UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 20
MARCH 29, 2003

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Let A, B, C, D be the vertices of a square, in clockwise order. Let P be a point inside the square such that the distance from P to A is 7, the distance from P to B is 9, the distance from P to C is 6. What is the distance from P to D?

Problem 2. Define

$$a_n = \sqrt{1^2 + \sqrt{2^2 + \sqrt{3^2 + \cdots + \sqrt{n^2}}}}.$$

Is the sequence $a_1, a_2, a_3, \ldots$ bounded?

Problem 3. You are playing a game. Your opponent chooses a polynomial P with non-negative integer coefficients: you don't know what it is. You are allowed to choose an integer a and ask for the value of $P(a)$. You may then choose an integer b and ask for the value of $P(b)$. After that, to win, you must determine what the polynomial is. Is there a foolproof strategy for winning this game?

Problem 4. A large collection of coins of varying weights is partitioned into n mutually disjoint subsets whose weights are $w_1 \leq w_2 \leq \cdots \leq w_n$. The same coins are then partitioned into n mutually disjoint subsets in another way so that their weights are $W_1 \geq W_2 \geq \cdots \geq W_n$. Show that for every k, $1 \leq k \leq n$,

$$W_1 + \cdots + W_k \geq w_1 + \cdots + w_k.$$

Problem 5. Given 4 points in Euclidean 3-space, not all lying in the same plane, how many planes are there such that the distance from the plane to each of the four points is the same?

Problem 6. Let $p(z)$ be a polynomial with complex coefficients satisfying

$$|p(j) - 3^j| < 1$$

for $j = 0, 1, 2, \ldots, n$. Show that $p(z)$ has degree at least n.

Problem 7. At time $t = 0$, n particles are at given positions on the unit circle (we will think of a particle as a point). Each particle moves at constant speed 1 over the unit circle, either in clockwise or in counterclockwise direction. If two particles meet, they will bounce in opposite direction again with speed 1. Show that after some time all particles will be in their original position again.

Problem 8. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that for every $y \in \mathbb{R}$, the equation $f(x) = y$ has exactly 2 distinct solutions for x. Show that f cannot be continuous.

Problem 9. Let $\alpha = 0.99$. Find $\epsilon_0, \epsilon_1, \ldots, \epsilon_7 \in \{-1, 1\}$ such that

$$|\epsilon_0 + \epsilon_1\alpha + \epsilon_2\alpha^2 + \cdots + \epsilon_7\alpha^7| < 0.000008 = 8 \cdot 10^{-6}.$$

Problem 10. An unbalanced penny and an unbalanced quarter, with probabilities of heads p for the penny and q for the quarter, are tossed together over and over. The probability that the penny shows heads (strictly) before the quarter is $3/5$, and the number of tosses required for both coins to show heads simultaneously has expected value exactly 4. Find the values of p and q.