UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 20
SOLUTIONS

HARM DERKSEN

Problem 1. Let A, B, C, D be the vertices of a square, in clockwise order. Let P be a point inside the square such that the distance from P to A is 7, the distance from P to B is 9, the distance from P to C is 6. What is the distance from P to D?

Solution: Let s, t, u, v be the distances from P to AB, BC, CD and DA respectively. We have

$$49 + 36 = |PA|^2 + |PC|^2 = (v^2 + s^2) + (t^2 + u^2) =$$

$$= (s^2 + t^2) + (u^2 + v^2) = |PB|^2 + |PD|^2 = 81 + |PD|^2.$$

It follows that $|PD|^2 = 49 + 36 - 81 = 4$ and $|PD| = \sqrt{4} = 2$.

Problem 2. Define

$$a_n = \sqrt{1^2 + \sqrt{2^2 + \sqrt{3^2 + \cdots + \sqrt{n^2}}}}.$$

Is the sequence $a_1, a_2, a_3, \ldots$ bounded?

Solution: Define

$$b_{m,n} = \sqrt{m^2 + \sqrt{(m+1)^2 + \cdots + \sqrt{n^2}}}.$$

By (decreasing) induction on m we prove that $b_{m,n} \leq m + 1$ if $1 \leq m \leq n$. The case $m = n$ is clear because $b_{n,n} = \sqrt{n^2} = n \leq n + 1$. Let us assume that $m < n$ and that $b_{m+1,n} < m + 2$. Then we get

$$b_{m,n} = \sqrt{m^2 + b_{m+1,n}} \leq \sqrt{m^2 + (m+2)} \leq \sqrt{m^2 + 2m + 1} = m + 1.$$

This proves that $b_{m,n} \leq m + 1$ for all m, n and in particular $a_n = b_{1,n} \leq 1 + 1 = 2$ for all n.

Problem 3. You are playing a game. Your opponent chooses a polynomial P with non-negative integer coefficients: you don't know what it is. You are allowed to choose an integer a and ask for the value of $P(a)$. You may then choose an integer b and ask for the value of $P(b)$. After that, to win, you must determine what the polynomial is. Is there a foolproof strategy for winning this game?

Solution:[1] Choose $a = 1$: then $P(a)$ is an upper bound for the coefficients. Choose $b > P(a)$, say $b = P(a) + 1$. The value of $P(b)$, written as an integer in base b, gives the sequence of coefficients of the polynomial.

Problem 4. A large collection of coins of varying weights is partitioned into n mutually disjoint subsets whose weights are $w_1 \leq w_2 \leq \cdots \leq w_n$. The same coins are then partitioned into n mutually disjoint subsets in another way so that their weights are $W_1 \geq W_2 \geq \cdots \geq W_n$. Show that for every k, $1 \leq k \leq n$,

$$W_1 + \cdots + W_k \geq w_1 + \cdots + w_k.$$

Solution: If $W_k \geq w_k$ then we have

$$W_1 + W_2 + \cdots + W_k \geq kW_k \geq kw_k \geq w_1 + w_2 + \cdots + w_k.$$

If $W_k < w_k$ then we have

$$W_{k+1} + W_{k+2} + \cdots + W_n \leq (n-k)W_k < (n-k)w_k \leq w_{k+1} + w_{k+2} + \cdots + w_n.$$

If we subtract this equation from

$$W_1 + W_2 + \cdots + W_n = w_1 + w_2 + \cdots + w_n$$

we get

$$W_1 + W_2 + \cdots + W_k > w_1 + w_2 + \cdots + w_k.$$

Problem 5. Given 4 points in Euclidean 3-space, not all lying in the same plane, how many planes are there such that the distance from the plane to each of the four points is the same?

Solution: Call the points A, B, C, D. Either (i) all four points are on one side of the plane, (ii) there is one point on one side of the plane and there are three points on the other side of the plane or (iii) there are two points on each side of the plane.

(i) If the four points lie on the same side of the plane and they have the same distance to the plane then the four points lie in a plane. Constradiction.

(ii) There are 4 ways of partitioning the four points into a group of three points and a single point. For each such partition there is exactly one plane with equal distance to the 4 points. For example, there is exactly one plane such that A is one one side, and B, C and D are on the other side. To see this, note that each such plane would have to go through the middle of AB and it would have to be parallel to the vectors $\vec{BC}$ and $\vec{BD}$. On the other hand, since the vectors $\vec{BC}$ and $\vec{BD}$ are linearly independent (because B, C, D are not on one line), there is exactly one plane going through the middle of AB which is parallel to $\vec{BC}$ and $\vec{BD}$.

(iii) There are 3 ways of partitioning the four points into two groups of two points. For each such partition there is exactly one plane with equal distance to the 4 points. For example, there is exactly one plane such that A and B are on one side of the plane and C and D are on the other side. To see this, note that each such plane would

[1] This is Mel Hochster's solution.

have to go through the middle of AC and it would have to be parallel to $\vec{AB}$ and $\vec{CD}$. On the other hand, since the vectors $\vec{AB}$ and $\vec{CD}$ are linearly independent (otherwise A, B, C, D would lie in a plane) there is exactly one plane through the middle of CD, parallel to $\vec{AB}$ and $\vec{CD}$.

The total number of planes with equal distance is $4 + 3 = 7$.

Problem 6. Let $p(z)$ be a polynomial with complex coefficients satisfying

$$|p(j) - 3^j| < 1$$

for $j = 0, 1, 2, \ldots, n$. Show that $p(z)$ has degree at least n.

Solution: We prove the statement by induction on n. Suppose $n = 0$. Then

$$|p(0) - 3^0| < 1$$

which implies that p is a nonzero polynomial of degree ≥ 0.

Suppose $n > 0$. Consider the polynomial $q(z) = (p(z+1) - p(z))/2$. of degree at most $\deg(p(z)) - 1$. Consider

$$|q(j) - 3^j| = |\tfrac{1}{2}q(j+1) - \tfrac{1}{2}q(j) - \tfrac{1}{2}3^{j+1} + \tfrac{1}{2}3^j| =$$

$$= |\tfrac{1}{2}(q(j+1) - 3^{j+1}) - \tfrac{1}{2}(q(j) - 3^j)| \leq \tfrac{1}{2}|q(j+1) - 3^{j+1}| + \tfrac{1}{2}|q(j) - 3^j| < \tfrac{1}{2} + \tfrac{1}{2} = 1$$

for $j = 0, 1, 2, \ldots, n-1$. By the induction hypothesis we have that $\deg(q(z)) \geq n - 1$, so $\deg(p(z)) \geq \deg(q(z)) + 1 \geq n$.

Problem 7. At time $t = 0$, n particles are at given positions on the unit circle (we will think of a particle as a point). Each particle moves at constant speed 1 over the unit circle, either in clockwise or in counterclockwise direction. If two particles meet, they will bounce in opposite direction again with speed 1. Show that after some time all particles will be in their original position again.

Solution: Let $P_i(t)$ be the position of particle i at time t. Suppose that we change the rules and that particles will not bounce when they meet but just move through each other. Let $Q_i(t)$ be the position of particle i at time t with these new rules (we assume the same initial positions and velocities). Note that

$$\{P_1(t), P_2(t), \ldots, P_n(t)\} = \{Q_1(t), Q_2(t), \ldots, Q_n(t)\}$$

for all t. The functions $Q_1, Q_2, \ldots, Q_n$ are clearly periodic with period 2π. We see that at time $t = 2\pi$ we get

$$\{P_1(2\pi), \ldots, P_n(2\pi)\} = \{Q_1(2\pi), \ldots, Q_n(2\pi)\} =$$

$$= \{Q_1(0), \ldots, Q_n(0)\} = \{P_1(0), \ldots, P_n(0)\}.$$

So $P_i(2\pi) = P_{\sigma(i)}(0)$ for $i = 1, 2, \ldots, n$ for some permutation σ. (Note also that particle P_i will have the same speed at $t = 2\pi$ as the speed of $P_{\sigma(i)}$ at $t = 0$.) Since the permutation σ has finite order, say N, we see that all the particles will be in their original position (and with their original speed) at time $t = 2\pi N$.

Problem 8. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that for every $y \in \mathbb{R}$, the equation $f(x) = y$ has exactly 2 distinct solutions for x. Show that f cannot be continuous.

Solution: Let us assume that $f(x)$ is continuous. The equation $f(x) = 0$ has two solutions, say $x = a$ and $x = b$. Now either $f(x) > 0$ on $[a, b]$ or $f(x) < 0$ on $[a, b]$.

Let us assume that $f(x) > 0$ on $[a, b]$. Since f is continuous, f will have a maximum on the interval $[a, b]$, say at $x = c$. the equation $f(x) = f(c) + 1$ has two solutions. Suppose $x = d$ is one of them. We have either $d < a$ or $d > b$. If $d < a$ then $f(x) = \frac{1}{2}f(c)$ has solutions on each interval (d, a), (a, c) and (c, b) by the intermediate value theorem. Contradiction! If $d > b$ then $f(x) = \frac{1}{2}f(c)$ has solutions on each interval (a, c), (c, b) and (b, d). Contradiction.

In the second case we have $f(x) < 0$ on $[a, b]$ .Now $g(x) = -f(x)$ is continuous, $g(a) = g(b) = 0$ and $g(x) > 0$ on $[a, b]$. A contradiction follows from the first case.

Problem 9. Let $\alpha = 0.99$. Find $\epsilon_0, \epsilon_1, \dots, \epsilon_7 \in \{-1, 1\}$ such that

$$|\epsilon_0 + \epsilon_1\alpha + \epsilon_2\alpha^2 + \dots + \epsilon_7\alpha^7| < 0.000008 = 8 \cdot 10^{-6}.$$

Solution: Using the Bernoulli inequality $(1 - \beta)^n \geq 1 - n\beta$ for $\beta = 0.01$ and $n = 2, 4$ we see that

$$0.000008 = 0.01 \cdot 0.02 \cdot 0.04 \leq |(1 - \alpha)(1 - \alpha^2)(1 - \alpha^4)| =$$

$$= |1 - \alpha - \alpha^2 + \alpha^3 - \alpha^4 + \alpha^5 + \alpha^6 - \alpha^7|$$

Problem 10. An unbalanced penny and an unbalanced quarter, with probabilities of heads p for the penny and q for the quarter, are tossed together over and over. The probability that the penny shows heads (strictly) before the quarter is $3/5$, and the number of tosses required for both coins to show heads simultaneously has expected value exactly 4. Find the values of p and q.

*Solution:*The probability that the penny shows head before the quarter is

$$p(1-q)(1 + (1-p)(1-q) + (1-p)^2(1-q)^2 + \cdots) = \frac{p(1-q)}{1 - (1-p)(1-q)} = \frac{p - pq}{p + q - pq}.$$

so we have

$$\frac{p - pq}{p + q - pq} = \frac{3}{5}.$$

The expected number of tosses needed for both coins to show head simultaneously is

$$pq + 2pq(1 - pq) + 3pq(1 - pq)^2 + \cdots = \frac{pq}{(1 - (1 - pq))^2} = \frac{pq}{(pq)^2} = \frac{1}{pq}.$$

We have

$$\frac{1}{pq} = 4.$$

We deduce $pq = \frac{1}{4}$. We get

$$\frac{4p - 4pq}{4p + 4q - 4pq} = \frac{4p - 1}{4p + 4q - 1} = \frac{3}{5}$$

so

$$20p - 5 = 12p + 12q - 3.$$

It follows that

$$8p - 12q - 2 = 0$$

so

$$\frac{2}{q} - 2 - 12q = 0$$

and

$$\frac{1}{q^2} - \frac{1}{q} - 6 = \left(\frac{1}{q} - 3\right)\left(\frac{1}{q} + 2\right) = 0.$$

Since $q > 0$ we must have $q = \frac{1}{3}$. Because $pq = \frac{1}{4}$ we get $p = \frac{3}{4}$.

REMARKS AND FURTHER QUESTIONS

Remark 1. In problem 2, $a_1, a_2, \ldots$ is an increasing bounded sequence. Therefore, the limit $a = \lim_{n\to\infty} a_n$ must exist.What is a?

Remark 2. Can one generalize problem 3 to a multivariate polynomial? Suppose that P is a polynomial in k variables with nonnegative integer coefficients. Your are allowed to choose an integer vector $a \in \mathbb{Z}^k$ and ask for $P(a)$. Then your allowed to choose an integer vector $b \in \mathbb{Z}^k$ and ask for $P(b)$. After this, you must determine all the coefficients of the polynomial P.

Remark 3. In problem 7, how will the particles permute at time $t = 2\pi$. Does this permutation only depend on the initial velocities?

Remark 4. Related to problem 8, one could ask if there exists a (discontinuous) function $f : \mathbb{R} \to \mathbb{R}$ such that the graph of f intersects the graph of $y = ax + b$ in exactly two points for all $a, b \in \mathbb{R}$.

THANKS

Thanks to Jason Bell, Bogdan Ion, Mel Hochster and Hugh Montgomery for submitting problems.