

University of Michigan
Undergraduate Mathematics Competition 13
April, 1996

Problem 1. A large (but finite) number of soldiers are arranged in an east-west line, and all the soldiers are facing north. The commander shouts “Right face!” One second later, all the soldiers ought to be facing east, but they have not completely mastered “right” and “left”, so some are facing east and some west. Any soldier who is face-to-face with his neighbor realizes that there was a mistake and turns 180° (disregarding the possibility that the mistake might have been the neighbor’s). One second later, when all these 180° turns have been completed, any soldier who is now face-to-face with a neighbor turns 180° (even if he had just turned at the previous step). The process repeats in the same manner. Prove that it stops after finitely many steps.

Problem 2. Prove that if a curve on a sphere is shorter than the equator then that curve must be contained in a hemisphere.

Problem 3. A *semigroup* is a set equipped with an associative operation, written $*$ in this problem. Prove that every finite semigroup has an idempotent element, i.e., an element x satisfying $x * x = x$.

Problem 4. Given $2n$ points in the plane, no three collinear, with n of the points colored red and the other n blue, prove that one can draw n line segments, each joining one of the red points to one of the blue points, in such a way that no two of the segments intersect.

Problem 5. The Duma (the Russian parliament) has 1994 members. Assume that each member has slapped the face of exactly one other member. Prove that one can form within the Duma a 665-member committee in which no member has slapped the face of another.

Problem 6. Let $\sum_{n=1}^{\infty} a_n$ be a divergent series with positive terms, and let s_n be the partial sum $\sum_{i=1}^n a_i$. Prove that $\sum_{n=1}^{\infty} (a_n/s_n)$ diverges.

Problem 7. Suppose that f has a continuous second derivative on $[0, 1]$, that $f(0) = f(1) = 0$, and that $|f''(x)| \leq 1$ for all $x \in [0, 1]$. Prove that

$$\left| \int_0^1 f(x) dx \right| \leq \frac{1}{12}.$$

Problem 8. Consider a checkerboard (8×8) minus its four corner squares. Show that it cannot be tiled with pieces of the shape $\begin{matrix} & & x & \\ & x & & \\ x & & & \\ & x & & \end{matrix}$, i.e., L-shaped pieces covering four squares.

Problem 9. One person in a ring of n people has a keg of beer. He takes a sip and passes the keg to the left or to the right with 50% probability. The recipient of the keg takes a sip and also passes the keg to the left or to the right with 50% probability. The process

is repeated until everyone has had at least one sip. What is the probability distribution of the final position of the keg?

Problem 10. What is the expected number of points into which a circular disk is divided by n randomly chosen chords? More precisely, let $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ be points on the circle, chosen at random, independently and uniformly with respect to arc length. Let C_i be the chord joining x_i and y_i . Let X be the number of connected components of the unit disk minus the C_i 's. What is the expectation of X ?