Extensions of irreducible representations of quaternion algebras over p-adic fields

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Abstract

We provide a classification of the irreducible mod-p representations of the quaternion algebra over the p-adic field $\mathbb{Q}_p$ when $p \geq 5$. We then determine all extensions of irreducible representations, and provide a basis for the space of extensions. We use analogous methods to classify extensions of irreducible representations of the quaternion algebra over an unramified extension of $\mathbb{Q}_p$, and discuss our progress towards the ramified case.

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1 Introduction

Representation Theory is the study of groups through their actions on vector spaces. Although representations of finite and compact groups over $\mathbb{C}$ are well-studied and understood, when we look at larger groups or over fields of positive characteristic, many of the nicer results begin to break down. In particular, for a fixed prime p, representations of a p-group, or pro-p-group, over a field of characteristic p can exhibit behavior which is not seen over fields of characteristic 0.

One such group is the multiplicative group of the quaternion algebra over p-adic fields, and representations of this group over $\overline{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$, are of interest. In particular, there exist more interesting ways to glue together irreducible representations than direct sums, known as extensions. Understanding the space of extensions can help us better understand how irreducible representations fit together.

Representations of these quaternion algebras also have ramifications in a broader context. The Local Langlands Conjectures give a hypothetical correspondence between n-dimensional mod-p representations of the absolute Galois group of $\mathbb{Q}_p$, the p-adic numbers, with certain mod-p representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. In the case where $n = 2$, we can look to understand representations of quaternion algebras over $\mathbb{Q}_p$, which are related to representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. In this paper, we discuss the irreducible mod-p representations of quaternion algebras over $\mathbb{Q}_p$ as well as a classification of their extensions.

2 Preliminaries

2.1 Representation Theory

We begin with some generalities from representation theory.

Definition 2.1.1. Let G be a group, and V be a vector space of dimension n over a field k, and let π be a group homomorphism

$$\pi : G \to \mathrm{GL}(V).$$

Then, (π, V) is said to be a **representation** of dimension n.

We sometimes abbreviate (π, V) to simply π or simply V, depending on context.

Definition 2.1.2. Let (π, V) be a representation. A subspace $W \subset V$ is said to be a **subrepresentation** of V provided that W is closed under the action of π. The non-zero representation V is said to be **irreducible** provided that the only subrepresentations are 0 and V.

When the ground field is $\mathbb{C}$ and the group G is finite, irreducible representations form the building blocks for all other representations.

Theorem 2.1.3 (Maschke)**.** Every finite-dimensional $\mathbb{C}$-representation of a finite group G can be written as the direct sum of irreducible representations.

However, when we take representations over fields of positive characteristic, representations do not always decompose nicely, and so in order to construct representations, we need a different way of putting irreducible representations together.

Definition 2.1.4. Let V_1 and V_2 be representations of G. Then, a representation V is said to be an **extension** of V_1 by V_2 provided that the sequence

$$0 \longrightarrow V_2 \hookrightarrow V \twoheadrightarrow V_1 \longrightarrow 0$$

of G-representations is exact. For two extensions V, V' of G, we say that V and V' are equivalent provided that there exists a linear map T such that the following diagram commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_2 & \longrightarrow & V & \longrightarrow & V_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & V_2 & \longrightarrow & V' & \longrightarrow & V_1 & \longrightarrow & 0
\end{array}$$

Note that since we are working in the category of G-representations, all maps are required to be G-equivariant. We denote the space of all equivalence classes of extensions by $\mathrm{Ext}^1_G(V_1, V_2)$.

Definition 2.1.5. Let G be a group, and let $H \leq G$ be a subgroup. Let (π, V) be a k-representation of H. Then, we define the **induction** of π to G by

$$\mathrm{Ind}_H^G(\pi) = k[G] \otimes_{k[H]} V$$

where the action of G is the natural action on the left, and we think of V as a $k[G]$-module. Note that this forms a representation of G of dimension $\frac{|G|}{|H|} \dim(V)$.

It is a natural question to ask why induction is defined in this manner. Consider the dual notion of taking a representation V of G and producing a representation of a subgroup H. We can simply restrict the action of G to only allow elements of H to act, producing the representation $\mathrm{Res}_H^G(V)$. Then, we have an adjunction, given by the following theorem of Frobenius.

Theorem 2.1.6 (Frobenius Reciprocity)**.** Let G be a group, and H a finite index subgroup of G. Let V be a representation of G, and let W be a representation of H. Then, we have that

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(W), V) \cong \mathrm{Hom}_H(W, \mathrm{Res}_H^G(V))$$

and similarly

$$\mathrm{Ext}^1_G(\mathrm{Ind}_H^G(W), V) \cong \mathrm{Ext}^1_H(W, \mathrm{Res}_H^G(V)).$$

This is also true on the other side, namely,

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W)$$

and similarly

$$\mathrm{Ext}^1_G(V, \mathrm{Ind}_H^G(W)) \cong \mathrm{Ext}^1_H(\mathrm{Res}_H^G(V), W).$$

Finally, we present a result which helps us compute inductions.

Theorem 2.1.7 (Mackey)**.** Let H be a normal subgroup of G, and let (τ, W) be a representation of H. Then, we have that

$$\mathrm{Ind}_H^G(W) = \bigoplus_{\gamma \in G/H} W^\gamma$$

where W^γ denotes the representation (τ^γ, W), and τ^γ is defined by

$$\tau^\gamma(h) = \tau(\gamma h \gamma^{-1}).$$

We also state a few lemmas which are of use.

Lemma 2.1.8. Let G be a (pro)-p group, and let V be an $\overline{\mathbb{F}}_p$ (smooth, in the pro-p case) representation of G. Define the space V^G as the set of vectors in V fixed by all elements of G, that is,

$$V^G = \{v \in V : g \cdot v = v\}.$$

Then, $V^G \neq 0$.

Lemma 2.1.9 (Schur's Lemma)**.** Let V and W be two irreducible representations of a group G over an algebraically closed field k. Then, the space of G-equivariant maps, $\mathrm{Hom}_G(V, W)$, is given by

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} k & V \cong W \\ 0 & \text{else} \end{cases}$$

2.2 p-adic Fields

We first define the field of p-adic numbers.

Definition 2.2.1. Fix a prime p. Given $0 \neq \frac{a}{b} \in \mathbb{Q}$, we can write

$$\frac{a}{b} = p^n \frac{a'}{b'}$$

where $a, a', b, b' \in \mathbb{Z}$, $b, b' \neq 0$, and p does not divide a' or b'. We then define the p-adic norm to be

$$\left|\frac{a}{b}\right|_p = p^{-n}$$

and furthermore define $|0|_p = 0$. Then, this defines a norm on $\mathbb{Q}$, which induces a metric, and we define $\mathbb{Q}_p$, the field of p-adic numbers, to be the completion of $\mathbb{Q}$ with respect to this metric.

Given a nonzero element $x \in \mathbb{Q}_p$, we can write it as a power series expansion in p as follows:

$$x = a_{-n}p^{-n} + a_{-n+1}p^{-n+1} + \cdots + a_{-1}p^{-1} + a_0 + a_1 p + a_2 p^2 + \cdots$$

where we have that $0 \leq a_i < p$ and $a_{-n} \neq 0$. Then, we define $\mathbb{Z}_p$, the p-adic integers, to be the set of $x \in \mathbb{Q}_p$ such that x contains only non-negative powers of p (together with 0). It is a fact that $\mathbb{Z}_p$ is the ring of integers in $\mathbb{Q}_p$. We can also define $\mathbb{Z}_p$ as the p-adic completion of $\mathbb{Z}$.

We give a consequence of Hensel's Lemma.

Theorem 2.2.2. There exists a group homomorphism $[-] : \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$, called the Teichmüller lift, and we set $[0] = 0$.

Thus, we can rewrite $x \in \mathbb{Q}_p$ in terms of an expansion using Teichmüller lifts, in the form

$$x = [a'_{-n}]p^{-n} + [a'_{-n+1}]p^{-n+1} + \cdots + [a'_{-1}]p^{-1} + [a'_0] + [a'_1]p + [a'_2]p^2 + \cdots$$

where each $a'_i \in \mathbb{F}_p$ and $a'_{-n} \neq 0$. Then, from this expansion, we obtain the decomposition

$$\mathbb{Q}_p^\times \cong p^{\mathbb{Z}} \times [\mathbb{F}_p^\times] \times 1 + p\mathbb{Z}_p \tag{2.2.3}$$

where we think of this as the power of p, the leading coefficient, and the remaining terms.

Given a finite-degree field extension $F/\mathbb{Q}_p$, we can perform a similar decomposition.

Remark 2.2.4. We define a function

$$|-| : F \to \mathbb{R}_{\geq 0}$$
$$x \mapsto |N_{F/\mathbb{Q}_p}(x)|_p^{\frac{1}{[F:\mathbb{Q}_p]}}$$

where N denotes the field extension norm. This is in fact a norm, and so we can define

$$\mathcal{O}_F = \{x \in F : |x| \leq 1\}$$

which is in fact a ring, since one can show that this norm satisfies the strong triangle inequality. $)_F$ has the maximal ideal

$$\mathfrak{p}_F = \{x \in \mathcal{O}_F : |x| < 1\}.$$

This ideal is principal, and generated by an element π, called the **uniformizer**. We then define the **residue field** to be

$$k_F = \mathcal{O}_F/\mathfrak{p}_F$$

and we similarly have a Teichmüller lift $[-] : k_F \to \mathcal{O}_F$, with respect to which we can write any element $x \in F^\times$ as

$$x = [a_{-n}]\pi^{-n} + [a_{-n+1}]\pi^{-n+1} + \cdots + [a_{-1}]\pi^{-1} + [a_0] + [a_1]\pi + [a_2]\pi^2 + \cdots$$

with each $a_i \in k_F$ and $a_{-n} \neq 0$.

Definition 2.2.5. It is a fact that the residue field k_F will always be finite, and thus we have that $k_F = \mathbb{F}_{p^f}$ for some positive integer f, called the **residue field degree** of F. Furthermore, we have that $p\mathcal{O}_F = \pi^e\mathcal{O}_F$ for some $e \geq 1$, called the **ramification degree** of F. We then have that

$$[F : \mathbb{Q}_p] = e \cdot f.$$

2.3 Quaternion Algebras

We define a quaternion algebra over a field k.

Definition 2.3.1. A **quaternion algebra** $D = D_{\alpha,\beta}$ over a field k is a 4-dimensional vector space over k, with basis $\{1, i, j, k\}$, given the structure of an algebra with the multiplication rules

$$i^2 = \alpha \qquad j^2 = \beta$$
$$ij = -\,ji = k$$

for some $\alpha, \beta \in k^\times$.

For the purposes of this paper, we choose $k = F$ to be a p-adic field as above, and α, β appropriately such that D is a division algebra. Let $D^\times$ denote the group of units in D under multiplication.

Remark 2.3.2. Let k_D/k_F be an extension of degree 2. There exists $\varpi \in D$, satisfying $\varpi^2 = \pi$, with respect to which every element $x \in D$ can be written as

$$x = [a_{-n}]\varpi^{-n} + [a_{-n+1}]\varpi^{-n+1} + \cdots + [a_{-1}]\varpi^{-1} + [a_0] + [a_1]\varpi + [a_2]\varpi^2 + \cdots$$

where $[-] : k_D \to \mathcal{O}_D$ is a Teichmüller lift, and $\mathcal{O}_D$ is the ring of integers in D. Note that as D is not commutative, ϖ will not necessarily commute with Teichmüller lifts, and in fact we have the relation

$$\varpi[x]\varpi^{-1} = [x^{p^f}]$$

where f is the residue field degree of F. Note that $\varpi^2 = \pi \in F$, which is the center of D, and thus the conjugation action of ϖ^2 is trivial. Letting $\mathfrak{p}_D = \varpi\mathcal{O}_D$, we have the decomposition

$$D^\times \cong \varpi^{\mathbb{Z}} \ltimes \left([k_D^\times] \ltimes (1 + \mathfrak{p}_D)\right) \tag{2.3.3}$$

where the semidirect product is given by the conjugation action of ϖ. The group $1 + \mathfrak{p}_D$ is pro-p, and the group $\mathcal{O}_D$ contains elements of D with only non-negative powers of ϖ.

Remark 2.3.4. When $F = \mathbb{Q}_p$, we have that $k_F = \mathbb{F}_p$ and $k_D = \mathbb{F}_{p^2}$. The conjugation action of ϖ is given by

$$\varpi[x]\varpi^{-1} = [x^p]$$

and we have the decomposition

$$D^\times \cong \varpi^{\mathbb{Z}} \ltimes \left(\left[\mathbb{F}_{p^2}^\times\right] \ltimes (1 + \varpi\mathcal{O})\right). \tag{2.3.5}$$

We can make a preliminary observation regarding representations of $D^\times$ when $F = \mathbb{Q}_p$.

Proposition 2.3.6. Irreducible representations of $D^\times$ correspond uniquely to representations of

$$D^\times/(1 + \varpi\mathcal{O}) \cong \varpi^{\mathbb{Z}} \ltimes \mathbb{F}_{p^2}^\times.$$

Proof. Let V be an irreducible representation of $D^\times$. Since $1 + \varpi\mathcal{O}$ is a pro-p group, by Lemma 2.1.8, we have that

$$V^{1+\varpi\mathcal{O}} = \{v \in V : g \cdot v = v \;\forall\; g \in 1 + \varpi\mathcal{O}\} \neq 0.$$

Furthermore, since $1 + \varpi\mathcal{O}$ is a normal subgroup of $D^\times$, we have that

$$g \cdot V^{1+\varpi\mathcal{O}} = V^{g(1+\varpi\mathcal{O})g^{-1}} = V^{1+\varpi\mathcal{O}}$$

and thus $V^{1+\varpi\mathcal{O}}$ is a nonzero subrepresentation of the irreducible representation V. Therefore we must have that

$$V = V^{1+\varpi\mathcal{O}}$$

and thus the $1+\varpi\mathcal{O}$ component acts trivially in any irreducible representation of $D^\times$. It therefore suffices to consider representations of the group

$$D^\times/(1+\varpi\mathcal{O}) \cong \varpi^{\mathbb{Z}} \ltimes \mathbb{F}_{p^2}^\times$$

which is our desired result. $\square$

We also introduce a computational tool for Teichmüller lifts.

Lemma 2.3.7 (Witt Vectors)**.** For the Teichmüller lifts above, the following property is satisfied.

$$[x] + [y] = [x+y] + O(p)$$

In particular, the error is of order at least $\varpi^{2e} = p$.

3 Irreducible Representations of $D^\times$

In this section, we give a classification of the smooth irreducible representations of $D^\times$ over $\overline{\mathbb{F}}_p$. We assume throughout this section that $p > 3$.

3.1 1-Dimensional Irreducible Representations

We first compute all 1-dimensional irreducible representations of $D^\times$, which are homomorphisms

$$\psi : D^\times \to \overline{\mathbb{F}}_p^\times.$$

Remark 3.1.1. Since $\overline{\mathbb{F}}_p^\times$ is abelian, any such map must factor through the abelianization of $D^\times$ as follows

$$\begin{array}{ccc} D^\times & \xrightarrow{\psi} & \overline{\mathbb{F}}_p^\times \\ \downarrow & \nearrow_{\overline{\psi}} & \\ D^\times/[D^\times, D^\times] & & \end{array}$$

Thus, it suffices to compute all maps

$$\overline{\psi} : D^\times/[D^\times, D^\times] \to \overline{\mathbb{F}}_p^\times$$

We first compute $[D^\times, D^\times]$. Let N denote the norm map on $D^\times$, defined by

$$N : D^\times \to \mathbb{Q}_p^\times$$
$$a + bi + cj + dk \mapsto a^2 - \alpha b^2 - \beta c^2 + \alpha\beta d^2$$

From Chapter 7.9 exercise 31 in [1], we know that $[D^\times, D^\times]$ is precisely the elements of norm 1 in $D^\times$ and thus we have that

$$D^\times/[D^\times, D^\times] \cong \mathbb{Q}_p^\times$$

We conclude from this calculation that a 1-dimensional irreducible representation of $D^\times$ corresponds to a group homomorphism between $\mathbb{Q}_p^\times$ and $\overline{\mathbb{F}}_p^\times$.

Proposition 3.1.2. Every irreducible 1-dimensional representation of $D^\times$ is given by a map

$$\psi : D^\times \to \overline{\mathbb{F}}_p^\times$$
$$(\varpi^n, [k], 1+\varpi x) \mapsto a^n k^{(p+1)b}$$

for some $a \in \overline{\mathbb{F}}_p^\times$ and $0 \le b < p-1$.

Proof. Using the decomposition of $\mathbb{Q}_p^\times$, we can examine homomorphisms out of a product as homomorphisms out of each component as follows

$$\mathrm{Hom}(\mathbb{Q}_p^\times, \overline{\mathbb{F}}_p^\times) = \mathrm{Hom}(p^{\mathbb{Z}}, \overline{\mathbb{F}}_p^\times) \times \mathrm{Hom}(\mathbb{F}_p^\times, \overline{\mathbb{F}}_p^\times) \times \mathrm{Hom}((1+p\mathbb{Z}_p), \overline{\mathbb{F}}_p^\times)$$

and we can now examine each component separately. We have that

- $\mathrm{Hom}(p^{\mathbb{Z}}, \overline{\mathbb{F}}_p^\times) \cong \overline{\mathbb{F}}_p^\times$ since $p^{\mathbb{Z}}$ is generated by the element p, and thus a choice of $p \mapsto \lambda$ for $\lambda \in \overline{\mathbb{F}}_p^\times$ uniquely defines a homomorphism $p^{\mathbb{Z}} \to \overline{\mathbb{F}}_p^\times$.
- $\mathrm{Hom}(\mathbb{F}_p^\times, \overline{\mathbb{F}}_p^\times)$ has size $p-1$ with maps $g \mapsto g^k$ for some integer $0 \leq k < p-1$.
- $\mathrm{Hom}((1+p\mathbb{Z}_p), \overline{\mathbb{F}}_p^\times)$ is trivial, since $1+p\mathbb{Z}_p$ is pro-p, and thus can only map trivially into $\overline{\mathbb{F}}_p^\times$, since $\overline{\mathbb{F}}_p$ has characteristic p.

Thus, for $a \in \overline{\mathbb{F}}_p^\times$ and $0 \leq b \leq p-2$, we can characterize every $\overline{\psi}_{a,b} \in \mathrm{Hom}(\mathbb{Q}_p^\times, \overline{\mathbb{F}}_p^\times)$ as follows.

$$\begin{aligned} \overline{\psi}_{a,b} : \mathbb{Q}_p^\times \cong p^{\mathbb{Z}} \times \mathbb{F}_p^\times \times (1+p\mathbb{Z}_p) &\to \overline{\mathbb{F}}_p^\times \\ (p^x, y, 1+z) &\mapsto a^x y^b \end{aligned}$$

Then, since $N : D^\times \to \mathbb{Q}_p^\times$ is the projection onto the abelianization, we have that every 1-dimensional irreducible representation ψ of $D^\times$ can be written as

$$\psi = \psi_{a,b} = \overline{\psi}_{(-1)^b a, b} \circ N$$

In particular, we can show that

$$N(\varpi) = -p \qquad N([k]) = \left[k^{(p+1)}\right]$$

and so if we allow

$$\psi(\varpi) = a \qquad \psi([k]) = k^{(p+1)b}$$

this determines a character, and all characters are of this form. □

3.2 Higher Dimensional Irreducible Representations

We now construct higher dimensional irreducible representations of $D^\times$. Recall that it suffices to consider representations of the group

$$D^\times/(1+\varpi\mathcal{O}) \cong \varpi^{\mathbb{Z}} \ltimes \mathbb{F}_{p^2}^\times$$

Proposition 3.2.1. Characters of $\varpi^{2\mathbb{Z}} \times \mathbb{F}_{p^2}^\times$ are given by

$$\begin{aligned} \chi_{a,b} &= \tau_a \otimes \lambda_b \\ \chi_{a,b}(\varpi^{2x}, [y]) &= \tau_a(x)\lambda_b(y) \\ &= a^x y^b \end{aligned}$$

for $a \in \overline{\mathbb{F}}_p$ and $0 \leq b \leq p^2 - 1$.

Proof. Since $\varpi^{2\mathbb{Z}} \times \mathbb{F}_{p^2}^\times$ is abelian, all irreducible representations are 1-dimensional, and are products of the characters of each component. Thus, for $a \in \overline{\mathbb{F}}_p^\times$ and $0 \leq b < p^2 - 1$, we have the characters

$$\begin{aligned} \tau_a : \varpi^{2\mathbb{Z}} &\to \overline{\mathbb{F}}_p^\times \\ \varpi^2 &\mapsto a \end{aligned}$$

and

$$\lambda_b : \mathbb{F}_{p^2}^\times \to \overline{\mathbb{F}}_p^\times$$

$$x \mapsto x^b$$

and characters of the product are products of these characters, given by

$$\begin{aligned}\chi_{a,b} &= \tau_a \otimes \lambda_b \\ \chi_{a,b}(\varpi^{2x}, [y]) &= \tau_a(x)\lambda_b(y) \\ &= a^x y^b\end{aligned}$$

which is our desired result. □

We then induce $\chi_{a,b}$ to a representation of $\varpi^{\mathbb{Z}} \ltimes \mathbb{F}_{p^2}^\times$. Let $G = \varpi^{\mathbb{Z}} \ltimes \mathbb{F}_{p^2}^\times$ and let $L = \varpi^{2\mathbb{Z}} \times \mathbb{F}_{p^2}^\times$. We identify $\varpi^{\mathbb{Z}}$ with $\mathbb{Z}$. Then, the induced representation

$$\mathrm{Ind}_L^G(\chi_{a,b}) = \overline{\mathbb{F}}_p[G] \otimes_{\overline{\mathbb{F}}_p[L]} \chi_{a,b}$$

has dimension 2 over $\overline{\mathbb{F}}_p$, with basis $(0,1)\otimes 1$ and $(1,1)\otimes 1$.

Proposition 3.2.2. Let $a \in \overline{\mathbb{F}}_p^\times$ and $0 \le b < p^2 - 1$. Then, if $b = c(p+1)$, we have that

$$\mathrm{Ind}_L^G(\chi_{a,c(p+1)}) = \psi_{a^{1/2},c} \oplus \psi_{-a^{1/2},c}$$

Otherwise, $\mathrm{Ind}_H^G(\chi_{a,b})$ is irreducible.

Proof. We consider the action of (x,y) on the induced representation, where $(x,y) \in \mathbb{Z} \times \mathbb{F}_{p^2}^\times$ corresponds to $[y]\varpi^x$. Note that we follow the semidirect product multiplication rule, where even integers act trivially and odd integers act by exponentiation by p, as is the case in $\mathbb{Z} \times \mathbb{F}_{p^2}^\times$, described as follows

$$(a,b)(c,d) = \begin{cases} (a+c, bd) & a \text{ is even} \\ (a+c, bd^p) & a \text{ is odd} \end{cases}$$

When x is even, we have that

$$\begin{aligned}(x,y)\cdot(0,1)\otimes 1 &= (x,y)\otimes 1 \\ &= (0,1)(x,y)\otimes 1 \\ &= (0,1)\otimes \chi_{a,b}(x,y) \\ &= \chi_{a,b}(x,y)((0,1)\otimes 1) \\ &= a^{x/2}y^b((0,1)\otimes 1)\end{aligned}$$

and similarly

$$\begin{aligned}(x,y)\cdot(1,1)\otimes 1 &= (x+1,y)\otimes 1 \\ &= (1,1)(x,p^{-1}y)\otimes 1 \\ &= (1,1)\otimes \chi_{a,b}(x,y^{p^{-1}}) \\ &= \chi_{a,b}(x,y^{p^{-1}})((1,1)\otimes 1) \\ &= a^{x/2}y^{p^{-1}b}((1,1)\otimes 1)\end{aligned}$$

When x is odd, we have that

$$\begin{aligned}(x,y)\cdot(0,1)\otimes 1 &= (x,y)\otimes 1 \\ &= (1,1)(x-1,y^{p^{-1}})\otimes 1 \\ &= (1,1)\otimes \chi_{a,b}(x-1,y^{p^{-1}})\end{aligned}$$

$$
\begin{aligned}
&= \chi_{a,b}(x-1, y^{p^{-1}})((1,1)\otimes 1) \\
&= a^{(x-1)/2} y^{p^{-1}b}((1,1)\otimes 1)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
(x,y)\cdot(1,1)\otimes 1 &= (x+1,y)\otimes 1 \\
&= (0,1)(x+1,y)\otimes 1 \\
&= (0,1)\otimes \chi_{a,b}(x+1,y) \\
&= \chi_{a,b}(x+1,y)((0,1)\otimes 1) \\
&= a^{(x+1)/2} y^{b}((0,1)\otimes 1)
\end{aligned}
$$

Then, in our chosen basis, and noting that $p = p^{-1}$ in $\mathbb{Z}/(p^2-1)\mathbb{Z}$, we can write down matrices for the action of (x,y) on $\mathrm{Ind}_L^G(\chi_{a,b})$.

$$
(x,y) \mapsto \begin{cases} \begin{pmatrix} a^{x/2}y^b & 0 \\ 0 & a^{x/2}y^{pb} \end{pmatrix} & x \text{ is even} \\ \begin{pmatrix} 0 & a^{(x-1)/2}y^{pb} \\ a^{(x+1)/2}y^b & 0 \end{pmatrix} & x \text{ is odd} \end{cases}
$$

Suppose that $a^{x/2}y^b = a^{x/2}y^{pb}$ for all y, or equivalently, $y^{b(p-1)} = 1$. This happens precisely when $b = c(p+1)$ for some integer c. When this is the case, the first matrix will be scalar in all bases, and in particular, in the basis which diagonalizes the second matrix. We compute eigenvalues to diagonalize the second matrix, and we obtain that in our new basis, we have that

$$
(x,y) \mapsto \begin{cases} \begin{pmatrix} a^{x/2}y^{c(p+1)} & 0 \\ 0 & a^{x/2}y^{c(p+1)} \end{pmatrix} & x \text{ is even} \\ \begin{pmatrix} a^{x/2}y^{c(p+1)} & 0 \\ 0 & -a^{x/2}y^{c(p+1)} \end{pmatrix} & x \text{ is odd} \end{cases}
$$

Note that even if x is odd, $a^{x/2}$ is defined, since all square roots exist in an algebraically closed field. Since the matrices are simultaneously diagonalizable, this representation is reducible, and using the $\psi_{a,b}$ notation as previously defined, this decomposes as

$$
\mathrm{Ind}_L^G(\chi_{a,c(p+1)}) = \psi_{a^{1/2},c} \oplus \psi_{-a^{1/2},c}
$$

When the first matrix is not a scalar multiple of the identity, we claim that $\mathrm{Ind}_L^G(\chi_{a,b})$ is irreducible. Suppose a nontrivial subrepresentation exists, which must have dimension 1. However, note that the first matrix has eigenvectors e_1 and e_2 associated with distinct eigenvalues, neither of which are fixed by the second matrix. Thus, no dimension 1 subspace is fixed by this representation, and therefore $\mathrm{Ind}_L^G(\chi_{a,b})$ is irreducible as desired. ☐

Proposition 3.2.3. All irreducible representations of $D^\times$ are of the form $\psi_{a,b}$ or $\mathrm{Ind}_L^G(\chi_{c,d})$, thought of as a representation of $D^\times$.

Proof. Since irreducible representations of G correspond uniquely to irreducible representations of $D^\times$, we consider instead the irreducible representations of G. Let V be an irreducible representation of G. Then, consider $V|_L$, which is a (possibly reducible) representation of L. Since V is finite dimensional, the representation $V|_L$ must contain an irreducible representation of L, so suppose that $\chi_{a,b}$ is contained in $V|_L$. Then, there exists a homomorphism of L-representations $\chi_{a,b} \hookrightarrow V|_L$, and so by Frobenius reciprocity, we have that

$$
0 \neq \mathrm{Hom}_L(\chi_{a,b}, V|_L) \cong \mathrm{Hom}_G(\mathrm{Ind}_L^G(\chi_{a,b}), V)
$$

If $\mathrm{Ind}_L^G(\chi_{a,b})$ is irreducible, then by Schur's lemma, $V \cong \mathrm{Ind}_L^G(\chi_{a,b})$. Otherwise, we write

$$
\mathrm{Hom}_G(\mathrm{Ind}_L^G(\chi_{a,c(p+1)}), V) = \mathrm{Hom}_G(\psi_{a^{1/2},c}, V) \oplus \mathrm{Hom}_G(\psi_{-a^{1/2},c}, V)
$$

where we think of $\psi_{\pm a^{1/2},c}$ as the character of $D^\times$ restricted to G. Since both $\psi_{\pm a^{1/2},c}$ are irreducible, and distinct for $p > 2$, by Schur's lemma, V cannot be isomorphic to both, and thus must be isomorphic to one of the two. We conclude that all irreducible representations of G are of the form $\psi_{a,b}$ or $\mathrm{Ind}_L^G(\chi_{a,b})$. □

4 Extensions of Irreducible Representations

We now compute all extensions of irreducible representations of $D^\times$. Let χ, χ' be irreducible 1-dimensional representations of $D^\times$. We then compute representations V which make the following sequence of $D^\times$-representations exact.

$$0 \longrightarrow \chi \hookrightarrow V \twoheadrightarrow \chi' \longrightarrow 0$$

Since we can tensor the sequence by χ'^*, the dual representation of χ', without loss of generality, we can take χ' to be the trivial character, denoted by $\mathbb{1}$. It then suffices to compute $\mathrm{Ext}^1_{D^\times}(\mathbb{1}, \chi)$. Fix a representation $V \in \mathrm{Ext}^1_{D^\times}(\mathbb{1}, \chi)$. In an appropriate basis, we can write the action of g on V as follows.

$$g \mapsto \begin{pmatrix} \chi(g) & c(g) \\ 0 & 1 \end{pmatrix} \tag{4.0.1}$$

where $c : D^\times \to \overline{\mathbb{F}}_p$ is not necessarily a group homomorphism, but has restrictions from the representation being a group homomorphism. A classification of functions c will describe all extensions, so we seek to understand all possible functions c. Since V is a representation, the above map must be a homomorphism, and thus we have that

$$\begin{pmatrix} \chi(gh) & c(gh) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \chi(g) & c(g) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi(h) & c(h) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \chi(g)\chi(h) & \chi(g)c(h) + c(g) \\ 0 & 1 \end{pmatrix}$$

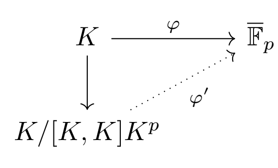
and so we obtain the multiplication rule for c, given by

$$c(gh) = c(g) + \chi(g)c(h) \tag{4.0.2}$$

Restricting c to the subgroup $K = 1 + \varpi\mathcal{O} \subset \ker(\chi)$, we observe that $c|_K$ is a group homomorphism from K to $\overline{\mathbb{F}}_p$ under addition. We classify all such homomorphisms.

4.1 Classification of Homomorphisms from $1 + \varpi\mathcal{O}$ to $\overline{\mathbb{F}}_p$

Remark 4.1.1. Since $\overline{\mathbb{F}}_p$ is abelian, and furthermore every element is p-torsion, any homomorphism $\varphi : K \to \overline{\mathbb{F}}_p$ must factor uniquely as in the diagram below.



where $[K,K]K^p$ is the Frattini subgroup, generated by all commutators and p-th powers in K.

We define

$$J = \left\{1 + [0]\varpi + [c]\varpi^2 + [a_3]\varpi^3 + \cdots : c, a_i \in \mathbb{F}_{p^2},\ \exists\, d \in \mathbb{F}_{p^2} \text{ such that } d^p - d = c\right\}$$

Proposition 4.1.2. $J = [K,K]K^p$.

Remark 4.1.3. Note that the set $\{d^p - d : d \in \mathbb{F}_{p^2}\}$ forms an $\mathbb{F}_p$-line in $\mathbb{F}_{p^2}$, since $x^p - x$ is $\mathbb{F}_p$-linear, and any choice of $d \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$ will satisfy $d^p - d \neq 0$.

Proof. In order to show $[K,K]K^p \subset J$, it suffices to show that K/J is abelian and p-torsion. Let L denote the subgroup $1+\varpi^3\mathcal{O}$ inside J. We then have that

$$\frac{K}{J} \cong \frac{K/L}{J/L} = \frac{\{1+[a]\varpi+[b]\varpi^2 \,:\, a,b\in\mathbb{F}_{p^2}\}}{\{1+[0]\varpi+[c]\varpi^2 \,:\, c=d^p-d\,,\, d\in\mathbb{F}_{p^2}\}}$$

We compute the commutator subgroup of K/L. We write

$$x = 1+[a_1]\varpi+[a_2]\varpi^2 \qquad y = 1+[b_1]\varpi+[b_2]\varpi^2$$

Taking all calculations modulo ϖ^3, we then have that

$$\begin{aligned} xyx^{-1}y^{-1} &= (1+[a_1]\varpi+[a_2]\varpi^2)(1+[b_1]\varpi+[b_2]\varpi^2)(1+[a_1]\varpi+[a_2]\varpi^2)^{-1}(1+[b_1]\varpi+[b_2]\varpi^2)^{-1} \\ &= (1+[a_1]\varpi+[a_2]\varpi^2)(1+[b_1]\varpi+[b_2]\varpi^2) \\ &\quad \left(1-[a_1]\varpi+\left(\left[a_1^{p+1}\right]-[a_2]\right)\varpi^2\right)\left(1-[b_1]\varpi+\left(\left[b_1^{p+1}\right]-[b_2]\right)\varpi^2\right) \\ &= 1+0\varpi+\left([a_1b_1^p]-[a_1^pb_1]\right)\varpi^2 \end{aligned}$$

Since $[x]+[y]=[x+y]+O(p)$, we can omit the brackets on the Teichmüller lifts in the above expression, and combined with the fact that $(a_1^pb_1)^p = a_1^{p^2}b_1^p = a_1b_1^p$, we obtain that all commutators are of the form

$$1+[0]\varpi+[c]\varpi^2 \text{ with } c=d^p-d\,,\, d\in\mathbb{F}_{p^2}$$

Note that these are elements of J/L, and thus we conclude that K/J is abelian.

We now compute n-th powers of elements of K/J. We have that

$$(1+[a]\varpi+[b]\varpi^2)^n = 1+(n[a])\varpi+\left(n[b]+\frac{n(n-1)}{2}[a^{p+1}]\right)\varpi^2$$

When $n=p$, we have that

$$\begin{aligned} (1+[a]\varpi+[b]\varpi^2)^p &= 1+(p[a])\varpi+\left(p[b]+\frac{p(p-1)}{2}[a^{p+1}]\right)\varpi^2 \\ &= 1+[a]\varpi^3+\left([b]+\frac{p-1}{2}[a^{p+1}]\right)\varpi^4 \\ &= 1 \end{aligned}$$

and thus every element of K/J is p-torsion. Furthermore, from our above definition, we see that K/L has order p^4, and since c comes from an $\mathbb{F}_p$-line, the group J/L has order p, and so the group K/J has order p^3. Thus, we conclude that K/J is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$. Since we have shown that K/J is both p-torsion and abelian, we must have that $[K,K]K^p \subset J$.

In the reverse direction, we show that $J \subset [K,K]K^p$. Fix $j\in J$, and write

$$j = 1+[0]\varpi+[c]\varpi^2+[a_3]\varpi^3+\cdots$$

where $c=d-d^p$ for some $d\in\mathbb{F}_{p^2}$. Then, taking $a=d$ and $b=1$, we compute that

$$\begin{aligned} [1+[d]\varpi, 1+[1]\varpi] &= 1+[0]\varpi+([d]-[d^p])\varpi^2+\cdots \\ &= 1+[0]\varpi+[c]\varpi^2+\cdots \end{aligned}$$

We can then write

$$j\,[1+[a]\varpi, 1+[b]\varpi]^{-1} = 1+[0]\varpi+[0]\varpi^2+[a_3']\varpi^3+\cdots$$

Lemma 4.1.4. For $p>3$, any element in $1+\varpi^3\mathcal{O}$ can be written as the p-th power of an element of $1+\varpi\mathcal{O}$.

Proof. We compute

$$(1+[a_1]\varpi+[a_2]\varpi^2+\cdots)^p = 1+p[a_1]\varpi+(p[a_2]+\xi_2(a_1))\,\varpi^2+(p[a_3]+\xi_3(a_1,a_2))\,\varpi^3+\cdots$$

where $\xi_n(a_1,\ldots,a_{n-1})$ denotes an expression involving only the variables $a_1,\ldots,a_{n-1}$. In particular, we can almost explicitly write a formula for ξ_n, given by

$$\xi_n(a_1,\ldots,a_{n-1}) = \left(\sum_{\substack{\sigma \text{ a non-increasing partition} \\ \text{of } n \text{ into } p \text{ non-negative integer parts}}} M_\sigma[a_{\sigma_1}^{k_1}a_{\sigma_2}^{k_2}\cdots a_{\sigma_p}^{k_p}]\right) - p[a_n]$$

where each k_i is produced by commuting each a_i with ϖ, $a_0=1$, and M_σ is produced as follows. Let m_i be the number of occurrences of i in the partition σ. Then, we have that

$$M_\sigma = \binom{p}{m_0,m_1,\ldots,m_n}$$

In particular, we notice that when σ is not a partition where every part has equal value, or alternatively $m_i \neq 0$ for more than one i, M_σ is divisible by p, and so we can write

$$p[a_n]+\xi_n(a_1,\ldots,a_{n-1}) = p\left([a_n]+\xi_n'(a_1,\ldots,a_{n-1})\right)$$

However, if σ is a partition in which every part has equal value, which can happen only if $n=\lambda p$ for some positive integer λ, we have that $M_\sigma=1$, and so we write

$$\xi_{\lambda p}(a_1,\ldots,a_{n-1}) = p\xi_{\lambda p}'(a_1,\ldots,a_{n-1})+[a_\lambda^{K_{\lambda p}}]$$

where $K_{\lambda p}$ is some integer representing the exponent on a_λ, coming from the conjugation action of ϖ. We can then simplify our above expression to the following.

$$\begin{aligned}(1+[a_1]\varpi+[a_2]\varpi^2+\cdots)^p &= 1+p[a_1]\varpi+(p[a_2]+\xi_2(a_1))\,\varpi^2+(p[a_3]+\xi_3(a_1,a_2))\,\varpi^3+\cdots \\ &= 1+[a_1]\varpi^3+([a_2]+\xi_2'(a_1))\,\varpi^4+\cdots \\ &\qquad \cdots+\left([a_{\lambda p-2}]+\xi_{\lambda p-2}'(a_1,\ldots,a_{\lambda p-2})+[a_\lambda^{K_{\lambda p}}]\right)\varpi^{\lambda p}+\cdots\end{aligned}$$

Then, in order to compute a p-th root, given a sequence of b_i with $b_1=b_2=0$, we would like to solve for a sequence of a_i for which the following relation is true

$$1+[b_3]\varpi^3+[b_4]\varpi^4+\cdots = (1+[a_1]\varpi+[a_2]\varpi^2+\cdots)^p$$

We have an expression for the right hand side, and can immediately equate coefficients to obtain $a_1=b_3$. We then repeat this process for each a_i iteratively, where at each step we can equate coefficients on powers of ϖ to obtain an equation relating $[a_i]$ to a combination of known quantities in a_j and b_k for $j,k<i$. Choose a_i such that this equation is satisfied up to order ϖ^0, and carry the error to the next term. We can repeat this process to compute a_j for any finite j, and since this converges in $D^\times$, we have constructed a p-th root. □

Thus, using Lemma 4.1.4, we can write

$$\begin{aligned}j\,[1+[a]\varpi,1+[b]\varpi]^{-1} &= 1+[0]\varpi+[0]\varpi^2+[a_3']\varpi^3+\cdots \\ &= (1+[0]\varpi+[0]\varpi^2+[b_3']\varpi^3+\cdots)^p \\ j &= (1+[0]\varpi+[0]\varpi^2+[b_3']\varpi^3+\cdots)^p\,[1+[a]\varpi,1+[b]\varpi]\end{aligned}$$

and thus $J\subset[K,K]K^p$ as desired, so we have that J is in fact equal to $[K,K]K^p$. □

Remark 4.1.5. This fails when $p=3$, as when $\lambda=1$, the quantity $\lambda p-2$ is equal to λ, and so the expression for the coefficient of ϖ^3 is a non-linear polynomial in a_1 and thus is not uniquely determined. However, for all other p and λ, this argument will hold.

Since $K/J \cong (\mathbb{Z}/p\mathbb{Z})^3$, we have that

$$\dim \operatorname{Hom}(K/J, \overline{\mathbb{F}}_p) = 3$$

as we make a choice of element in $\overline{\mathbb{F}}_p$ for each generator of $(\mathbb{Z}/p\mathbb{Z})^3$. We now produce an explicit basis for this vector space.

Proposition 4.1.6. The maps

$$\begin{aligned} \varphi : K/J &\to \overline{\mathbb{F}}_p \\ 1 + [a]\varpi + [b]\varpi^2 &\mapsto a \end{aligned}$$

$$\begin{aligned} \varphi' : K/J &\to \overline{\mathbb{F}}_p \\ 1 + [a]\varpi + [b]\varpi^2 &\mapsto a^p \end{aligned}$$

$$\begin{aligned} \psi : K/J &\to \overline{\mathbb{F}}_p \\ 1 + [a]\varpi + [b]\varpi^2 &\mapsto \operatorname{Tr}(b) - N(a) \end{aligned}$$

form a basis for the space $\operatorname{Hom}(K, \overline{\mathbb{F}}_p)$.

Proof. Note first that φ and φ' are group homomorphisms, and furthermore, are linearly independent, since if $\varphi = \lambda\varphi'$, then the polynomial $a^p - \lambda a$ would be identically 0 and thus have $p^2 > p$ roots. We produce a third basis vector, involving the value b. Recall that for two elements $x, y \in K/J$, with

$$x = 1 + [a_1]\varpi + [a_2]\varpi^2 \qquad y = 1 + [b_1]\varpi + [b_2]\varpi^2$$

we have that $x = y$ if and only if $a_2 - b_2 = c^p - c$ for some $c \in \mathbb{F}_{p^2}$. Consider the Frobenius map on $\mathbb{F}_{p^2}$, given by

$$\begin{aligned} F : \mathbb{F}_{p^2} &\to \mathbb{F}_{p^2} \\ x &\mapsto x^p \end{aligned}$$

Note that F is $\mathbb{F}_p$-linear, and since $p > 2$, F has eigenspace $V_1 = \mathbb{F}_p$ associated with eigenvalue 1, and eigenspace $V_{-1} = \langle c^p - c \rangle$ associated with eigenvalue -1. Thus, we can project $\mathbb{F}_{p^2}$ onto $V_1 = \mathbb{F}_p$, given by

$$\operatorname{proj}_{V_1}(x) = \frac{x + F(x)}{2} = \frac{1}{2}(x + x^p) = \frac{1}{2}\operatorname{tr}(x)$$

Note that this construction is well defined on b, since multiples of $c^p - c$ are sent to 0 under the projection. Thus, up to scaling, we have that our third basis element must map

$$\psi : 1 + [0]\varpi + [b]\varpi^2 \mapsto \operatorname{Tr}(b)$$

We then consider how it maps an element with non-zero coefficient on ϖ. Suppose that

$$\psi : 1 + [a]\varpi + [0]\varpi^2 \mapsto f(a)$$

Then, we have that

$$\begin{aligned} f(a) + f(b) &= \psi(1 + [a]\varpi) + \psi(1 + [b]\varpi) \\ &= \psi((1 + [a]\varpi)(1 + [b]\varpi)) \\ &= \psi(1 + [a+b]\varpi + [ab^p]\varpi^2) \\ &= \psi((1 + [a+b]\varpi)(1 + [ab^p]\varpi^2)) \\ &= \psi(1 + [a+b]\varpi) + \psi(1 + [ab^p]\varpi^2) \\ &= f(a+b) + \operatorname{Tr}(ab^p) \end{aligned}$$

$$= f(a+b) + ab^p + a^p b$$

We notice that the map

$$f(x) = -x^{p+1}$$

satisfies this condition, and this is in fact, up to sign, the field norm map of $\mathbb{F}_{p^2}/\mathbb{F}_p$. Thus, up to scaling, we have that our third basis element is given by

$$\begin{aligned} \psi : K/J &\to \overline{\mathbb{F}}_p \\ 1 + [a]\varpi + [b]\varpi^2 &\mapsto \mathrm{Tr}(b) - N(a) \end{aligned}$$

and since this involves b whereas the others do not, these maps are linearly independent, and so together the set $\{\varphi, \varphi', \psi\}$ forms a basis for the space $\mathrm{Hom}(K/J, \overline{\mathbb{F}}_p) = \mathrm{Hom}(K, \overline{\mathbb{F}}_p)$ as desired. □

4.2 Extensions of Characters

Throughout this discussion, let $\chi = \psi_{\gamma,\delta}$ be a character of $D^\times$. Consider the function c from equation 4.0.1. We recall the multiplication rule in equation 4.0.2. Taking $g \in 1 + \varpi\mathcal{O} \subset \ker(\chi)$, and for general $h \in D^\times$, we can write

$$\begin{aligned} c(h) + c(g) &= c(gh) \\ &= c(h(h^{-1}gh)) \\ &= \chi(h)c(h^{-1}gh) + c(h) \end{aligned}$$

and thus we obtain the relationship

$$c(g) = \chi(h)c(h^{-1}gh) \tag{4.2.1}$$

and since $1+\varpi\mathcal{O}$ is normal in $D^\times$, and we understand c on $1+\varpi\mathcal{O}$, we can obtain conditions c must satisfy. From Proposition 4.1.6, we have that

$$c|_{1+\varpi\mathcal{O}} = x\varphi + y\varphi' + z\psi$$

Proposition 4.2.2. Let $\psi_{\gamma,\delta}$ be a character of $D^\times$, and let c be the associated function described above. Then, we have that c is of the form

$$c|_{1+\varpi\mathcal{O}} = \begin{cases} z\psi & (\gamma,\delta) = (1,0) \\ 0 & (\gamma,\delta) \neq (1,0) \end{cases}$$

Proof. Let $h = \varpi$. Then, taking

$$g = 1 + [a_1]\varpi + [a_2]\varpi^2 + \cdots$$

we compute

$$\begin{aligned} h^{-1}gh &= \varpi^{-1}(1 + [a_1]\varpi + [a_2]\varpi^2 + \cdots)\varpi \\ &= \varpi^{-1}(\varpi + [a_1]\varpi^2 + [a_2]\varpi^3 + \cdots) \\ &= \varpi^{-1}(\varpi + \varpi[a_1^p]\varpi + \varpi[a_2^p]\varpi^2 + \cdots) \\ &= 1 + [a_1^p]\varpi + [a_2^p]\varpi^2 + \cdots \end{aligned}$$

and applying formula 4.2.1, we have that

$$xa_1 + ya_1^p + z(a_2 + a_2^p - a_1^{p+1}) = \gamma(xa_1^p + ya_1 + z(a_2^p + a_2 - a_1^{p+1})) \tag{4.2.3}$$

When $\gamma = 1$, we can choose $z \in \overline{\mathbb{F}}_p$ freely, and equation 4.2.3 simplifies to

$$\begin{aligned} xa_1 + ya_1^p &= xa_1^p + ya_1 \\ x(a_1 - a_1^p) + y(a_1^p - a_1) &= 0 \end{aligned}$$

$$(x-y)(a_1 - a_1^p) = 0$$

and since this must hold for all g, and $a_1 - a_1^p$ is not identically 0 on $\mathbb{F}_{p^2}$, we must have that $x - y = 0$ and thus $x = y$.

When $\gamma \neq 1$, we must have that $z = 0$, and furthermore, since we can choose a_1 freely, consider the case where $a_1 \in \mathbb{F}_p$. Then, equation 4.2.3 simplifies to

$$\begin{aligned} xa_1 + ya_1^p &= \gamma(xa_1^p + ya_1) \\ xa_1 + ya_1 &= \gamma(xa_1 + ya_1) \\ 0 &= a_1(\gamma - 1)(x + y) \end{aligned}$$

and since we can choose $a_1 \neq 0$, and we are in the case where $\gamma \neq 1$, we conclude that $x = -y$. Then, allowing a_1 to be in $\mathbb{F}_{p^2}$, we have that

$$\begin{aligned} xa_1 + ya_1^p &= \gamma(xa_1^p + ya_1) \\ xa_1 - xa_1^p &= \gamma(xa_1^p - xa_1) \\ 0 &= x(\gamma - 1)(a_1^p - a_1) \end{aligned}$$

and again since $a_1^p - a_1$ is not identically 0, we conclude that $x = y = 0$ when $\gamma \neq 1$.

If we instead take $h = [k]$ for some $k \in \mathbb{F}_{p^2}^\times$, we compute

$$\begin{aligned} h^{-1}gh &= [k^{-1}](1 + [a_1]\varpi + [a_2]\varpi^2 + \cdots)[k] \\ &= [k^{-1}](1 + [a_1]\varpi[k] + [a_2]\varpi^2[k] + \cdots) \\ &= [k^{-1}](\varpi + \varpi[a_1 k^p]\varpi + \varpi[a_2 k]\varpi^2 + \cdots) \\ &= 1 + [a_1 k^{p-1}]\varpi + [a_2]\varpi^2 + \cdots \end{aligned}$$

Then, using formula 4.2.1, we have that

$$xa_1 + ya_1^p + z(a_2 + a_2^p - a_1^{p+1}) = k^{(p+1)\delta}(xa_1^p k^{p-1} + ya_1 k^{1-p} + z(a_2^p + a_2 - a_1^{p+1})) \tag{4.2.4}$$

When $\delta = 0$, we have that z can be chosen freely in $\overline{\mathbb{F}}_p$, and from above we recall that $x = y$. Choosing $a_1 \in \mathbb{F}_p^\times$, equation 4.2.4 simplifies to

$$\begin{aligned} 0 &= x(a_1 + a_1^p - a_1 k^{1-p} - a_1^p k^{p-1}) \\ &= xa_1(1 + 1 - k^{1-p} - k^{p-1}) \end{aligned}$$

Since the polynomial $k^{2p-2} - 2k^{p-1} + 1$ is of degree $2p - 2 < p^2 - 1$, we can choose k to be a non-root, and since $a_1 \neq 0$, we have that $x = y = 0$ is forced.

When $\delta \neq 0$, we see that $z = 0$, and as above, we can similarly obtain $x = y$ and $a_1 \in \mathbb{F}_p$, and thus equation 4.2.4 reduces to

$$xa_1(2 - k^{1-p+(p-1)\delta} - k^{p-1+(p+1)\delta}) = 0.$$

Since $0 < \delta \leq p - 2$, note that the polynomial

$$k^{p-1+(p+1)\delta} + k^{1-p+(p+1)\delta} - 2 = 0$$

has positive powers of p, since $\delta > 0$, and has degree at most $p^2 - 3 < p^2 - 1$, and thus we can choose k to be a non-root, forcing $x = y = 0$. We conclude that

$$c|_{1+\varpi\mathcal{O}} = \begin{cases} z\psi & (\gamma, \delta) = (1, 0) \\ 0 & (\gamma, \delta) \neq (1, 0) \end{cases}$$

□

Proposition 4.2.5. Let $\chi = \psi_{\gamma,\delta}$ be a character of $D^\times$, and let c be the associated function described above. Then,

- When $\delta \neq 0$, a choice of $c([k]) \in \overline{\mathbb{F}}_p$ for a generator $k \in \mathbb{F}_{p^2}^\times$ completely determines the function c.
- When $\delta = 0$ and $\gamma \neq 1$, a choice of $c(\varpi) \in \overline{\mathbb{F}}_p$ completely determines the function c.
- When $\delta = 0$ and $\gamma = 1$, a choice of $c(\varpi) \in \overline{\mathbb{F}}_p$ together with a choice of coefficient $z \in \overline{\mathbb{F}}_p$ for $c|_{1+\varpi\mathcal{O}} = z\psi$ completely determine the function c.

Proof. We examine $c([k])$ for some $k \in \mathbb{F}_{p^2}^\times$. Using the multiplication rule in equation 4.0.2, we compute

$$c([k]^m) = c([k]) \left(\sum_{n=0}^{m-1} \chi([k])^n \right) \tag{4.2.6}$$

When $m = p$, if $\chi([k]) \neq 1$, we sum over a subgroup of $(p-1)$-th roots of unity an integer number of times, together with the $n = 0$ case, and thus this sum is 1, and therefore $c([k]^p) = c([k])$. When $\chi([k]) = 1$, we have that this sum is $p = 0$.

When $\delta = 0$, we have that for all choices of k, $\chi([k]) = 1$, and so we have that

$$c([k^p]) = c([k]^p) = pc([k]) = 0 \tag{4.2.7}$$

and since $k \mapsto k^p$ is an automorphism of $\mathbb{F}_{p^2}$, we have that $c|_{\mathbb{F}_{p^2}^\times} = 0$. Note that we can compute $c(g^{-1})$, using the multiplication rule.

$$\begin{aligned} 0 &= \chi(g)c(g^{-1}) + c(g) \\ c(g^{-1}) &= -\frac{c(g)}{\chi(g)} \end{aligned}$$

Then, when $\delta \neq 0$, and k a generator (so $\chi([k]) \neq 1$), we have that

$$\begin{aligned} c([k]) &= c([k^p]) \\ &= c(\varpi[k]\varpi^{-1}) \\ &= \chi(\varpi)c([k]\varpi^{-1}) + c(\varpi) \\ &= \chi(\varpi)(\chi([k])c(\varpi^{-1}) + c([k])) + c(\varpi) \\ &= \chi(\varpi)(\chi([k])\chi(\varpi)^{-1}c(\varpi) + c([k])) + c(\varpi) \\ &= \chi([k])c(\varpi) + \chi(\varpi)c([k]) + c(\varpi) \\ 0 &= (1 - \chi([k]))c(\varpi) + (\chi(\varpi) - 1)c([k]) \end{aligned}$$

Thus, we can write

$$c(\varpi) = \left(\frac{\chi(\varpi) - 1}{\chi([k]) - 1} \right) c([k])$$

and thus $c(\varpi)$ is given by a scalar multiple of $c([k])$. However, when $\delta = 0$, the conjugation relation between ϖ and $[k]$ gives no restriction on $c(\varpi)$. Thus, we have found all restrictions of c on each component of $D^\times$, and obtain our desired conclusion. □

Then, in order to compute the dimension of $\mathrm{Ext}^1_{D^\times}(\mathbb{1}, \psi_{\gamma,\delta})$, it suffices to impose the equivalence relation on extensions.

Theorem 4.2.8. For a character $\psi_{\gamma,\delta}$ of $D^\times$, we have that

$$\dim \mathrm{Ext}^1_{D^\times}(\mathbb{1}, \psi_{\gamma,\delta}) = \begin{cases} 2 & (\gamma,\delta) = (1,0) \\ 0 & (\gamma,\delta) \neq (1,0) \end{cases}$$

Proof. Let $\chi = \psi_{\gamma,\delta}$. In order for two extensions V_1 and V_2 to be equivalent, we must have a linear map T such that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \chi & \longrightarrow & V_1 & \longrightarrow & \mathbb{1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & \chi & \longrightarrow & V_2 & \longrightarrow & \mathbb{1} & \longrightarrow & 0 \end{array}$$

In terms of matrices, since T must restrict to the identity in both χ and $\mathbb{1}$, we have that T is of the form

$$T = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

and furthermore we must have that T is $D^\times$-equivariant. We check this condition for each choice of χ.

- When $\delta \neq 0$, suppose we have that $c_1([k]) = a$ and $c_2([k]) = b$, where k is a generator of $\mathbb{F}_{p^2}^\times$. Then, we must have
$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^{(p+1)\delta} & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k^{(p+1)\delta} & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$
and so we have that
$$x + a = k^{(p+1)\delta} x + b.$$
Since $\delta \neq 0$, we can write
$$x = \frac{a-b}{k^{(p+1)\delta} - 1}$$
and thus such an x, and therefore such a T, exists. Furthermore, since $c(\varpi)$ is a scalar multiple of $c([k])$, this T will also commute with the matrix corresponding to ϖ. Since c is identically 0 on the other components of $D^\times$, the matrix of the generators will be the identity, and thus will always commute with T. Since we have checked equivariance on generators, we conclude that every extension is equivalent.

- When $\delta = 0$ and $\gamma \neq 1$, suppose we have that $c_1(\varpi) = a$ and $c_2(\varpi) = b$. Then, we must have
$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$
and so we have that
$$x + a = \gamma x + b.$$
Since $\gamma \neq 1$, we can write
$$x = \frac{a-b}{\gamma - 1}$$
and thus such an x, and therefore such a T, exists. Since c is identically 0 on the other components of $D^\times$, the matrix of the generators will be the identity, and thus will always commute with T. Since we have checked equivariance on generators, we conclude that every extension is equivalent.

- When $\delta = 0$ and $\gamma = 1$, suppose that $c_1(\varpi) = a_1$ and $c_2(\varpi) = a_2$, and furthermore $c_1|_{1+\varpi\mathcal{O}} = z_1\psi$ and $c_2|_{1+\varpi\mathcal{O}} = z_2\psi$. Then, we have that
$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1\psi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_2\psi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$
and so we obtain
$$x + a_1 = x + a_2$$
$$x + z_1\psi = x + z_2\psi$$
and thus we conclude that if such a T is equivariant, we must have that $a_1 = a_2$ and $z_1 = z_2$.

We conclude that

$$\dim \operatorname{Ext}^1_{D^\times}(\mathbb{1}, \chi_{\gamma,\delta}) = \begin{cases} 2 & (\gamma,\delta) = (1,0) \\ 0 & (\gamma,\delta) \neq (1,0) \end{cases}$$

□

Remark 4.2.9. Note that in particular, when $(\gamma, \delta) = (1, 0)$, $\chi = \mathbb{1}$, and we know that

$$\operatorname{Ext}^1_{D^\times}(\mathbb{1}, \mathbb{1}) \cong \operatorname{Hom}(D^\times, \overline{\mathbb{F}}_p)$$

which is consistent with $D^\times/[D^\times, D^\times] \cong \mathbb{Q}_p^\times$.

4.3 Extensions of Higher Dimensional Representations

We now consider extensions involving higher dimensional representations.

Proposition 4.3.1. In order to compute extensions of higher dimensional representations, it suffices to compute extensions of characters in the group $H = \varpi^{2\mathbb{Z}} \times (\mathbb{F}_{p^2} \ltimes 1 + \varpi\mathcal{O})$.

Proof. Let $\psi_{\gamma,\delta}$ be a character of $D^\times$, and let $V = \operatorname{Ind}_H^{D^\times}(\chi_{a,b})$ be an irreducible representation. Then, by Frobenius reciprocity, we have that

$$\begin{aligned} \operatorname{Ext}^1_{D^\times}(\psi_{\gamma,\delta}, V) &\cong \operatorname{Ext}^1_{D^\times}(\psi_{\gamma,\delta}, \operatorname{Ind}_H^{D^\times}(\chi_{a,b})) \\ &\cong \operatorname{Ext}^1_H(\operatorname{Res}_H \psi_{\gamma,\delta}, \chi_{a,b}) \\ &\cong \operatorname{Ext}^1_H(\chi_{\gamma^2,(p+1)\delta}, \chi_{a,b}) \\ &\cong \operatorname{Ext}^1_H(\mathbb{1}, \chi_{\frac{a}{\gamma^2}, b-(p+1)\delta}) \end{aligned}$$

and we obtain a similar result for

$$\operatorname{Ext}^1_{D^\times}(\operatorname{Ind}_H^{D^\times}(\chi_{a,b}), \psi_{\gamma,\delta}) \cong \operatorname{Ext}^1_H(\mathbb{1}, \chi_{\frac{\gamma^2}{a},(p+1)\delta-b}).$$

Furthermore, if $V_1 = \operatorname{Ind}_H^{D^\times}(\chi_{a,b})$ and $V_2 = \operatorname{Ind}_H^{D^\times}(\chi_{c,d})$ are irreducible representations of $D^\times$, then using Frobenius Reciprocity and Mackey's formula (Theorem 2.1.7), we obtain

$$\begin{aligned} \operatorname{Ext}^1_{D^\times}(V_1, V_2) &= \operatorname{Ext}^1_{D^\times}(\operatorname{Ind}_H^{D^\times}(\chi_{a,b}), \operatorname{Ind}_H^{D^\times}(\chi_{c,d})) \\ &\cong \operatorname{Ext}^1_H(\chi_{a,b}, \operatorname{Res}_H \operatorname{Ind}_H^{D^\times}(\chi_{c,d})) \\ &\cong \operatorname{Ext}^1_H(\chi_{a,b}, \chi_{c,d} \oplus \chi_{c,d}^{\varpi}) \\ &\cong \operatorname{Ext}^1_H(\chi_{a,b}, \chi_{c,d} \oplus \chi_{c,pd}) \\ &\cong \operatorname{Ext}^1_H(\chi_{a,b}, \chi_{c,d}) \oplus \operatorname{Ext}^1_H(\chi_{a,b}, \chi_{c,pd}) \\ &\cong \operatorname{Ext}^1_H(\mathbb{1}, \chi_{\frac{c}{a},d-b}) \oplus \operatorname{Ext}^1_H(\mathbb{1}, \chi_{\frac{c}{a},pd-b}). \end{aligned}$$

In both cases, we find that it suffices to understand the spaces $\operatorname{Ext}^1_H(\mathbb{1}, \chi)$ for some irreducible character χ of H. □

As before, characters are given by products of characters of the components, and so we have that all characters of H are of the form

$$\begin{aligned} \chi_{m,n} : \varpi^{2\mathbb{Z}} \times (\mathbb{F}_{p^2}^\times \ltimes 1 + \varpi\mathcal{O}) &\to \overline{\mathbb{F}}_p^\times \\ (\varpi^{2x}, y, 1+z) &\mapsto m^x y^n \end{aligned}$$

Just as in $D^\times$, an extension of $\mathbb{1}$ by χ must be of the form

$$h \mapsto \begin{pmatrix} \chi(h) & c(h) \\ 0 & 1 \end{pmatrix}$$

for $h \in H$. We recall that c still satisfies the multiplication rule in equation 4.0.2 restricted to H, and furthermore we still have a classification of the form that $c|_{1+\varpi\mathcal{O}}$ can take. Taking $g \in 1 + \varpi\mathcal{O}$ as before, we still have the conjugation formula given in equation 4.2.1.

Proposition 4.3.2. Let $\chi_{m,n}$ be a character of H, and let c be the function associated to an extension of $\mathbb{1}$ by $\chi_{m,n}$. Then, c is of the form

$$c|_{1+\varpi\mathcal{O}} = \begin{cases} z\psi & m=1 \text{ and } n=0 \\ y\varphi' & m=1 \text{ and } n=p-1 \\ x\varphi & m=1 \text{ and } n=1-p \\ 0 & \text{else} \end{cases}$$

Proof. Taking $h = \varpi^2$ in equation 4.2.1, we have that

$$c(g) = \chi(\varpi^2)c(g)$$

and thus we have that for $\chi = \chi_{m,n}$, if $m \neq 1$, then we have that $c|_{1+\varpi\mathcal{O}} = 0$. Taking $h = [k]$ for some $k \in \mathbb{F}_{p^2}^\times$, we have that

$$h^{-1}gh = 1 + [a_1k^{p-1}]\varpi + [a_2]\varpi^2 + \cdots$$

as before, and thus equation 4.2.1 above gives us

$$xa_1 + ya_1^p + z(a_2 + a_2^p - a_1^{p+1}) = k^n(xa_1k^{p-1} + ya_1^pk^{-(p-1)} + z(a_2 + a_2^p - a_1^{p+1})) \tag{4.3.3}$$

We observe that when $n = 0$, we can choose z freely in $\overline{\mathbb{F}}_p$, and furthermore, if we restrict $a_1 \in \mathbb{F}_p$, equation 4.3.3 gives

$$\begin{aligned} xa_1 + ya_1^p &= xa_1k^{p-1} + ya_1^pk^{-(p-1)} \\ xa_1 + ya_1 &= xa_1k^{p-1} + ya_1k^{-(p-1)} \\ a_1(x(1-k^{p-1}) + y(1-k^{1-p})) &= 0 \\ a_1(k^{p-1}-1)\left(-x + \frac{1}{k^{p-1}}y\right) &= 0 \end{aligned}$$

Since this must hold for all k, we can choose k such that $k^{p-1} \neq 1$, and so we must have that

$$\left(-x + \frac{1}{k^{p-1}}y\right) = 0$$

and choosing k_1, k_2 such that $k_1^{p-1} \neq k_2^{p-1}$ implies that $x = y = 0$.

When $n \neq 0$, we can immediately conclude that $z = 0$. Furthermore, we can take $a_1 \in \mathbb{F}_p$, and thus equation 4.3.3 gives

$$\begin{aligned} xa_1 + ya_1^p &= xa_1k^{n+p-1} + ya_1^pk^{n-(p-1)} \\ xa_1 + ya_1 &= xa_1k^{n+p-1} + ya_1k^{n-(p-1)} \\ a_1(x(1-k^{n+p-1}) + y(1-k^{n-(p-1)})) &= 0 \end{aligned}$$

When $n = p-1$, we obtain

$$a_1x(1-k^{2(p-1)}) = 0 \tag{4.3.4}$$

and since $2(p-1) < p^2-1$, we can choose k such that $k^{2(p-1)} \neq 1$ and thus $x = 0$. Similarly, when $n = -(p-1)$, we obtain

$$a_1y(1-k^{-2(p-1)}) = 0 \tag{4.3.5}$$

and thus $y = 0$. When $n \neq 0, \pm(p-1)$, the exponent on k is not p^2-1, and so for any $x, y \in \overline{\mathbb{F}}_p$, we have a polynomial in k of degree less than p^2-1, but since this polynomial must equal 0 for all $k \in \mathbb{F}_{p^2}^\times$, the polynomial must be identically 0, and so we must have that $x = y = z = 0$. We conclude that

$$c|_{1+\varpi\mathcal{O}} = \begin{cases} z\psi & m=1 \text{ and } n=0 \\ y\varphi' & m=1 \text{ and } n=p-1 \\ x\varphi & m=1 \text{ and } n=1-p \\ 0 & \text{else} \end{cases}$$

which is our desired result. ☐

Lemma 4.3.6. Let $\chi_{m,n}$ be a character of H, and let c be the function associated to an extension of $\mathbb{1}$ by $\chi_{m,n}$. Then,

- When $n = 0$, we have that $c([k]) = 0$ and we have no restriction on $c(\varpi^2)$.
- When $n \neq 0$, we have no restriction on $c([k])$ and $c(\varpi^2)$ is determined by $c([k])$.

Proof. By the multiplication rule, it suffices to understand $c([k])$ for some generator k. Then, repeatedly using the multiplication rule in equation 4.0.2, we obtain the identity

$$c([k]^\ell) = c([k]) \left(\sum_{j=0}^{\ell-1} \chi([k])^j \right) \tag{4.3.7}$$

Then, we have that

$$0 = c([1]) = c([k]^{p^2-1}) = c([k]) \left(\sum_{j=0}^{p^2-2} \chi([k])^j \right) \tag{4.3.8}$$

When $\chi([k]) = 1$, and thus χ is identically 1 on $\mathbb{F}_{p^2}^\times$, corresponding to the case of $n = 0$, we have that

$$0 = c([k])(p^2 - 1)$$

and thus $c([k]) = 0$. When $n \neq 0$, equation 4.3.8 provides no restriction on $c([k])$.

We then consider the case of $c(\varpi^2)$. We examine the conjugation relation, which in this case must be trivial, as $\varpi^2 = p$ is central in $D^\times$. We have that

$$\begin{aligned}
c([k]) &= c(\varpi^2[k]\varpi^{-2}) \\
&= \chi(\varpi^2)c([k]\varpi^{-2}) + c(\varpi^2) \\
&= \chi(\varpi^2)(\chi([k])c(\varpi^{-2}) + c([k])) + c(\varpi^2) \\
&= \chi(\varpi^2)(\chi([k])\chi(\varpi^2)^{-1})c(\varpi^2) + c([k])) + c(\varpi^2) \\
&= \chi([k])c(\varpi^2) + \chi(\varpi^2)c([k]) + c(\varpi^2) \\
0 &= (1 - \chi([k]))c(\varpi^2) + (\chi(\varpi^2) - 1)c([k])
\end{aligned}$$

When $n \neq 0$, we can write

$$c(\varpi^2) = \frac{\chi(\varpi^2) - 1}{\chi([k]) - 1} c([k]) \tag{4.3.9}$$

and thus $c(\varpi^2)$ is determined by $c([k])$. When $n = 0$, $c(\varpi^2)$ has no restriction. ☐

We now examine when two such functions c produce equivalent extensions. We use the following notation.

$$c|_{1+\varpi\mathcal{O}} = x\varphi + y\varphi' + z\psi \qquad c([k]) = s \qquad c(\varpi^2) = t$$

Theorem 4.3.10. Let $\chi_{m,n}$ be a character of H. Then, we have that

$$\dim \operatorname{Ext}^1_H(\mathbb{1}, \chi_{m,n}) = \begin{cases} 2 & (m,n) = (1,0) \\ 1 & (m,n) = (1, \pm(p-1)) \\ 0 & \text{else} \end{cases}$$

Proof. When $m \neq 1$, recall that $c|_{1+\varpi\mathcal{O}} = 0$, and furthermore, using equation 4.3.9, we can write

$$c([k]) = \frac{\chi([k]) - 1}{\chi(\varpi^2) - 1} c(\varpi^2)$$

and thus a choice of $t = c(\varpi^2)$ determines c on all of H. If we choose two extensions parameterized by t_1 and t_2, and examine the image of ϖ^2, we have that

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m & t_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m & t_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

and we obtain the relation

$$u + t_1 = mu + t_2 \tag{4.3.11}$$

Since $m \neq 1$, choosing

$$u = \frac{t_1 - t_2}{m - 1}$$

produces the desired intertwining map, and thus all extensions in this case are equivalent.

When $m = 1$, note that equation 4.3.11 requires that $t_1 = t_2$ when a choice of t is permitted, and puts no restriction on u. Examining the image of $[k]$, where k is a generator of $\mathbb{F}_{p^2}^\times$, gives us the relation

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^n & s_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k^n & s_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

and thus we obtain the relation

$$u + s_1 = k^n u + s_2$$

and when $n = 0$, these extensions are equivalent if and only if $s_1 = s_2$. Otherwise, we can choose

$$u = \frac{s_1 - s_2}{k^n - 1}$$

and thus these extensions are equivalent. Finally, we check that

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_1\varphi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_2\varphi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \implies u + x_1\varphi = u + x_2\varphi \implies x_1 = x_2$$
$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y_1\varphi' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y_2\varphi' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \implies u + y_1\varphi' = u + y_2\varphi' \implies y_1 = y_2$$
$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1\psi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_2\psi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \implies u + z_1\psi = u + z_2\psi \implies z_1 = z_2$$

Thus, from lemma 4.3.6 we conclude the following:

- When $m \neq 1$, all choices of parameters in c are equivalent.
- When $m = 1$ and $n = 0$, choices of z and t are inequivalent.
- When $m = 1$ and $n = p - 1$, choices of y are inequivalent.
- When $m = 1$ and $n = 1 - p$, choices of x are inequivalent.
- When $m = 1$ and n is anything else, no choices are available.

Thus, we obtain the result:

$$\dim \operatorname{Ext}^1_H(\mathbb{1}, \chi_{m,n}) = \begin{cases} 2 & (m,n) = (1,0) \\ 1 & (m,n) = (1, \pm(p-1)) \\ 0 & \text{else} \end{cases}$$

□

We now relate these to extensions of $D^\times$.

Corollary 4.3.12. Previously, we had shown that

$$\mathrm{Ext}^1_{D^\times}(\psi_{\gamma,\delta}, \mathrm{Ind}_H^{D^\times}(\chi_{a,b})) = \mathrm{Ext}^1_H(\mathbb{1}, \chi_{\frac{a}{\gamma^2}, b-(p+1)\delta})$$

and thus we have that

$$\dim \mathrm{Ext}^1_{D^\times}(\psi_{\gamma,\delta}, \mathrm{Ind}_H^{D^\times}(\chi_{a,b})) = \begin{cases} 2 & a = \gamma^2, b = (p+1)\delta \\ 1 & a = \gamma^2, b-(p+1)\delta = \pm(p-1) \\ 0 & \text{else} \end{cases}.$$

Similarly, we have that

$$\mathrm{Ext}^1_{D^\times}(\mathrm{Ind}_H^{D^\times}(\chi_{a,b}), \mathrm{Ind}_H^{D^\times}(\chi_{c,d})) = \mathrm{Ext}^1_H(\mathbb{1}, \chi_{\frac{c}{a}, d-b}) \oplus \mathrm{Ext}^1_H(\mathbb{1}, \chi_{\frac{c}{a}, pd-b})$$

and so, removing the cases where the induction is reducible, we conclude

$$\mathrm{Ext}^1_{D^\times}(\mathrm{Ind}_H^{D^\times}(\chi_{a,b}), \mathrm{Ind}_H^{D^\times}(\chi_{c,d})) = \begin{cases} 3 & a = c, (b,d) \in S \\ 2 & a = c, b = d \text{ or } b = pd \text{ (excluding above)} \\ 1 & a = c, b-d = \pm(p-1) \text{ or } b - pd = \pm(p-1) \text{ (excluding above)} \\ 0 & \text{else} \end{cases}$$

where

$$\begin{aligned} S = \{&(\lambda(p+1)+p, \lambda(p+1)+1), \\ &(\lambda(p+1)-p, \lambda(p+1)-1), \\ &(\lambda(p+1)+1, \lambda(p+1)+1), \\ &(\lambda(p+1)-1, \lambda(p+1)-1)\} \end{aligned}$$

for some integer $0 \le \lambda < p-1$.

5 Quaternion Algebras over Field Extensions of $\mathbb{Q}_p$

Let F be a p-adic field with degree n, ramification degree e, and residue field degree f. Throughout this section, we assume $p > 2e+1$ (this implies $1 + \varpi\mathcal{O}$ is torsion-free).

5.1 Unramified Extensions

In this section, we assume that $e = 1$, which means that F is unramified over $\mathbb{Q}_p$. Let D denote the quaternion algebra over F, with units $D^\times$. Then, much of our work follows almost identically from that of the case where $F = \mathbb{Q}_p$. We obtain the following results.

Proposition 5.1.1. Let K denote $1 + \varpi\mathcal{O} \subset D^\times$. Then, we have that

$$\dim \mathrm{Hom}(K, \overline{\mathbb{F}}_p) = 3f$$

and has basis $\{\varphi_1, \ldots, \varphi_f, \varphi'_1, \ldots, \varphi'_f, \psi_1, \ldots, \psi_f\}$, defined by

$$\begin{aligned} \varphi_i : K &\to \overline{\mathbb{F}}_p \\ 1 + [a_1]\varpi + [a_2]\varpi^2 + \cdots &\mapsto a_1^{p^i} \end{aligned}$$

$$\begin{aligned} \varphi'_i : K &\to \overline{\mathbb{F}}_p \\ 1 + [a_1]\varpi + [a_2]\varpi^2 + \cdots &\mapsto a_1^{p^{f+i}} \end{aligned}$$

$$\begin{aligned} \psi_i : K &\to \overline{\mathbb{F}}_p \\ 1 + [a_1]\varpi + [a_2]\varpi^2 + \cdots &\mapsto (\mathrm{Tr}(a_2) - N(a_1))^{p^i} \end{aligned}$$

for $0 \le i < f$. For convenience, we denote $\varphi'_i = \varphi_{f+i}$.

Proof. Let J denote the subgroup

$$J = \{1 + [0]\varpi + [c]\varpi^2 + [a_3]\varpi^3 + \cdots : c, a_i \in k_D \,,\ \exists\, d \in k_D \text{ such that } d^p - d = c\}$$

Then, using a nearly identical method as in Proposition 4.1.2, replacing p with p^f, we can show that $J = [K,K]K^p$. Then, truncating at the ϖ^3 term, the order of $J/(1+\varpi^3\mathcal{O})$ is p^f, and so the order of K/J is p^{3f}. Since this is a p-group, and every non-identity element is of order p, K/J is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{3f}$ and thus we have that

$$\dim \operatorname{Hom}(K, \overline{\mathbb{F}}_p) = 3f.$$

Furthermore, since Galois conjugates are linearly independent, we can begin with φ_0, φ_0', and ψ_0 from above, which vaguely resemble the analogs from Proposition 4.1.6, and examine their Galois conjugates. Since φ_0 and φ_0' are conjugates under $\operatorname{Gal}(k_D/k_F)$, we can apply one of the f Galois automorphisms of $k_F/\mathbb{F}_p = \mathbb{F}_{p^f}/\mathbb{F}_p$, to produce a set of $2f$ conjugates of φ_0. Then, since the image of ψ_0 is in k_F, we can apply one of the f Galois automorphisms of $k_F/\mathbb{F}_p$ to produce a set of f conjugates. Thus, we have produced $3f$ elements which form a basis for $\operatorname{Hom}(K, \overline{\mathbb{F}}_p)$. ☐

Note that characters of $D^\times$ are of the form

$$\begin{aligned}\psi_{\gamma,\delta} : \varpi^{\mathbb{Z}} \ltimes (k_D^\times \ltimes 1 + \varpi\mathcal{O}) &\to \overline{\mathbb{F}}_p^\times \\ (\varpi^x, y, 1+z) &\mapsto \gamma^x y^{(p^f+1)\delta}\end{aligned}$$

where $\gamma \in \overline{\mathbb{F}}_p^\times$ and $0 \le \delta \le p^f - 1$

Proposition 5.1.2. Let $\psi_{\gamma,\delta}$ be a character of $D^\times$, and let c be the function associated to an extension of $\mathbb{1}$ by $\chi_{m,n}$. Then, we have that

$$c|_{1+\varpi\mathcal{O}} = \begin{cases} \sum_{i=0}^{f-1} y_i\psi_i & (\gamma,\delta) = (1,0) \\ 0 & \text{else} \end{cases}$$

Proof. We proceed analogously to Proposition 4.2.2. From above, we know that

$$c|_{1+\varpi\mathcal{O}} = \sum_{i=0}^{2f-1} x_i\varphi_i + \sum_{i=0}^{f-1} y_i\psi_i$$

Using the formula in equation 4.2.1, with $h = \varpi$, we have

$$\begin{aligned} &x_0a_1 + x_1a_1^p + \cdots + x_{2f-1}a_1^{p^{2f-1}} + y_0(a_2 + a_2^{p^f} - a_1^{p^f+1}) \\ &\quad + y_1(a_2 + a_2^{p^f} - a_1^{p^f+1})^p + \cdots + y_{f-1}(a_2 + a_2^{p^f} - a_1^{p^f+1})^{p^{f-1}} \\ &= \gamma\Big(x_0a_1^{p^f} + x_1a_1^{p^{f+1}} + \cdots + x_{2f-1}a_1^{p^{f-1}} + y_0(a_2 + a_2^{p^f} - a_1^{p^f+1}) \\ &\quad + y_1(a_2 + a_2^{p^f} - a_1^{p^f+1})^p + \cdots + y_{f-1}(a_2 + a_2^{p^f} - a_1^{p^f+1})^{p^{f-1}}\Big) \end{aligned}$$

and since each ψ_i is invariant under the action of ϖ, we see that when $\gamma = 1$, we can choose each of the y_i freely in $\overline{\mathbb{F}}_p$, and when $\gamma \ne 1$, each y_i is forced to be 0. Furthermore, we can pair the terms $a_i^{p^i}$ with $a_i^{p^{f+i}}$, and apply the argument in the proof of Proposition 4.2.2 pairwise to show that each x_i must be 0. Applying a similar argument to conjugation by $h = [k]$ yields the desired result. ☐

Theorem 5.1.3. Let $\psi_{\gamma,\delta}$ be a character of $D^\times$. Then, we have that

$$\dim \operatorname{Ext}^1_{D^\times}(\mathbb{1}, \psi_{\gamma,\delta}) = \begin{cases} f+1 & (\gamma,\delta) = (1,0) \\ 0 & (\gamma,\delta) \ne (1,0) \end{cases}$$

Proof. We proceed analogously to the proof of Theorem 4.2.8, applying the reasoning pairwise, however instead we make a choice of coefficient for each y_i when $(\gamma, \delta) = (1, 0)$, and each choice is inequivalent, so we have dimension $f + 1$. ☐

Let $H = \varpi^{2\mathbb{Z}} \times (k_D^\times \ltimes 1 + \varpi\mathcal{O})$. Then, characters of H are of the form

$$\begin{aligned} \chi_{m,n} : \varpi^{2\mathbb{Z}} \times (k_D^\times \ltimes 1 + \varpi\mathcal{O}) &\to \overline{\mathbb{F}}_p^\times \\ (\varpi^x, y, 1 + z) &\mapsto m^x y^n \end{aligned}$$

where $m \in \overline{\mathbb{F}}_p^\times$ and $0 \leq \delta \leq p^{2f} - 1$

Theorem 5.1.4. Let $\chi_{m,n}$ be a character of H. Then, we have that

$$\dim \operatorname{Ext}^1_H(\mathbb{1}, \chi_{m,n}) = \begin{cases} f+1 & (m,n) = (1,0) \\ 1 & (m,n) = (1, p^i(p^f - 1)),\ 0 \leq i < 2f \\ 0 & \text{else} \end{cases}$$

Proof. We proceed analogously to the proof of Theorem 4.3.10, applying the reasoning pairwise, however, we obtain more distinguished values of n for which one element of the pair is automatically 0, as in equations 4.3.4 and 4.3.5, and thus we can choose that coefficient freely. ☐

Corollary 5.1.5. Let $\psi_{\gamma,\delta}$ be a character of $D^\times$, and let $\chi_{a,b}$ be a character of H. Then,

$$\dim \operatorname{Ext}^1_{D^\times}(\psi_{\gamma,\delta}, \operatorname{Ind}_H^{D^\times}(\chi_{a,b})) = \begin{cases} f+1 & a = \gamma^2, b = (p^f+1)\delta \\ 1 & a = \gamma^2, b - (p^f+1)\delta = p^i(p^f - 1),\ 0 \leq i < 2f \\ 0 & \text{else} \end{cases}$$

Similarly, let $\chi_{c,d}$ be a character of H. Removing the inductions which are reducible, we have that

$$\operatorname{Ext}^1_{D^\times}(\operatorname{Ind}_H^{D^\times}(\chi_{a,b}), \operatorname{Ind}_H^{D^\times}(\chi_{c,d})) = \begin{cases} f+2 & a = c, b = d \text{ and } b - p^f d = p^i(p^f-1),\ 0 \leq i < 2f \\ f+2 & a = c, b = p^f d \text{ and } b - d = p^i(p^f-1),\ 0 \leq i < 2f \\ f+1 & a = c, b = d \text{ (but not above)} \\ f+1 & a = c, b = p^f d \text{ (but not above)} \\ 1 & a = c, b - d = p^i(p^f-1) \text{ or } b - p^f d = p^j(p^f-1) \ (i = j \text{ reducible}) \\ 0 & \text{else} \end{cases}$$

Proof. We apply Theorem 5.1.4 analogously as in the proof of Corollary 4.3.12. ☐

5.2 Ramified Extensions and Next Steps

When $e > 1$, much of our prior work does not translate as well.

Remark 5.2.1. When $e > 1$, Lemma 4.1.4 fails, and can be replaced with the weaker condition that any element of $1 + \varpi^{2e+1}\mathcal{O}$ can be written as a p-th power of an element of $1 + \varpi\mathcal{O}$.

Thus, in order for our classification of homomorphisms from $1 + \varpi\mathcal{O}$ to be applicable, we must examine which elements of $1 + \varpi\mathcal{O}/1 + \varpi^{2e+1}\mathcal{O}$ can be written as commutators.

Proposition 5.2.2. For odd $k > 1$, we can produce a commutator of the form $1 + [a_k]\varpi^k + \cdots$.

Proof. We compute

$$\begin{aligned} [1 + [a]\varpi^{k-1}, 1 + [b]\varpi] &= (1 + [a]\varpi^{k-1})(1 + [b]\varpi)(1 + [a]\varpi^{k-1})^{-1}(1 + [b]\varpi)^{-1} \\ &= 1 + [0]\varpi + \cdots + [0]\varpi^{k-1} + [b(a - a^{p^f})]\varpi^k + \cdots \end{aligned}$$

Choosing any a such that $a - a^{p^f} \neq 0$, we can take $b = a_k(a - a^{p^f})^{-1}$ to produce our desired commutator. ☐

If we could also do this for even k, then we could construct our desired elements by using commutators to iteratively reduce the error, until it is of order ϖ^{2e+1}. Unfortunately, this proof does not work for even k, as the formula for the coefficient of ϖ^k when k is even produces a k_F-line inside of k_D. Through computational experiments and brute-force checking for small cases, we hypothesize that in the coefficient of ϖ^4, of the p^{2f} elements possible, exactly $\binom{p^f+1}{2}$ are attained by commutators. This is reason to suspect that the structure of the commutator subgroup is more interesting than one would expect.

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References

[1] John Voight *Quaternion algebras*, `https://math.dartmouth.edu/ jvoight/quat-book.pdf`