Stochastic Volatility Modeling: From Rough Heston Model to Affine Volterra Equatio

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Abstract

In the famous Heston model, where the variance satisfies a CIR process, the characteristic function of the log-price can be calculated explicitly by the Riccati equation. When replacing the Brownian motion by the fractional Brownian motion, one now faces the Rough Heston model and difficulties arise because of the lack of Markovian structure. In this paper, we will first study the mechanism of calculating the characteristic function of the Rough Heston model using the convergence of a suitable class of Hawkes processes. Then, we will study the optimal consumption and stochastic control problems in the Heston framework and Affine Volterra framework by some specific examples.

Contents

1 Introduction

In financial mathematics, the Black-Scholes Equation is a well-known partial differential equation (PDE) that governs the evolution of price of a European call or put under Black-Scholes model. The assumptions under the model state that the price of assets follows a geometric Brownian motion with constant drift and volatility. The Black-Scholes PDE is

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = rU, \tag{1.1}$$

where U is the price of the option as a function of stock price S and time t, r is the risk-free interest rate, and σ is the volatility of the stock.
The derivation of this PDE can be obtained using pricing by replication, where we have assumed that the market is complete under Black-Scholes model. Here we are going to use 1 option and ϕ stocks to replicate cash X, which has the risk-free rate r. In continuous time, we have

$$X_t = u(t, S_t) + \phi S_t, \tag{1.2}$$

$$dX_t = rX_t dt. \tag{1.3}$$

Under Black-Scholes model, we also have $dS_t = rS_t dt + \sigma S_t dW_t$. Applying Ito's formula to X_t, we have

$$\begin{aligned} &dx_t \\ &= u_t dt + uxdS_t + \frac{1}{2}\sigma^2 S_t^2 dt + \phi dS_t \\ &= (u_t + \frac{1}{2}\sigma^2 S_t^2)dt + (u_x + \phi)(rS_t dt + \sigma S_t dW_t) \\ &= rX_t dt \end{aligned} \tag{1.4}$$

Now, we need to match the coefficients before dt and dW_t:

- Coefficient before dW_t: $u_x + \phi = 0$,
- Coefficient before d_t: $u_t + \frac{1}{2}\sigma^2 S_t^2 = rX_t = r(u(t, S_t) + \phi S_t)$.

Solving for ϕ and plug back to the second equation, we obtain the famous Black-Scholes PDE in equation(1.1).
However, due to the stock market crash in 1987, the classical Black-Scholes model began to lose its validity. Traders realized that extreme events could happen and markets have a significant skew. As a result, the possibility for extreme events needed to be factored into option pricing, and volatility smiles started occurring. It shows that implied volatility increases or decreases as options move more in-the-money (ITM) or out-of-the-money (OTM).
In order to incorporate real-world situations like non-constant volatility into our volatility model, Dupire Local Volatility and Heston model was introduced. Moreover, rough Heston model will also be discussed in the following passage.

2 Local Volatility

The concept of a local volatility was developed when Bruno Dupire, Emanuel Derman and Iraj Kani noted that there is a unique diffusion process consistent with the risk neutral densities derived from the market prices of European options. They have observed that volatility could be a function of stock price and time. Thus, the Dupire Local Volatility in its most general form is(assume there is no dividend):

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K}. \tag{2.5}$$

The underlying S_t follows the process

$$dS_t = r_t S_t dt + \sigma(S_t, t) S_t dW_t. \tag{2.6}$$

We need the following preliminaries:

- Discount factor $P(t,T) = exp(-\int_t^T r_s ds)$.
- Fokker-Planck Equation. Denote $f(S_t, t)$ the probability density function of the underlying price S_t at time t. Then f satisfies the following equation:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial S}\left[\mu S f(S,t)\right] + \frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2 f(S,t)\right]. \tag{2.7}$$

  The implication of applying Fokker-Planck Equation in our model is that there are lots of liquid options (vanilla options) in the market, and we can use the price of the options to derive the volatility function, which is consistent with the option price.
- time-t price of European call with strike K, denoted by $C = C(S, K)$:

$$C = P(t,T)E[(S_T - K)^+] = P(t,T)\int_K^\infty (S_T - K) f(S,T) dS. \tag{2.8}$$

We need the following derivatives of the call $C(S_t, t)$, which can be easily derived using equation(2.8):

$$\frac{\partial C}{\partial K} = -P(t,T)\int_K^\infty f(S,T) dS, \tag{2.9}$$

$$\frac{\partial^2 C}{\partial K^2} = P(t,T) f(K,T), \tag{2.10}$$

$$\frac{\partial C}{\partial T} = -r_T C + P(t,T)\int_K^\infty (S_T - K)\frac{\partial}{\partial T}[f(S,T)] dS. \tag{2.11}$$

Substituting the Fokker-Planck equation for $\frac{\partial f}{\partial t}$at t=T, we have:

$$\frac{\partial C}{\partial T} + r_T C = P(t,T)\int_K^\infty (S_T - K)(-\frac{\partial}{\partial S}\left[\mu_T S f(S,T)\right] + \frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2 f(S,T)\right]) dS. \tag{2.12}$$

We need to evaluate two integrals from the above equation:

$$I_1 = \mu_T \int_K^\infty (S_T - K)\frac{\partial}{\partial S}\left[S f(S,T)\right] dS, \tag{2.13}$$

$$I_2 = \int_K^\infty (S_T - K)\frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2 f(S,T)\right]) dS. \tag{2.14}$$

In order to solve these two integrals, we first need to utilize two identities, namely

$$\int_K^\infty S_T f(S,T) dS = \frac{C}{P(t,T)} - \frac{K}{P(t,T)}\frac{\partial C}{\partial K}, \tag{2.15}$$

$$f(K,T) = \frac{1}{P(t,T)}\frac{\partial^2 C}{\partial K^2}. \tag{2.16}$$

These two integrals can be easily evaluated by using these two identities and the integral by parts technique. Finally, we have obtained the solutions of the two integrals:

$$I_1 = \frac{-\mu_T C}{P(t,T)} + \frac{\mu_T K}{P(t,T)}\frac{\partial C}{\partial K}, \tag{2.17}$$

$$I_2 = \frac{\sigma^2 K^2}{P(t,T)} \frac{\partial^2 C}{\partial K^2}. \tag{2.18}$$

Plugging them back to equation(2.12), we have achieved the Dupire Local Volatility PDE. Moreover, we have solved a function of local volatility. Assuming zero interest rate and zero dividend yield, we have:

$$\sigma^2(K,T) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}}. \tag{2.19}$$

A limitation for local volatility is that it makes the assumption that the volatility must has something to do with the stock price and time. Research works find that the volatility is more likely to move in a random process. So Heston model was introduced, in which the volatility follows stochastic process.

3 Heston model

3.1 Heston Dynamics

The Heston model assumes that S_t follows a Black-Sholes type stochastic process, while the stochastic variance v_t follows a Cox, Ingersoll, Ross process. Thus

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dW_{1,t} \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_{2,t} \\ E[dW_{1,t} dW_{2,t}] &= \rho\, dt \end{aligned} \tag{3.20}$$

3.2 The Heston PDE

Under the Heston model, the market is no longer complete as we assumed in the Black-Sholes model. To replicate an option, we need to construct a portfolio consisting of one option $V = V(S,v,t)$, Δ units of the stock S, and ϕ units of another option $U = U(S,v,t)$. The value of the portfolio is:

$$\Pi = V + \Delta\, S + \phi\, U.$$

Assuming the portfolio is self-financing, the change in the value of the portfolio is:

$$d\Pi = dV + \Delta\, dS + \phi\, dU. \tag{3.21}$$

Apply Itô's Lemma to dV and dU, differentiating with respect to the variables t, S and v. Plug the results into (3.21), we get:

$$\begin{aligned} d\Pi =& \Big\{ \frac{\partial V}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma\, vS \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2}\sigma^2 v \frac{\partial^2 V}{\partial v^2} \Big\} dt + \\ & \phi \Big\{ \frac{\partial U}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma\, vS \frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2}\sigma^2 v \frac{\partial^2 U}{\partial v^2} \Big\} dt + \\ & \Big\{ \frac{\partial V}{\partial S} + \phi \frac{\partial U}{\partial S} + \Delta \Big\} dS + \Big\{ \frac{\partial V}{\partial v} + \phi \frac{\partial U}{\partial v} \Big\} dv. \end{aligned} \tag{3.22}$$

In order for the portfolio to be hedged against movements and volatility in the stock, the last two terms of equation (3.22) containing dS and dv must be zero. Thus the hedge parameters are:

$$\begin{aligned} \phi &= -\frac{\frac{\partial V}{\partial v}}{\frac{\partial U}{\partial v}}, \\ \Delta &= -\phi \frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}. \end{aligned} \tag{3.23}$$

Plug the values in equation (3.23) into equation (3.22) gives us a form of $d\Pi$ without unknown parameters, which we write in a simplified form as $d\Pi = (A + \phi\, B)dt$. On the other hand, the portfolio must earn the risk free rate, r. Hence $d\Pi = r\Pi\, dt$. Combining the two equations of $d\Pi$, we get:

$$\frac{A - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial v}} = \frac{B - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}}, \tag{3.24}$$

where

- $A = \frac{\partial V}{\partial t} + \frac{1}{2}\, vS^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma\, vS \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2}\, \sigma^2 v \frac{\partial^2 V}{\partial v^2}$,
- $B = \frac{\partial U}{\partial t} + \frac{1}{2}\, vS^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma\, vS \frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2}\, \sigma^2 v \frac{\partial^2 U}{\partial v^2}$.

By observation, both sides of equation (3.24) can be written as a function $f(S, v, t)$ of S, v, and t. Following Heston assumption, specify the function as $f(S, v, t) = -\kappa(\theta - v) + \lambda(S, v, t)$, where $\lambda(S, v, t)$ is the price of volatility risk. Substitute $f(S, v, t)$ into the left-hand side of equation (3.23) and substitute for B. To get the log price, let $x = lnS$. We get the Heston PDE of the log price version.

$$\frac{\partial U}{\partial t} + \frac{1}{2} v \frac{\partial^2 U}{\partial x^2} + (r - \frac{1}{2} v)\frac{\partial U}{\partial x} + \rho\sigma\, v \frac{\partial^2 U}{\partial v \partial x} + \frac{1}{2}\sigma^2 v \frac{\partial U}{\partial v^2} - rU + [\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} = 0, \tag{3.25}$$

where the market price of the risk is a linear function of the volatility, following Heston. So that $\lambda(S, v, t) = \lambda v$.

3.3 The Call Price

The form of call price is

$$C_t(K) = e^{x_t} P_1(x, v, \tau) - e^{-r\tau} K P_2(x, v, \tau), \tag{3.26}$$

where $P_j(x, v, \tau)$ each represents the probability of the call expiring in-the-money, conditional on x_t and v_t at time t, and $\tau = T - t$ is the time to expiration.
The call price in equation (3.26) follows the PDE in equation (1.1). Compute the derivatives of call price in equation (3.26) and substitute into the PDE for C,
Under Heston assumption, the characteristic functions for $x = lnS_T$ have the following affine structure:

$$f(\phi; x, v) = exp(C(\tau, \phi) + D(\tau, \phi)v + i\phi x), \tag{3.27}$$

where C and D are coefficients.
Applying two-dimension Feynman-Kac theorem, the characteristic function in (3.27) will follow the PDE equation

$$-\frac{\partial f}{\partial \tau} + (r - \frac{1}{2}v)\frac{\partial f}{\partial x} + \kappa(\theta - v)\frac{\partial f}{\partial v} + \frac{1}{2}v\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 f}{\partial v^2} + \sigma\rho v_t \frac{\partial^2 f}{\partial x \partial v} = 0. \tag{3.28}$$

Compute the derivatives in equation (3.28), drop the f terms, and re-arrange to obtain two differential equations

$$\begin{aligned} \frac{\partial D}{\partial \tau} &= \rho\sigma i\phi D - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D^2 - \frac{1}{2}\phi - (\kappa + \lambda)D, \\ \frac{\partial C}{\partial \tau} &= ri\phi + aD. \end{aligned} \tag{3.29}$$

The first formula in equation (3.29) fulfills the Riccati equation. This simplifies the problem since Riccati equation is solvable. We solve for the equation and get the call price

$$C = r\phi\tau + \frac{a}{\sigma^2}[(b - \rho\sigma i\phi + d)\tau - 2ln(\frac{1 - ge^{d\tau}}{1 - g})], \tag{3.30}$$

where

- $a = \kappa\theta$,
- $d = \sqrt{(\rho\sigma\phi - b)^2 - \sigma^2(2u\phi - \phi^2)}$, and
- $g = \frac{b - \rho\sigma\phi + d}{b - \rho\sigma\phi - d}$.

4 Rough Heston Model

Since Heston model failed to capture the heavy tail property in financial market, we want to calculate the characteristic function of rough heston model. However, due to lack of the Markovian structure of the rough heston model, we are not able to calculate its characteristic function directly. Therefore, we will rely on the characteristic function of Hawkes process, which is a well-known imgrant process, and investigate the correspondance between Hawkes process and rough heston model. We want to find a suitable class of Hawkes process that will converge in law under skorokhod topology to rough heston model. Combining all together, we are able to obtain the characteristic function for the log-price in rough heston model.

4.1 Relationship between Hawkes process and Rough Heston model

We work on a sequence of probability spaces$(\Omega^T, \mathcal{F}^T, \mathcal{P}^T)$, indexed by $T > 0$ (going to infinity), on which $N^T = (N^{T,+}, N^{T,-})$ is a bidimensional Hawkes process with intensity

$$\lambda_t^T = \begin{pmatrix} \lambda_t^{T,+} \\ \lambda_t^{T,-} \end{pmatrix} = \hat{\mu}_T(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t \phi^T(t-s) dN_s^T \tag{4.31}$$

Note that $\lambda_t^{T,+} = \lambda_t^{T,-}$. This is because that the stock price can jump up or down, and we assume that the density of jumping up equals to the desity of jumping down.
For a given T, the probability space is equipped with the filtration $(F_t^T)_{t \geq 0}$, where (F_t^T) is the σ-algebra generated by $(N_s^T)_{s \leq t}$.
We have made a particular choice for the heavy-tailed functions defining ϕ^T as follows.
Definition 4.1.1 There exists $\beta > 0$,$1/2 < \alpha < 1$ and $\gamma > 0$ such that

$$a_T = 1 - \gamma T^{-\alpha}, \ \phi^T = \varphi^T \chi \tag{4.32}$$

where

$$\chi = \frac{1}{\beta + 1} \begin{pmatrix} 1 & \beta \\ 1 & \beta \end{pmatrix}, \quad \varphi^T = a_T \varphi, \quad \varphi = f^{\alpha,1} \tag{4.33}$$

with $f^{\alpha,1}$ the Mittag-Leffler density function that captures the heavy-tail property.

Now we spilt Hawkes process into drift and diffusion term, let us write

$$M_t^T = (M_t^{T,+}, M_t^{T,-}) = N_t^T - \int_0^t \lambda_s^T ds \tag{4.34}$$

for martingale associated to the point process N_t^T. We easily obtain

$$\lambda_t^{T,+} = \hat{\mu}_T(t) + \int_0^t \phi^T(t-s)\lambda_s^{T,+}ds + \frac{1}{1+\beta}\int_0^t \phi^T(t-s)(dM_s^{T,+} + \beta dM_s^{T,-}) \tag{4.35}$$

Now let

$$\psi^T = \sum_{k\geq 1}(\phi^T)^{*k}, \tag{4.36}$$

where $(\phi^T)^{*1} = \phi^T$ and for $k > 1$, $(\phi^T)^{*k}(t) = \int_0^t \phi^T(s)(\phi^T)^{*(k-1)}(t-s)ds$. The point for introducing convolution is to convert it into multiplication by applying Laplace transform. So we get

$$\lambda_t^{T,+} = \hat{\mu}_T(t) + \int_0^t \psi^T(t-s)\hat{\mu}_T(s)ds + \frac{1}{1+\beta}\int_0^t \psi^T(t-s)(dM_s^{T,+} + \beta dM_s^{T,-}). \tag{4.37}$$

The inhomogeneous intensity $\hat{\mu}_T(t)$ should be of order μ_t with $\mu_T = \mu T^{\alpha-1}$, where μ is some positive constant. Define renormalized intensity $C_t^T = \frac{1-a_T}{\mu_T}\lambda_{tT}^{T,+}$.

After some computations, we have

$$C_t^T = \frac{1-a_T}{\mu_T}\hat{\mu}_T(tT) + \int_0^t T(1-a_T)\psi^T(T(t-s))\frac{\hat{\mu}_T(Ts)}{\mu_T}ds + \upsilon\int_0^t T(1-a_T)\psi^T(T(t-s))\sqrt{C_s^T}dB_s^T, \tag{4.38}$$

where

$$B_t^T = \int_0^{tT}\frac{dM_s^{T,+} + \beta dM_s^{T,-}}{\sqrt{T(\lambda_s^{T,+} + \beta^2\lambda_s^{T,-})}},\ \upsilon = \sqrt{\frac{1+\beta^2}{\gamma\mu(1+\beta)^2}}. \tag{4.39}$$

Using Laplace transform, we have the following expression for C^T:

$$C_t^T = \frac{1-a_T}{\mu_T}\hat{\mu}_T(tT) + \int_0^t a_T f^{\alpha,\gamma}(t-s)\frac{\hat{\mu}_T(Ts)}{\mu_T}ds + \upsilon\int_0^t a_T f^{\alpha,\gamma}(t-s)\sqrt{C_s^T}dB_s^T, \tag{4.40}$$

From the computations in the proof of Theorem 4.2.1, we derive the right choice of $\hat{\mu_T}$.
Definition 4.1.2 The baseline intensity $\hat{\mu}_T$ is given by

$$\hat{\mu}_T(t) = \mu_T + \xi\mu_T(\frac{1}{1-a_T}(1-\int_0^t \phi^T(s)ds) - \int_0^t \phi^T(s)ds), \tag{4.41}$$

with $\xi > 0$ and $\mu_T = \mu T^{\alpha-1}$ for some $\mu > 0$.

4.2 The rough limits of Hawkes processes

We now state the limiting behavior of our specified bidimensional nearly unstable Hawkes processes with heavy tails. For $t \in [0,1]$, we define

$$X_t^T = \frac{1-a_T}{T^\alpha\mu}N_{tT}^T,\ \Lambda_t^T = \frac{1-a_T}{T^\alpha\mu}\int_0^{tT}\lambda_s^T ds,\ Z_t^T = \sqrt{\frac{T^\alpha\mu}{1-a_T}}(X_t^T - \Lambda_t^T). \tag{4.42}$$

Here, X_t^T is the rescaled Hawkes process, Λ_t^T is the rescaled drift term, and Z_t^T is the rescaled diffusion term.
Theorem 4.2.1 As $T \longrightarrow \infty$, under definition 4.1.1. and 4.1.2., the process $(\Lambda_t^T, X_t^T, Z_t^T)_{t\in[0,1]}$ converges in law for the Skorokhod topology to (Λ, X, Z), where

$$\Lambda_t = X_t = \int_0^t Y_s ds\begin{pmatrix}1\\1\end{pmatrix},\ Z_t = \int_0^t \sqrt{Y_s}ds\begin{pmatrix}dB_s^1\\dB_s^2,\end{pmatrix} \tag{4.43}$$

and Y is the unique solution of the rough stochastic differential equation

$$Y_t = \xi + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma(1 - Y_s) ds + \gamma \sqrt{\frac{1+\beta^2}{\gamma\mu(1+\beta^2)}} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sqrt{Y_s} dB_s, \quad (4.44)$$

where

$$B = \frac{B^1 + \beta B^2}{\sqrt{1+\beta^2}} \quad (4.45)$$

and (B^1, B^2) is a bidimensional Brownian motion. Furthermore, for any $\epsilon > 0$, Y has Holder regularity $\alpha - 1/2 - \epsilon$. Thanks to Theorem 4.2.1, we are now able to build such microscopic processes converging to the log-price. More precisely, for $\theta > 0$, let us define:

$$P^T = \sqrt{\frac{\theta}{2}} \sqrt{\frac{1-a_T}{T^\alpha \mu}} (N_T^{T,+} - N_T^{T,-}) - \frac{\theta}{2} \frac{1-a_T}{T^\alpha \mu} N_T^{T,+}. \quad (4.46)$$

We have the following corollary of Theorem 4.2.1.
Corollary 4.2.2 As $T \longrightarrow \infty$, under definition 4.1.1. and 4.1.2., the sequence of processes $(P_t^T)_{t\in[0,1]}$ converges in law for the skorokhod topology to

$$P_t = \int_0^t \sqrt{V_s} dWs - \frac{1}{2} \int_0^t V_s ds, \quad (4.47)$$

where V is the unique solution of the rough stochastic differential equation

$$Y_t = \theta\xi + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma(\theta - V_s) ds + \gamma \sqrt{\frac{\theta(1+\beta^2)}{\gamma\mu(1+\beta^2)}} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sqrt{V_s} dB_s, \quad (4.48)$$

with (W,B) a correlated bidimensional Brownian motion whose bracket satisfies

$$d < W, B >_t = \frac{1-\beta}{\sqrt{2(1+\beta^2)}} dt. \quad (4.49)$$

Thus, we have successfully built a sequence of microscopic process P^T, defined by (4.47), which converges to the logarithm of our rough Heston process of interest.

5 Merton Problem

Merton problem is a decision made by an investor to choose how much to consume and also allocates his invest portfolio, given a known amount of wealth at the beginning. The investor gains utility from both the consumption activity and the investment returns. Mathematically, Merton problem is an optimal control problem in which an investor is given the state variable (X_t) and control variables (π, C). The objective for Merton problem is to maximize expected utility.

Here we write out the Supreme expectation (the objective function) for the investor:

$$\max_{(\pi,c) \in \Pi\times C} \quad \mathbb{E}[\int_0^\infty e^{-\beta t} U(c_t) dt], \quad (5.50)$$

where the utility function is: $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$. $\Pi \times C$ represents the product space of admissible strategies for the investor, where Π represents all of the possible investment portfolios, and C

represents the possible range of consumption.
Note that in our model, we assume that the utility function is of power form. There are also other forms of utility function, including natural log, exponential, etc.

We apply the Heston assumptions in the Merton problem. Under Heston assumption (stochastic volatility), the SED for the stock is:

$$\begin{aligned} dS_t^0 &= rS_t^0 dt, \\ dS_t &= rS_t dt + \sqrt{v_t} S_t dW_{1,t}, \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_{2,t}. \end{aligned} \tag{5.51}$$

The portfolio process is:

$$dX_t = rX_t dt + \pi_t(\mu - r)dt + \pi_t \sigma dW_t - c_t dt. \tag{5.52}$$

What left for us to do is to apply dynamic programming principle to solve for the HJB equation. Then do the verification part. It is the extension of Feynman-Kac Theorem. Due to the limitation of time, we don't have time to finish this part. Hopefully we would complete it in the future.

6 Future Work

Our research could also be extended as studied under affine Volterra process. The concept of Affine Volterra process is developed by Jaber, Larsson, and Pulido (2019), which is defined as solutions of certain stochastic convolution equations with affine coefficients. In the model, we need to switch the kernel of fractional Brownian motion to a more general form.
Affine Voleterra processes are neither semimartingales, nor Markov processes, making it impossible to solve for the exponential function as we did in precious parts. To overcome these obstacles, we would introduce the idea of limit process. Also, because of the time limitation, we can't finish this part.

7 References

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