UNIVERSITY OF MICHIGAN UNDERGRADUATE MATHEMATICS COMPETITION 19 SOLUTIONS

Problem 1. *Given a set S with n elements (n is a positive integer), what is the number of subsets of subsets of S? (More precisely, we want to count the number of pairs of subsets (X, Y) with $X \subseteq Y \subseteq S$.)*

Solution: For every element $a \in S$, there are three possibilities, namely $a \in X$, $a \in Y \setminus X$ or $a \in S \setminus Y$. Since S has n elements, there are 3^n possibilities in total.

Problem 2. *What is the smallest positive integer k such that the sum of the decimal digits of $k(10^{2002} - 1)$ is not equal to 18018?*

Solution: The number $10^{2002} - 1$ consists of 2002 9's. The sum of the digits of this number is equal to $9 \cdot 2002 = 18018$. If $k \leq 10^{2002}$ is a positive integer, then the sum of the digits of $10^{2002} - k = (10^{2002} - 1) - (k - 1)$ is $18018 - s$ where s is the sum of the digits of $k - 1$. Therefore the sum of the digits of

$$k(10^{2002} - 1) = (k - 1)10^{2002} + ((10^{2002} - 1) - (k - 1))$$

is equal to $a + 18018 - a = 18018$. The sum of the digits of

$$(10^{2002} + 1)(10^{2002} - 1) = 10^{4004} - 1$$

is equal to 36036 because this number consists of 4004 9's and $9 \cdot 4004 = 36036$. We have shown that $k = 10^{2002} + 1$ is the smallest positive integer k such that the sum of the digits of $k \cdot (10^{2002} - 1)$ is not equal to 36036.

Problem 3. *Suppose that $f \in C^\infty(-1, 1)$, and suppose that there exist points $x_1, x_2, x_3, \ldots$ in $(-1, 1)$ with $\lim_{n\to\infty} x_n = 0$ and $f(x_n) = 0$ for all n. Prove that every derivative of f vanishes at $x = 0$. (By $f \in C^\infty(-1, 1)$ we mean that f is a real valued function on the interval $(-1, 1)$ whose k-th derivative exists for any k.)*

Solution: By induction on k we will show that $f^{(k)}(0) = 0$ ($f^{(k)}(x)$ is the k-th derivative). Since f is continuous, we get $f(0) = \lim_{n\to\infty} f(x_n) =$

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0. If $x_i < x_{i+1}$ then by the mean value theorem there exists a $y_i \in (x_i, x_{i+1})$ such that

$$f'(y_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = 0.$$

Similarly if $x_i > x_{i+1}$ we can find a $y_i \in (x_{i+1}, x_i)$ with

$$f'(y_i) = \frac{f(x_i) - f(x_{i+1})}{x_i - x_{i+1}} = 0.$$

Since $|y_i| \leq \max\{|x_i|, |x_{i+1}|\}$ we get $\lim_{n\to\infty} y_n = 0$. We apply the induction hypothesis to f' to see that $f^{(k+1)}(0) = (f')^{(k)}(0) = 0$.

Problem 4. *Start with the sequence*

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$$

(each number n appears n times) and form its partial sums

$$1, 3, 5, 8, 11, 14, 18, 22, 26, 30, \ldots$$

Identify all the prime numbers in the latter sequence.

Solution: All the numbers in the sequence are of the form

$$1^2 + 2^2 + \cdots + n^2 + (n+1)k \tag{1}$$

with $0 \leq k \leq n$. It is known (or one can prove by induction) that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

so (1) is equal to

$$\frac{(2n^2 + n + 6k)(n+1)}{6} = \left(\frac{n+1}{a}\right)\left(\frac{a(2n^2 + n + 6k)}{6}\right)$$

where a is the greatest common divisor of $n+1$ and 6. If $n \geq 6$ then $(n+1)/a$ and $a(2n^2 + n + 6k)/6$ are both integers > 1 and (1) cannot be a prime number. The only possible prime numbers are the cases where $n \leq 5$ and $0 \leq k \leq n$. We find $3, 5, 11, 61, 67, 73, 79$.

Problem 5. *A permutation of the set $X = \{1, 2, \ldots, 2n\}$ is called complementing if there exists an n-element subset $Y \subset X$ such that $\pi(Y)$ is the complement of Y. Show that the number of complementing permutations is a square.*

Solution: Let N be the number of ways to partition a set of $2n$ elements, which we will assume is $\{1, 2, \ldots, 2n$, into n pairs, each with two elements. This gives a non-oriented graph of degree 1 at every vertex. Note that

$$N = \frac{\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2}}{n!} = 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

Given an ordered pair of such graphs, of which there are N^2, the superposition gives a graph with $2n$ edges in which every vertex has degree 2. As such, this graph is a union of disjoint cycles. (There may be instances in which two vertices are connected by two distinct edges.) We can think of edges coming from the first graph as colored blue and those from the second graph as colored red. Then the new graph breaks up into disjoint cycles of even length in which the edges alternate in color in any given cycle. We get a unique permutation from such a cycle decomposition by letting the least element in any given cycle map to the element to which it is connected via a blue edge – that orients the whole cycle.

Conversely, a complementing permutation π gives an ordered pair of partitions: the first pairs the least element u in each cycle with its image under the permutation, and also pairs $\pi^{2i}(u)$ with $\pi^{2i+1}(u)$, and the second pairs u with $\pi^{-1}(u)$ and also pairs $\pi^{-2i}(u)$ with $\pi^{-2i-1}(u)$.

Problem 6. *Show that if s is a real number, $s > 1$, then*

$$\log \frac{s}{s-1} = \int_1^\infty \frac{1-1/x}{\log x} x^{-s}\, dx.$$

(In the formula, log *stands for the natural logarithm.)*

Solution: We have

$$\frac{x^{-s}}{\log x} = \int_s^\infty x^{-t}\, dt.$$

Using this we see that

$$\int_1^\infty \frac{1-1/x}{\log x} x^{-s}\, dx = \int_1^\infty \int_s^\infty (1-1/x) x^{-t}\, dt\, dx =$$
$$= \int_s^\infty \int_1^\infty x^{-t} - x^{-1-t}\, dx\, dt = \int_s^\infty \frac{1}{t-1} - \frac{1}{t}\, dt =$$
$$= -\log(s-1) + \log(s) = \log \frac{s}{s-1}.$$

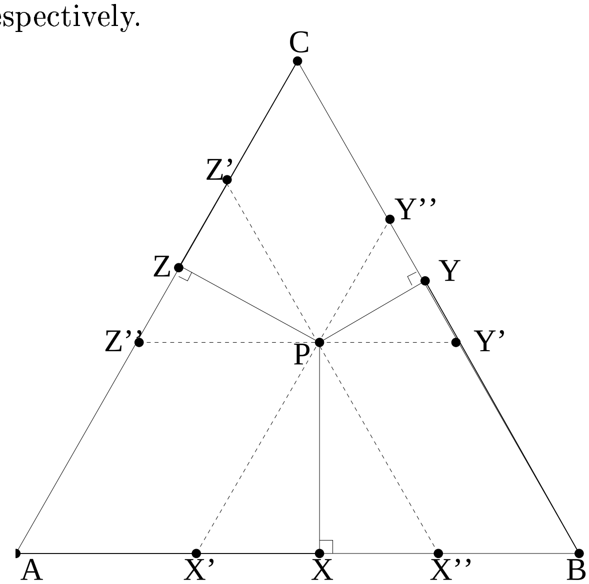
(By absolute convergence, we were allowed to interchange the order of integration.)

Problem 7. *Is there a binary operation $*$ on a set S that consists of three distinct elements that is commutative, i.e., $x * y = y * x$ for all $x, y \in S$, and also satisfies $x * (x * y) = y$ for all $x, y \in S$?*

Solution: Let $S = \{a, b, c\}$ be a set with three elements and define $*$ on S by $x * x = x$ for all $x \in S$ and $x * y = z$ whenever $x, y, z \in S$ are pairwise distinct. We check that $x * (x * x) = x * x = x$ for all $x \in S$ and $x * (x * y) = x * z = y$ if $x, y, z \in S$ are pairwise distinct. So indeed, $x * (x * y) = y$ for *all* $x, y \in S$.

Problem 8. *An interior point P in an equilateral triangle ABC is connected to the vertices, and perpendiculars are dropped to the sides, hitting AB, BC, and CA at X, Y, and Z respectively. Is it necessarily true that the sum of the lengths of AX, BY, and CZ is equal to half the perimeter of the triangle?*

Solution: The answer is yes. Suppose that the line through P parallel to AB intersects BC and CA in Y' and Z'' respectively, the line through P parallel to BC intersects CA and AB in Z' and X'' respectively and the line through P parallel to CA intersects AB and BC in X' and Y'' respectively.



It is immediately clear now that $|AX'| = |CY''|$, $|BY'| = |AZ''|$, $|CZ'| = |BX''|$, $|XX'| = |XX''|$, $|YY'| = |YY''|$ and $|ZZ'| = |ZZ''|$. It follows that

$$|AX|+|BY|+|CZ| = |AX'|+|BY'|+|CZ'|+|XX'|+|YY'|+|ZZ'| =$$
$$\tfrac{1}{2}(|AX'| + |CY''| + |BY'| + |AZ''| + |CZ'| + |BX''|+$$
$$+|XX'| + |XX''| + |YY'| + |YY''| + |ZZ'| + |ZZ''|)$$

is exactly half the perimeter of the triangle.

Second solution: Write $|PA| = a$, $|PB| = b$, $|PC| = c$, $|PX| = x$, $|PY| = y$, $|PZ| = z$, $|AX| = p$, $|BY| = q$ and $|CZ| = r$. Without loss of generality we may assume that the sidelength of the equilateral triangle is 1. By Pythagoras we have $p^2 = a^2 - x^2$, $q^2 = b^2 - y^2$,$r^2 = c^2 - z^2$, $(1-p)^2 = b^2 - x^2$, $(1-q)^2 = c^2 - y^2$ and $(1-r)^2 = a^2 - z^2$. Adding these gives

$$p^2+q^2+r^2 = (a^2+b^2+c^2)-(x^2+y^2+z^2) = (1-p)^2+(1-q)^2+(1-r)^2.$$

After expanding and cancellation we get $p+q+r=\frac{3}{2}$ which is half of the perimeter.

Problem 9. *Let $n \geq 2$ be a positive integer. Show that every complex number c with $|c| \leq n$ can be written as $c = a_1 + a_2 + \cdots + a_n$ where $|a_j| = 1$ for every j.*

Solution: Define $b_k(t) = e^{ikt} = \cos(kt) + i\sin(kt)$ for all i where $0 \leq t \leq \frac{2\pi}{n}$. Also define

$$B(t) = b_1(t) + b_2(t) + \cdots + b_n(t).$$

We have $|B(0)| = n$ and $|B(\frac{2\pi}{n})| = 0$, because the vectors

$$b_1(\tfrac{2\pi}{n}), b_2(\tfrac{2\pi}{n}), \cdots, b_n(\tfrac{2\pi}{n})$$

form a regular n-gon. The function $|B(t)|$ is continuous, so by the intermediate value theorem there exists a $s \in [0, \frac{2\pi}{n}]$ such that $|B(s)| = |c|$. If we define

$$a_i = \frac{b_i(s)c}{B(s)}$$

for $i = 1, \ldots, n$, then

$$a_1 + a_2 + \cdots + a_n = c.$$

Problem 10. *Consider the sequence of first digits in the successive powers of 2:*

$$2, 4, 8, 1, 3, 6, 1, \ldots$$

Does one of the digits 7 and 8 appear more often in the sequence than the other one? (We say for example that 5 appears more often than 6 in the sequence if there exists a positive integer N such that for all $n \geq N$, 5 appears more often than 6 among the first n terms of the sequence.)

Solution: The digit 7 will appear more often. To prove this, we make the following claims:

Claim 1: There exists a positive integer n such that 2^n starts with the digits 875.

Claim 2: There are infinitely many integers k such that 2^k starts with the digits 79.

From the claims follows that 7 appears more often. Indeed, assume that 2^n starts with 875, say $.875 \cdot 10^a < 2^n < .876 \cdot 10^a$. If 2^m starts with an 8, say $.8 \cdot 10^b < 2^m < .9 \cdot 10^b$, then

$$.7 \cdot 10^{a+b} = (.875 \cdot 10^a)(.8 \cdot 10^b) < 2^{m+n} < (.876 \cdot 10^a)(.9 \cdot 10^b) < .7885 \cdot 10^{a+b},$$

so 2^{m+n} will start with a 7 and will not start with the digits 79. Let A be the number of elements in $\{2, 2^2, \ldots, 2^m\}$ starting with an 8 and let B be the number of elements in $\{2, 2^2, \ldots, 2^m\}$ starting with a 7. Let A' be the number of elements in $\{2, 2^2, \ldots, 2^{m-n}\}$ starting with an 8 and let B' be the number of elements in $\{2, 2^2, \ldots, 2^m\}$ starting with the digits 79. If we take m large enough, then $B' > n$. Then we have

$$A \leq A' + n \leq (B - B') + n < B.$$

If $\alpha = \log_{10} 2$ were rational, say $\alpha = p/q$ with p, q positive integers, then $2^p 5^p = 10^p = 2^q$ which contradicts the unique factorization into primes. Therefore, α is irrational. To prove the two claims we need the following result from diomphantine approximation:

Claim 3: For every real number β and every $\varepsilon > 0$ there exist infinitely many positive integers p, q such that $|p\alpha - \beta - q| < \varepsilon$.

Proof. First we do the case $\beta = 0$. Assume $N > 1/\varepsilon$ is an integer and consider $a_k = k\alpha - \lfloor k\alpha \rfloor \in [0, 1]$ for $k = 0, 1, \ldots, N$. The interval $[0, 1]$ can be divided into N intervals

$$[0, \tfrac{1}{N}], [\tfrac{1}{N}, \tfrac{2}{N}], \ldots, [\tfrac{N-1}{N}, 1]$$

Then there must be at least 2 numbers, say a_k and a_l with $0 \leq k < l \leq N$ wich lie in the same interval. In particular

$$|(l - k)\alpha - (\lfloor l\alpha \rfloor - \lfloor k\alpha \rfloor)| = |a_l - a_k| \leq \frac{1}{N} < \varepsilon$$

so we can take $p = l - k$ and $q = \lfloor l\alpha \rfloor - \lfloor k\alpha \rfloor$.

Now suppose $\beta > 0$ (the case $\beta < 0$ then also follows). We already have found positive integers p and q such that $0 < \delta := |p\alpha - q| < \varepsilon$. By allowing p and q to be negative we may assume that $\delta = p\alpha - q$. Choose an positive integer N such that $N \leq \beta/\delta < N + 1$. Then $\beta < (N + 1)p\alpha - (N + 1)q - \beta < \beta + \delta$ so

$$|(N + 1)p\alpha - (N + 1)q - \beta| < \delta < \varepsilon.$$

□

Proof of Claim 1. If we take $\beta = \log_{10} 875.5$ then Claim 3 guarantees the existence of (infinitely many) positive integers n and m such that

$$|n\alpha - m - \beta| < \varepsilon$$

for any $\varepsilon > 0$. By taking ε small enough we get

$$\log_{10} 875 < \log_{10} 2^n = n\alpha - m < \log_{10} 876$$

which implies that 2^n starts with the digits 875. □

Proof of Claim 2. Claim 2 follows similarly from Claim 3. □

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