & 1 & 0 \\ 0 & 1 & a & 1 \\ 0 & 0 & 1 & a \end{pmatrix}.$$

Prove that $\det(A_n)$ is negative for infinitely many n.

Solution: Expanding the determinant along the first column of the matrix yields the recurrence relation $\det(A_{n+1}) = a\det(A_n) - \det(A_{n-1})$. We define $x_n = \det(A_n)$. Then we have $x_0 = 1$, $x_1 = a$ and $x_{n+1} = ax_n - x_{n-1}$ for $n \geq 1$. We apply standard techniques for solving recurrence relations. The characteristic equation of the recurrence relation is

$$\lambda^2 - a\lambda + 1 = 0.$$

The two solutions of this equation are

$$\lambda_1 = \frac{a + i\sqrt{4-a^2}}{2} \text{ and } \lambda_2 = \frac{a - i\sqrt{4-a^2}i}{2}.$$

The general solution for the recurrence relation is

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n$$

for some complex numbers c_1, c_2. The initial conditions give us

$$1 = c_1 + c_2 \text{ and } a = c_1\lambda_1 + c_2\lambda_2.$$

We solve for c_1 and c_2 and obtain

$$c_1 = \frac{\lambda_2 - a}{\lambda_2 - 1} \text{ and } c_2 = \frac{\lambda_1 - a}{\lambda_2 - 1}.$$

Note that λ_2 is the complex conjugate of λ_1 and c_2 is the complex conjugate of c_1. Also, λ_1 is not real and has absolute value 1. It follows that

$$x_n = \frac{1}{2}\,\mathrm{Re}(c_1\lambda_1^n)$$

where $\mathrm{Re}(z)$ denotes the real part of a complex number z. We can write $c_1 = Ce^{i\gamma}$ and $\lambda_1 = e^{i\alpha}$ where $C \neq 0$ and $0 < \alpha < \pi$, so

$$x_n = \frac{1}{2}C\,\mathrm{Re}(e^{i(\gamma+n\alpha)}).$$

Suppose we are given a positive integer n. We can write $\gamma+n\alpha = 2\pi k+\beta$ where $-\frac{1}{2}\pi \leq \beta < \frac{3}{2}\pi$. There exists a nonnegative integer $m \geq n$ such that $\frac{1}{2}\pi < \gamma+m\alpha-2\pi k < \frac{3}{2}\pi$. It follows that x_m is negative. We have proven that there are infinitely many positive integers m for which x_m is negative.

Problem 6. Show that for every positive integer k there is an integer n_k whose decimal expansion uses only the digits 1 and 2, such that 2^k divides n_k. For example $2 \mid 2$, $4 \mid 12$ and $8 \mid 112$.

Solution: We prove by induction on k that there exists an integer n_k with *exactly* k digits wich are all either 1 or 2, such that 2^k divides n_k. The cases $k = 1, 2, 3$ already have been done. Suppose that n_k is a k-digit number consisting of 1's and 2's and 2^k divides n_k. If 2^{k+1} divides n_k then 2^{k+1} divides $n_{k+1} := 2\cdot 10^k + n_k$ (which is the decimal number n_k preceded by a 2). If 2^{k+1} does not divide n_k then $5^k + n_k/2^k$ is even, so 2^{k+1} divides $n_{k+1} := 2^k(5^k + n_k/2^k) = 10^k + n_k$ (which is n_k preceded by a 1).

Problem 7. Show that for any polynomial $p(z) \in \mathbb{R}[z]$ there is a nonzero polynomial $q(z) \in \mathbb{R}[z]$ such that $p(z)q(z) = \sum_k c_k z^k$ has the property that if c_k is nonzero then k is prime. For example, if we are given $p(z) = 1 + 2z + 3z^2$, then we can take $q(z) = 2z^2 - 3z^3$, for then $p(z)q(z) = 2z^2 + z^3 - 9z^5$ has the required form.

Proof: The quotient ring $\mathbb{R}[z]/(p(z))$ is a d-dimensional vector space where d is the degree of $p(z)$. If $r_0, r_1, \ldots, r_d$ are $d+1$ distinct primes then the images of $z^{r_0}, z^{r_1}, \ldots, z^{r_d}$ in $\mathbb{R}[z]/(p(z))$ will be dependent over $\mathbb{R}$. This means that there exist real numbers $a_0, a_1, \ldots, a_d \in \mathbb{R}$ for which $a_0 z^{r_0} + a_1 z^{r_1} + \cdots + a_d^{r_d}$ is divisible by the polynomial $p(z)$.

Problem 8. Given a natural number n let $\mathcal{S}_n$ denote the set of all integers m such that $\{n/m\} \geq 1/2$, where $\{x\} = x - \lfloor x \rfloor$ denotes the "fractional part" of x. Prove that

$$\sum_{m\in\mathcal{S}_n} \phi(m) = n^2,$$

where $\phi(m)$ is Euler's ϕ-function (that is, the number of integers k in $\{1, 2, \ldots, m\}$ that are coprime to m). For example $\mathcal{S}_6 = \{4, 7, 8, 9, 10, 11, 12\}$, and we see that $\phi(4)+\phi(7)+\phi(8)+\phi(9)+\phi(10)+\phi(11)+\phi(12) = 2+6+4+6+4+10+4 = 36 = 6^2$.

Solution: Let $\lfloor x \rfloor$ denote the integer part of the positive real number x. Note that $\lfloor 2x \rfloor - 2\lfloor x \rfloor = 1$ if $\{x\} \geq 1/2$ and is 0 otherwise. Thus

$$\sum_{m \in \mathcal{S}_n} \phi(m) = \sum_m \phi(m)\Big(\Big\lfloor \frac{2n}{m} \Big\rfloor - 2\Big\lfloor \frac{n}{m} \Big\rfloor\Big). \tag{1}$$

Now observe that

$$\sum_{n \leq k} \sum_{d|n} \phi(d) = \sum_d \phi(d) \sum_{n \leq k, d|n} 1 = \sum_d \phi(d) \Big\lfloor \frac{k}{d} \Big\rfloor,$$

and as $\sum_{d|n} \phi(d) = n$ we see that the left-hand side also equals

$$\sum_{n \leq k} n = \frac{k(k+1)}{2}.$$

Using this in (1) we get that

$$\sum_{m \in \mathcal{S}_n} \phi(m) = \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} = n^2.$$

Problem 9. Suppose that you are playing the following game. First a random number x_0 is chosen from the interval $[0,1]$ (with a uniform distribution). In round 1, a second number x_1 will be chosen randomly from the interval $[0,1]$ — but before this, you have to guess if this number is going to be *higher* or *lower* than the previous number x_0. If you were wrong, then the game is over. Otherwise you will proceed to round 2. The other rounds proceed similarly: In round k you first guess "higher" or "lower". Then a random number x_k is chosen from the interval $[0,1]$. If you said "higher" and $x_k > x_{k-1}$ or you said "lower" and $x_k < x_{k-1}$ then you proceed to round $k+1$. Otherwise the game will end in round k. Assume that you are using a strategy that in each round maximalizes the probability to proceed to the next round.

(a) What is the probability that the game will last 3 or more rounds? In other words, what is the probability that the first two guesses will be right?
(b) What is the expected number of rounds that will be played?

Solution: The best strategy of course is the say "higher" in round k if $x_{k-1} < \frac{1}{2}$ and "lower" in round k if $x_{k-1} > \frac{1}{2}$. If $0 \leq y \leq \frac{1}{2}$, let $f_k(y)$ be the probability that the game lasts at least $k+1$ rounds, given that $x_k = y$. By symmetry $f_k(1-y)$ is the probability that the game last at least $k+1$ rounds if $x_k = y$ and $\frac{1}{2} \leq y \leq 1$. Also note that $f_k(\frac{1}{2})$ is the probabilty that the game lasts at least k rounds (if a game lasts k rounds and $x_k = \frac{1}{2}$, then the k-th guess will be right with probability 1). We try to find a recursive formula for $f_{k+1}(y)$. Suppose that $x_{k+1} = y$ with $0 \leq y \leq \frac{1}{2}$ and

$x_k = z$. The game lasts at least $k+2$ rounds if it lasts at least $k+1$ rounds and either $z \le \frac{1}{2}$ and $z \le y$ or $z \ge \frac{1}{2}$. It follows that

$$f_{k+1}(y) = \int_0^z f_k(y)\,dy + \int_{\frac{1}{2}}^1 f_k(1-y)\,dy = \int_0^z f_k(y)\,dy + \int_0^{\frac{1}{2}} f_k(y)\,dy$$

Also note that $f_0(y) = 1$. We compute $f_1(y) = y + \frac{1}{2}$, $f_2(y) = \frac{1}{2}y^2 + \frac{1}{2}y + \frac{3}{8}$ and $f_3(y) = \frac{1}{6}y^3 + \frac{1}{4}y^2 + \frac{3}{8}y + \frac{13}{48}$. The probability that the game lasts at least 3 rounds is $f_3(\frac{1}{2}) = \frac{13}{24}$. This solves (a).

We create a generating function

$$F(x,y) = \sum_{k=0}^{\infty} f_k(y)x^k.$$

We see that

$$\frac{\partial F(x,y)}{\partial y} = xF(x,y)$$

because $f_k(y)' = f_{k-1}(y)$ for all $k \ge 1$ and $f_0(y)' = 0$. Solution to this differential equation are of the form

$$F(x,y) = A(x)e^{xy}.$$

for some function $A(x)$. Note that $f_k(\frac{1}{2}) = 2f_k(0)$ for $k \ge 1$ and $f_k(\frac{1}{2}) = 2f_k(0) - 1$ for $k = 0$. It follows that

$$F(x, \frac{1}{2}) = 2F(x,0) - 1.$$

So we have

$$A(x)e^{\frac{1}{2}x} = 2A(x) - 1.$$

We solve

$$A(x) = \frac{1}{2 - e^{\frac{1}{2}x}}.$$

The expected number of rounds is

$$\sum_{k=1} \text{probability of} \ge k \text{ rounds} = \sum_{k=1}^{\infty} f_k(\tfrac{1}{2}) = F(x, \tfrac{1}{2}) - 1 = \frac{e^{\frac{1}{2}x}}{2 - e^{\frac{1}{2}x}} - 1 = \frac{2\sqrt{e} - 2}{2 - \sqrt{e}}.$$

This solves (b).

Problem 10. Suppose that $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are real numbers such that

$$y_1 \ge y_2 \ge \cdots \ge y_n > 0$$

and

$$x_1 x_2 \cdots x_k \ge y_1 y_2 \cdots y_k, \qquad \text{for } k = 1, 2, \ldots, n.$$

Prove that

$$x_1 + x_2 + \cdots + x_n \ge y_1 + y_2 + \cdots + y_n.$$

Solution: We prove the statement by induction on n, the case $n = 1$ being trivial. Choose some constant $C > y_1 + y_2 + \cdots + y_n$ and define the compact set

$$S = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 x_2 \cdots x_k \geq y_1 y_2 \cdots y_k \text{ for all } k, \text{ and } s \leq C\},$$

where $s = x_1 + x_2 + \cdots + x_n$ is the sum function. Let $(x_1, x_2, \ldots, x_n) \in S$ be a point where s is minimal. This point must lie on the boundary of S. There are three cases:

case 1: $s = x_1 + x_2 + \cdots + x_n = C$, but this is absurd. If we take $x_i = y_i$ for all i we would get a smaller value for s.

case 2: $x_1 x_2 \cdots x_k = y_1 y_2 \cdots y_k$ for some $k < n$. By induction we already have

$$x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k. \tag{2}$$

Also note that

$$x_{k+1} x_{k+2} \cdots x_{k+l} \geq y_{k+1} y_{k+2} \cdots y_{k+l}, \quad \text{for } l = 1, 2, \ldots, n-k.$$

Again by induction we have that

$$x_{k+1} + x_{k+2} + \cdots + x_n \geq y_{k+1} + y_{k+2} + \cdots + y_n$$

Combined with (2) this gives

$$s = x_1 + x_2 + \cdots + x_n \geq y_1 + y_2 + \cdots + y_n.$$

case 3: $x_1 x_2 \cdots x_n = y_1 y_2 \cdots y_n$ and $x_1 x_2 \cdots x_k > y_1 y_2 \cdots y_k$ for $k < n$. The theory of Lagrange multiplyiers tells us that

$$\nabla s = (1, 1, \ldots, 1)$$

and

$$\nabla p = \left(\frac{p}{x_1}, \frac{p}{x_2}, \ldots, \frac{p}{x_n}\right)$$

are linearly dependent, where $p = x_1 x_2 \cdots x_n$. It follows that $x_1 = x_2 = \cdots = x_n$ and

$$x_1 \geq y_1 \geq y_2 \geq \cdots \geq y_n = \frac{y_1 y_2 \cdots y_n}{y_1 y_2 \cdots y_{n-1}} \geq \frac{x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_{n-1}} = x_n = x_1.$$

So we have

$$x_1 = x_2 = \cdots = x_n = y_1 = y_2 = \cdots = y_n$$

and

$$s = x_1 + x_2 + \cdots + x_n \geq y_1 + y_2 + \cdots + y_n.$$