University of Michigan
Undergraduate Math Competition 16

April, 1999

Problem 1 Find the last six digits of 1999^{1999}.

Problem 2 For any integer $n \geq 2$, factor it into primes. Then add up all the prime factors (including duplicates) and call this $f(n)$. Then create the sequence $\{a_n : n \geq 2\}$ with $a_n = 1/f(n)$. For example, a portion of this sequence would be calculated like this:

$$\begin{array}{llllll} n = 14 = 2 \times 7 & \Rightarrow & f(14) = 2 + 7 = 9 & \Rightarrow & a_{14} = 1/9 \\ n = 15 = 3 \times 5 & \Rightarrow & f(15) = 3 + 5 = 8 & \Rightarrow & a_{15} = 1/8 \\ n = 16 = 2 \times 2 \times 2 \times 2 & \Rightarrow & f(16) = 2 + 2 + 2 + 2 = 8 & \Rightarrow & a_{16} = 1/8 \\ n = 17 & \Rightarrow & f(17) = 17 & \Rightarrow & a_{17} = 1/17 \\ n = 18 = 2 \times 3 \times 3 & \Rightarrow & f(18) = 2 + 3 + 3 = 8 & \Rightarrow & a_{18} = 1/8. \end{array}$$

The sequence $\{a_n : n \geq 2\}$ begins

$$\left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{7}, \frac{1}{6}, \frac{1}{6}, \frac{1}{7}, \frac{1}{11}, \frac{1}{7}, \frac{1}{13}, \frac{1}{9}, \frac{1}{8}, \frac{1}{8}, \frac{1}{17}, \frac{1}{8}, \frac{1}{19}, \frac{1}{9}, \frac{1}{10}, \frac{1}{13}, \frac{1}{23}, \frac{1}{9}, \frac{1}{10}, \frac{1}{15}, \ldots \right\}.$$

Prove that $\lim_{n \to \infty} a_n = 0$. Does the series $\sum_{n=1}^{\infty} a_n$ converge?

Problem 3 How many solutions in non-negative integers m and n are there to the equation $6m + 11n = 1999$?

Problem 4 (a) If each of three circles of radius 1 is (externally) tangent to the other two, find the radius of the little circle inscribed in the space among them.

(b) If each of 2000 spheres of radius 1 in 1999-dimensional space is (externally) tangent to all of the others, find the radius of the little sphere inscribed in the space in the midst of them.

Problem 5 Suppose you have nine coins and a balance. Eight of the coins weigh the same and the ninth is lighter. How can you tell which coin is the light one by using the balance only twice? (One use of the balance means choosing two disjoint subsets of the coins and finding out which has the smaller total weight.)

Problem 6 A floor is tiled with squares of side 1, in the usual checkerboard fashion. A small square of side $L \leq 1/\sqrt{2}$ is thrown at random on the floor. What is the probability that it does not land on any cracks?

Problem 7 Find all positive integer solutions of $a^b = b^a$. (Be sure to prove that you have all the solutions.)

Problem 8 A set S has two binary operations $\#$ and $*$ on it, and the following axioms hold:

1. There is an element z in S such that $z\#s = s$ for all $s \in S$.
2. For all $s,\ t, u \in S$ if $s\#u = t\#u$ then $s = t$.
3. For all $s,\ t \in S$ if $z * s = z * t$ then $s = t$.
4. For all $s,\ t,\ u \in S$, $(s\#t) * u = (s * u)\#(t * u)$.

Prove that $S = \{z\}$.

Problem 9 Assume:

1. $f(x)$ is a real-valued function defined on the whole real line.
2. $f(x)$ is bounded on each finite interval.
3. The difference $f(x+1) - f(x)$ approaches a limit l as $x \to \infty$.

Prove that the ratio $f(x)/x$ also approaches the same limit l as $x \to \infty$.

Problem 10 Take a regular icosahedron inscribed in a unit sphere with center O, and let A and B be two vertices connected by an edge. Calculate $\cos(\angle AOB)$. You may use the fact that

$$\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}.$$