QUIVER POLYNOMIALS AND BUMPLESS PIPE DREAMS

ALEXANDER C. VIDINAS AND ZIXUAN XU

1. Introduction

A **quiver** is a finite directed graph. In this paper, we study the *quiver polynomials* of [BF99] in terms of *bumpless pipe dreams*, new combinatorial objects introduced in [LLS18]. Quiver polynomials arise as multi-degrees of the natural action of a product of general linear groups on the *representation space* of a quiver. Our main focus is on type A equioriented quivers, those which are simple directed paths. Numerous positive formulas have been given to describe quiver polynomials, notably in work of Knutson, Miller, and Shimozono [KMS06]. In particular, they presented a solution to the factor sequence conjecture of [BF99], which represents quiver polynomials as sums of products of specialized Schur functions. Outside of the equioriented case, there are generalizations of these formulas (see [BFR05; BR07; KKR19]). However, an analogue for the factor sequence formula is unknown. By using bumpless pipe dreams to study equioriented quiver polynomials, we find an intuitive means of computing quiver polynomials and their equivalent formula. With this, we hope we hope to gain insight into studying quiver polynomials arising from the non-equioriented case.

Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 is the set of vertices and Q_1 the set of directed edges. Fixing a dimension vector $\mathbf{d} \in \mathbb{N}^{|Q_0|}$, we can construct the **representation space** of a quiver as

$$\mathrm{Rep}_Q(\mathbf{d}) := \bigoplus_{x \xrightarrow{\alpha} y \in Q_1} \mathrm{Mat}(d_x, d_y),$$

where $\mathrm{Mat}(d_x, d_y)$ denotes the space of $d_x \times d_y$ matrices.

Define the group

$$\mathrm{GL}_Q(\mathbf{d}) := \prod_{x \in Q_0} \mathrm{GL}(d_x),$$

where $\mathrm{GL}(d_x)$ is the set of $d_x \times d_x$ invertible matrices. We take a natural conjugation action of $\mathrm{GL}_Q(\mathbf{d})$ on $\mathrm{Rep}_Q(\mathbf{d})$, and we call the orbit closures of $\mathrm{Rep}_Q(\mathbf{d})$ under this action *quiver loci*. We investigate *quiver polynomials*, which are the multidegrees of quiver loci under a specific $\mathbb{Z}^d$-grading for $d = \sum_{x \in Q_0} d_x$. In particular, we will focus on *Type A_{m+1} equioriented quivers*, whose underlying graph is a directed path with $m+1$ vertices.

For much of this paper, we use Q to denote a type A_{m+1} equioriented quiver. We label the vertices of Q by $0, \ldots, m$ in order and dimension vectors of representations of Q by $\mathbf{d} = (d_0, \ldots, d_m)$. Fixing $\mathbf{d}$, $\mathbf{x_d}$ denotes the sequence of alphabets $\mathbf{x}^0_{\mathbf{d}}, \ldots, \mathbf{x}^m_{\mathbf{d}}$, where each $\mathbf{x}^i_{\mathbf{d}} = (x^i_1, \ldots, x^i_{d_i})$ has cardinality d_i. We use $\mathbf{x}$ to denote the sequence of variables $\mathbf{x}^0, \ldots, \mathbf{x}^m$ and $\mathring{\mathbf{x}}$ to denote the block reversal of $\mathbf{x_d}$, which is the sequence of variables $\mathbf{x}^m, \ldots, \mathbf{x}^0$.

Date: October 20, 2020.

Consider a generic quiver representation $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_m)$ of Q, where $\Phi_i : V_{i-1} \to V_i$ are $r_{i-1} \times r_i$ matrices filled by the variables $f^i_{\alpha\beta}$. Then the (**ordinary**) **quiver polynomial** is defined as the *multidegree*[1]

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) = \mathcal{C}(\Omega_r; \mathbf{x})$$

of the quiver locus Ω_r in $\mathrm{Rep}_Q(\mathbf{d})$, under the $\mathbb{Z}^d$-grading in which $\deg(f^i_{\alpha\beta}) = x^{i-1}_\alpha - x^i_\beta$. However, this expression has many equivalent forms, expressed in terms of Schubert polynomials. More specifically, for type A equioriented quivers, we can identify orbit closures of $\mathrm{Rep}_Q(\mathbf{d})$ with **rank arrays r** that record ranks of successive matrix products. Additionally, using a rank array, we may define a unique **Zelevinsky Permutation**, $v(\mathbf{r})$. For the ordinary quiver polynomial associated to the quiver locus corresponding to a rank array $\mathbf{r}$, we take the following **ratio formula** given in [KMS06] as definition:

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) = \frac{\mathfrak{S}_{v(\mathbf{r})}(\mathbf{x} - \mathring{\mathbf{x}})}{\mathfrak{S}_{v(Hom)}(\mathbf{x} - \mathring{\mathbf{x}})}.$$

This expression naturally extends to a **double ratio formula**, defined as

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) = \frac{\mathfrak{S}_{v(\mathbf{r})}(\mathbf{x} - \mathring{\mathbf{y}})}{\mathfrak{S}_{v(Hom)}(\mathbf{x} - \mathring{\mathbf{y}})}.$$

In addition to this formula, Buch and Fulton [BF99] found a second expansion for quiver polynomials – a sum of products of specialized Schur functions.

Theorem 1.1 ([BF99])**.** *There exist unique integers $c_{\boldsymbol{\lambda}}(\mathbf{r})$ called **quiver constants** such that*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) = \sum_{\boldsymbol{\lambda}} c_{\boldsymbol{\lambda}}(\mathbf{r}) s_{\boldsymbol{\lambda}}(\mathbf{x} - \mathring{\mathbf{x}}),$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ are sequences of partitions, and

$$s_{\boldsymbol{\lambda}}(\mathbf{x} - \mathring{\mathbf{x}}) = s_{\lambda_1}(\mathbf{x}^0_{\mathbf{d}} - \mathbf{x}^1_{\mathbf{d}}) s_{\lambda_2}(\mathbf{x}^1_{\mathbf{d}} - \mathbf{x}^2_{\mathbf{d}}) \cdots s_{\lambda_m}(\mathbf{x}^{m-1}_{\mathbf{d}} - \mathbf{x}^m_{\mathbf{d}}).$$

Furthermore, as described in [Buc05] and remarked in [KMS06], this formula has a doubled form, making use of double Schur functions. The following theorem is a specialization of Buch's K-theoretic result to the cohomological narrative.

Theorem 1.2 ([Buc05; KMS06])**.** *Given a rank array $\mathbf{r}$ with quiver constants $c_{\boldsymbol{\lambda}}(\mathbf{r})$, the double quiver polynomial of $\mathbf{r}$ can be written as*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) = \sum_{\boldsymbol{\lambda}} c_{\boldsymbol{\lambda}}(\mathbf{r}) s_{\boldsymbol{\lambda}}(\mathbf{x} - \mathring{\mathbf{y}}),$$

where $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ are sequences of partitions and

$$s_{\boldsymbol{\lambda}}(\mathbf{x} - \mathring{\mathbf{y}}) = s_{\lambda^{(1)}}(\mathbf{x}^0 - \mathbf{y}^1) s_{\lambda^{(2)}}(\mathbf{x}^1 - \mathbf{y}^2) \cdots s_{\lambda^{(m)}}(\mathbf{x}^{m-1} - \mathbf{y}^m).$$

It is conjectured in [BF99] that the quiver constants count the number of sequences of tableaux $\mathbf{T} = (T_1, \ldots, T_m)$ for $\mathbf{r}$, known as **r-factor sequences**, such that $\mathrm{shape}(\mathbf{T}) = (\mathrm{shape}(T_1), \ldots, \mathrm{shape}(T_m)) = \boldsymbol{\lambda}$, where $\mathrm{shape}(T)$ denotes the underlying partition shape of a tableau T. The conjecture was later proved by Knutson, Miller, and Shimozono [KMS06] for Type A equioriented quivers, and we refer to it as the **factor sequence formula**. Let $\mathrm{FS}(\mathbf{r})$ denote the set of $\mathbf{r}$-factor sequences and let $\mathrm{shape}(T)$ denote the underlying partition shape of a tableau T. Formally, [KMS06] shows the following theorem.

[1]For more information on multidegrees, refer to Chapter 17 in [MS05].

Theorem 1.3 (Tableau Formula ([KMS06]))**.** *Given a rank array* $\mathbf{r}$*, the quiver polynomial can be written as*

$$\mathcal{Q}_{\mathbf{r}} = \sum_{\mathbf{T} \in \mathrm{FS}(\mathbf{r})} c_{\boldsymbol{\lambda}}(\mathbf{r}) s_{\boldsymbol{\lambda}}(\mathbf{x} - \mathring{\mathbf{y}}),$$

where $\mathbf{T} = (T_1, \ldots, T_m) \in \mathrm{FS}(\mathbf{r})$ *and* $\boldsymbol{\lambda} = \mathrm{shape}(\mathbf{T}) = (\mathrm{shape}(T_1), \ldots, \mathrm{shape}(T_m))$*.*

We present two propositions in which we frame the preceding results in terms of bumpless pipe dreams. The first expresses double quiver polynomials as sums of weights of bumpless pipe dreams. For a rank array $\mathbf{r}$, we let $\mathrm{BPD}(v(\mathbf{r}))$ denote the set of BPDs corresponding to the permutation $v(\mathbf{r})$.

Proposition 1.4 (Bumpless Pipe Formula)**.** *Let* $P(Hom)$ *denote the Rothe bumpless pipe dream corresponding to the permutation* $v(Hom)$*. Then given a rank array* $\mathbf{r}$ *we have*

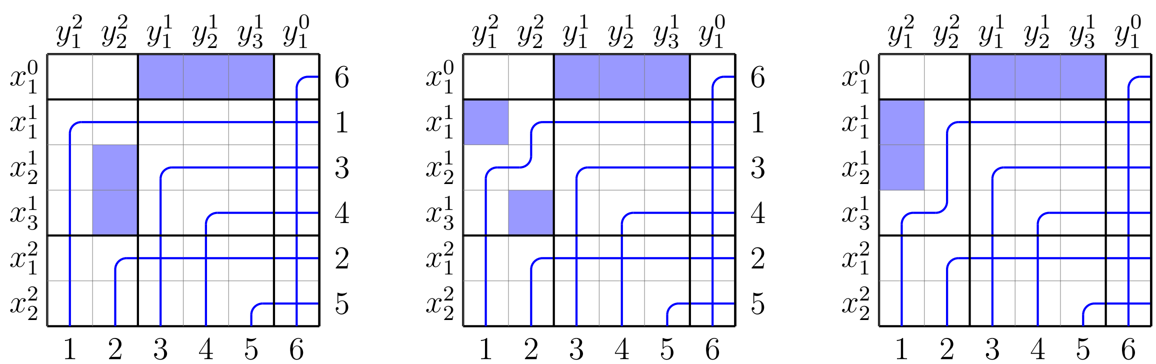
$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) = \sum_{P \in \mathrm{BPD}(v(\mathbf{r}))} (\mathbf{x} - \mathring{\mathbf{y}})^{P \setminus P(Hom)},$$

where $(\mathbf{x} - \mathring{\mathbf{y}})^{P \setminus P(Hom)}$ *denotes the product over the weights of all empty tiles* (i, j) *which lie in* P *but not* $P(Hom)$*.*

Example 1.5. *Let* Q *be a type* A_3 *equioriented quiver and fix* $\mathbf{d} = (1, 3, 2)$*. Given the rank array*

$$\mathbf{r} = \begin{array}{ccc|c} 2 & 1 & 0 & \\ \hline & & 1 & 0 \\ & 3 & 0 & 1 \\ 2 & 1 & 0 & 2 \end{array}.$$

It has Zelevinsky permutation $v(\mathbf{r}) = (6, 1, 3, 4, 2, 5)$ *with the following bumpless pipe dreams.*



For each bumpless pipe dream P*, the tiles which lie in* P *but not* $P(Hom)$ *are colored blue. Each such tile* (i, j) *contributes weight* $(x_i - y_j)$*. Hence by Proposition 1.4, we have*

$$\begin{aligned} \mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) &= (x_1^0 - y_1^1)(x_1^0 - y_2^1)(x_1^0 - y_3^1)(x_2^1 - y_2^2)(x_3^1 - y_2^2) \\ &+ (x_1^0 - y_1^1)(x_1^0 - y_2^1)(x_1^0 - y_3^1)(x_1^1 - y_1^2)(x_3^1 - y_2^2) \\ &+ (x_1^0 - y_1^1)(x_1^0 - y_2^1)(x_1^0 - y_3^1)(x_1^1 - y_1^2)(x_2^1 - y_1^2). \end{aligned}$$

The next result allows us to reinterpret the *factor sequence formula* in terms of bumpless pipe dreams. Based on the work of [Buc+06], we can compute quiver constants from Edelman-Greene bumpless pipe dreams (EG-BPDs), which are bumpless pipe dreams with all empty tiles top-left justified. For an EG-BPD P, let $\mathrm{shape}(P)$ denote the partition shape formed by the top-left justified empty tiles in P. We define a truncation map

$\tau_{\mathbf{d}} : \lambda \mapsto \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ according to $\mathbf{d}$. Specifically, we divide P into blocks according to $\mathbf{d}$, and $\tau_{\mathbf{d}}(\text{shape}(P))$ is the sequence of partition shapes in the blocks on the main block super-antidiagonal from right to left.The map is illustrated in Example 1.7. Formally, we have the following proposition.

Proposition 1.6. *Given a rank array* $\mathbf{r}$*, the quiver constant* $c_{\boldsymbol{\lambda}}(\boldsymbol{r})$ *is the number of Edelman-Greene bumpless pipe dreams* P *for* $v(\mathbf{r})$ *such that* $\tau_{\mathbf{d}}(\text{shape}(P)) = \boldsymbol{\lambda}$*, i.e.*

$$c_{\boldsymbol{\lambda}}(\mathbf{r}) = |\{P \in \text{EGBPD}(v(\mathbf{r})) : \tau_{\mathbf{d}}(\text{shape}(P)) = \boldsymbol{\lambda}\}|.$$

Example 1.7. *Let* Q *be a type* A_3 *equioriented quiver and fix* $\mathbf{d} = (2, 3, 2)$*. Consider the orbit closure with representative*

$$\mathbf{M} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \in \text{Rep}_Q(\mathbf{d}).$$

It has corresponding Zelevinsky permutation $v(\mathbf{r}) = (3, 4, 1, 2, 6, 5, 7)$.

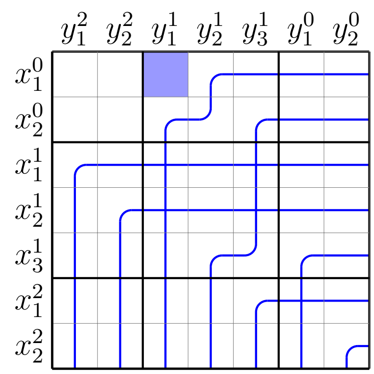


FIGURE 1. A bumpless pipe dream P for $v(\mathbf{r})$.

Here, P *is an Edelman-Greene bumpless pipe dream, and we have*

$$\boldsymbol{\lambda} = \tau_{\mathbf{d}}(\text{shape}(P)) = \tau_{\mathbf{d}}\left(\ydiagram{3,2}\right) = (\ydiagram{1}, \varnothing) = ((1), \varnothing).$$

Since P *is the unique EG-BPD that yields* $\boldsymbol{\lambda}$*, the corresponding quiver constant is* $c_{\boldsymbol{\lambda}} = 1$.

This report is organized as follows. In Section 2, we recall necessary information on Schubert polynomials, Schur functions, bumpless pipe dreams and quiver polynomials. Then in Section 3, we prove Proposition 1.4 and Proposition 1.6. In Section 4, we present some partial progress to proving the equivalence between Proposition 1.4 and Proposition 1.6 using techniques in bumpless pipe dreams. Finally, in Section 5, we present a conjecture that we are working towards proving.

2. Background

2.1. **Schubert Polynomials.** Let $\mathbb{Z}[\mathbf{x}] := \mathbb{Z}[x_1, \ldots, x_n]$ and let S_n denote the group of permutations of the elements in $[n] := \{1, 2, \ldots, n\}$. Note that S_n acts on $\mathbb{Z}[\mathbf{x}]$ by permuting variables, i.e. for $w = (w_1, \ldots, w_n) \in S_n$, $f \in \mathbb{Z}[\mathbf{x}]$,

$$w \cdot f = f(x_{w_1}, \ldots, x_{w_n}).$$

1	1	2	3	5
2	3	3		
3	4	5		
4				

FIGURE 2. Example of a semi-standard Young tableau of shape $\lambda = (5, 3, 3, 1)$

For $i \in [n-1]$, we define the **divided difference operator** $\partial_i : \mathbb{Z}[\mathbf{x}] \to \mathbb{Z}[\mathbf{x}]$ as

$$\partial_i(f) := \frac{f - s_i \cdot f}{x_i - x_{i+1}},$$

where $s_i \in S_n$ denotes the simple transposition $(i, i+1)$. Let $w_0 = (n, n-1, \ldots, 1)$ denote the longest permutation in S_n. For $w \in S_n$, the **Schubert polynomial** $\mathfrak{S}_w \in \mathbb{Z}[\mathbf{x}]$ is defined recursively as

$$\mathfrak{S}_{w_0}(\mathbf{x}) := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1,$$
$$\mathfrak{S}_w(\mathbf{x}) := \partial_i(\mathfrak{S}_{ws_i}(\mathbf{x})) \text{ for any } i \text{ with } w_i > w_{i+1}.$$

It is easy to check that the divided difference operator satisfies the following braid relations:

$$\partial_i^2 = 0,$$
$$\partial_i \partial_j = \partial_j \partial_i \quad \text{for} \quad |i-j| > 1, \text{ and}$$
$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}.$$

This makes the recursive definition for Schubert polynomials well-defined.

We may also define **double Schubert polynomials**. Let $\mathbb{Z}[\mathbf{x}, \mathbf{y}] = \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and let the divided difference operators ∂_i for $i > 0$ act on $\mathbb{Z}[\mathbf{x}, \mathbf{y}]$ by acting only on the x-variables. We define the double Schubert polynomial $\mathfrak{S}_w(\mathbf{x}; \mathbf{y})$ for $w \in S_n$ similarly. Let

$$\mathfrak{S}_{w_0}(\mathbf{x}; \mathbf{y}) = \prod_{i+j \leq 1} (x_i - y_j),$$

$$\mathfrak{S}_w(\mathbf{x}; \mathbf{y}) := \partial_i(\mathfrak{S}_{ws_i}(\mathbf{x}; \mathbf{y})) \text{ for any } i \text{ with } ws_i > w.$$

Note that we can recover the single Schubert polynomial by setting $\mathbf{y} = 0$, i.e. $\mathfrak{S}_w(\mathbf{x}; 0) = \mathfrak{S}_w(\mathbf{x})$. For more information on Schubert polynomials, the reader may refer to [Man01].

2.2. **Schur Functions.** In this section, we recall the combinatorial definition of Schur functions in terms of semi-standard Young tableaux and describe another combinatorial formula for Schur functions. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a sequence of nonnegative integers. We say λ is a **partition** if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. A **Young diagram** is a diagram of top-left justified boxes with the number of boxes of each row weakly decreasing from top to bottom. We say that a Young diagram is of shape λ if there are λ_i number of boxes in row i. A **semi-standard Young tableaux (SSYT)** is a filling of a Young diagram of shape λ with positive integers such that the labels are weakly increasing along each row and strictly increasing along each column. We use $\mathrm{SSYT}(\lambda, n)$ to denote the set of all semi-standard Young tableaux of shape λ with labels in $[n]$ and $\mathrm{SSYT}(\lambda)$ denotes the set of all semi-standard Young tableaux of shape λ with no restrictions on the labels.

Given $T \in \mathrm{SSYT}(\lambda)$, we define its weight as

$$\mathrm{wt}(T) = \prod x_i^{\#\text{ of labels } i \text{ in } T}.$$

Then we can define the **Schur polynomial** for $\lambda = (\lambda_1, \dots, \lambda_k)$ as

$$s_\lambda(x_1, \dots, x_k) = \sum_{T \in \mathrm{SSYT}(\lambda, k)} \mathrm{wt}(T),$$

and we define the **Schur function** as the formal power series

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathrm{wt}(T).$$

Note that Schur functions can be specialized by setting all but finitely many variables as 0; in doing so, we recover Schur polynomials. We remark that the set of Schur functions is a basis of the ring of symmetric functions.

Similarly, we can define the *double Schur functions* by setting

$$\mathrm{wt}(T) = \prod_{(i,j) \in \lambda} (x_{T_{ij}} - y_{T_{ij}+j-i}),$$

where the product is over all positions in the Young diagram of shape λ. Then the **double Schur function** is defined as

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathrm{wt}(T).$$

Example 2.1. *Let $\lambda = (2,1)$ and we have*

$$\mathrm{SSYT}(\lambda, 2) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \right\}$$

Therefore we have the Schur polynomial

$$s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2.$$

2.2.1. *Plus Diagrams.* Another way to compute Schur functions is through plus diagrams by [KMY09], [IN09] and [Kre06]. We define a **plus diagram** to be a set of tuples of positive integers $D = \{(i,j) : i, j > 0\}$, which we realize by setting plus signs, whose positions are specified by D, in a northwest-justified grid.

Example 2.2. *The following is an example of a plus diagram for $D = \{(1,1), (2,3), (3,2)\}$.*

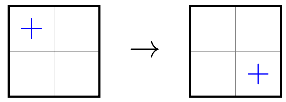
	y_1	y_2	y_3	y_4	y_5
x_1	+				
x_2			+		
x_3		+			
x_4					
x_5					

For a plus diagram D, we define the weight of D on the sequence of alphabets $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ as

$$(\mathbf{x} - \mathbf{y})^D = \prod_{(i,j) \in D} (x_i - y_j).$$

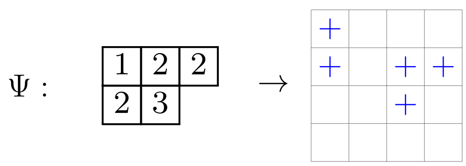
Therefore, the plus diagram above has weight $(x_1 - y_1)(x_2 - y_3)(x_3 - y_2)$.

These diagrams are subject to operations called local moves. A **local move** on a plus diagram moves a plus sign one step along its diagonal when there are no plus signs immediately to the right or below it, as illustrated below.



Formally, a local move on a plus diagram moves $(i,j) \in D$ to $(i+1,j+1)$ when $(i+1,j), (i,j+1), (i+1,j+1) \notin D$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we say that D is a plus diagram of **shape** λ if it can be obtained through a sequence of local moves on the plus diagram $D_\lambda = \{(i,j) : 1 \leq i \leq k, 1 \leq j \leq \lambda_i\}$, the plus diagram with plus signs on the positions of the boxes in the Young diagram of shape λ, top-left left in the grid. We us $\mathcal{D}(\lambda)$ to denote the set of plus diagrams of shape λ.

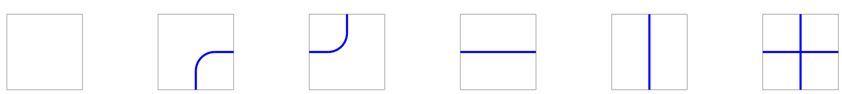
There is a natural weight-preserving bijection between SSYT of shape λ and plus diagrams of shape λ (see e.g. [KMY09]). We define the map $\Psi : \mathrm{SSYT}(\lambda) \to \mathcal{D}(\lambda)$ as follows: for $T \in \mathrm{SSYT}(\lambda)$, construct $\Psi(T)$ as $\{(T_{ij}, T_{ij}+j-i) : \forall (i,j) \in \lambda\}$. The following is an example of the bijection.



As an immediate result, the Schur function for a partition λ can be expressed as

$$s_\lambda(\mathbf{x} - \mathbf{y}) = \sum_{D \in \mathcal{D}(\lambda)} (\mathbf{x} - \mathbf{y})^D. \tag{2.2.1}$$

2.3. **Bumpless Pipe dreams.** Bumpless pipe dreams were first defined and studied by Lam, Lee and Shimozono in [LLS18]. Here we briefly recall the definition of bumpless pipe dreams and their useful properties. Start with the following six tiles.



We interpret these tiles as forming a network of pipes in a grid and we consider the "crossing" tile as a place where two pipe cross, one vertically and one horizontally. A **bumpless pipe dream** is a tiling of the $n \times n$ grid using the tiles in Figure 2.3 such that

(1) there are n pipes in total,
(2) each pipe starts vertically in the bottom edge of the grid and ends horizontally at the right edge of the grid, and
(3) pairwise, pipes cross at most once.

We use matrix coordinates for tiles in the grid, i.e. row coordinates increase from top to bottom and column coordinates increase from left to right. The tuple (i,j) indicates the cell in row i and column j.

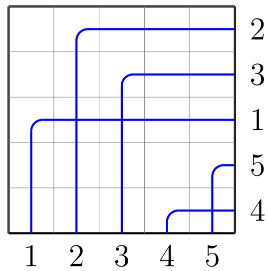
For a permutation $w = (w_1, \ldots, w_n) \in S_n$, a w-bumpless pipe dream is a bumpless pipe dream in the $n \times n$ grid such that the pipe starting from row i on the right edge ends at column w_i on the bottom edge. We write $\mathrm{BPD}(w)$ to denote the set of bumpless pipe dreams for w. We define the weight of a bumpless pipe dream P on the sequences of alphabets

$\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$ as the product

$$(\mathbf{x} - \mathbf{y})^P := \prod (x_i - y_j)$$

over all the blank tiles (i, j).

Example 2.3. *The following is a bumpless pipe dream for the permutation $w = (2, 3, 1, 5, 4) \in S_5$.*



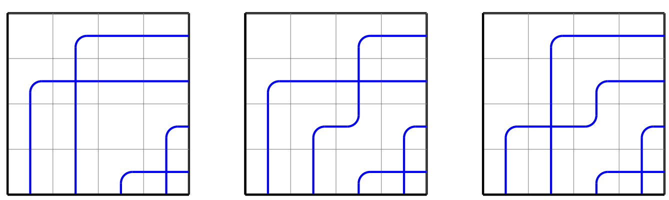
The weight of this bumpless pipe dream is $(x_1 - y_1)(x_2 - y_1)(x_4 - y_4)$.

The following theorem gives a combinatorial interpretation for double Schubert polynomials.

Theorem 2.4 ([LLS18])**.** *Let $w \in S_n$ be a permutation, then*

$$\mathfrak{S}_w(\mathbf{x} - \mathbf{y}) = \sum_{P \in \mathrm{BPD}(w)} (\mathbf{x} - \mathbf{y})^P.$$

Example 2.5. *The following are the three BPDs for the permutation $w = (2, 1, 4, 3)$:*



By Theorem 2.4,

$$\mathfrak{S}_w(\mathbf{x} - \mathbf{y}) = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_2)(x_1 - y_2) + (x_1 - y_1)(x_2 - y_1)$$

2.3.1. *Rothe Pipe Dreams and Drooping.* Since a w-bumpless pipe dream is uniquely determined by the positions of the elbow tiles and every pipe must turn at least once, we can obtain a bumpless pipe dream by putting an SE elbow in positions $\{(w_i^{-1}, i) \mid 1 \leq i \leq n\}$ and extend pipes vertically and horizontally from the elbows. We call this the **Rothe pipe dream** $D(w)$, since the positions of blank tiles in this pipe dream correspond to the positions of blank tiles in the Rothe diagram of w. Note that the bumpless pipe dream in Example 2.3 is in fact a Rothe pipe dream for the permutation $w = (2, 3, 1, 5, 4)$.

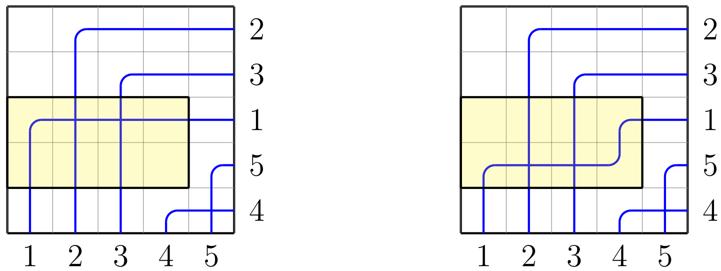
We now define local moves called **droop moves** on bumpless pipe dreams, moves which modify bumpless pipe dreams while preserving their corresponding permutation. Let P be a BPD for a permutation w. Consider the rectangular region S enclosing an empty tile e and a pipe p; with southeast elbow t. We enforce the following conditions on S to allow a valid droop move

(1) the left most column and the top row of S contains p,

(2) there are no elbow tiles in S other than t,

We then swap the SE elbow t with e. After the droop, the pipe p travels along the bottom row and right most column of S; a NW elbow occupies the position of e and a blank tile occupies the position of T. If such a move results in a valid BPD, the droop is allowed.

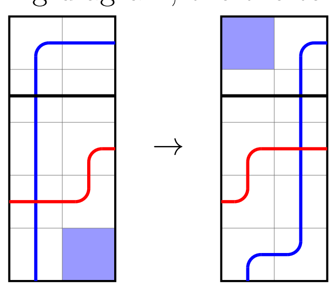
Example 2.6. *The following diagram is an example of drooping the NW elbow tile in position* $(3,1)$ *with the blank tile in position* $(4,4)$*, obtaining another bumpless pipe dream for the permutation* $(2,3,1,5,4)$.



The following proposition describes an important property regarding droop moves and Rothe pipe dreams.

Proposition 2.7 ([LLS18])**.** *Every w-bumpless pipe dream can be obtained from the Rothe pipe dream $D(w)$ of w by a sequence of droop moves.*

2.3.2. *Column moves.* In addition to droop moves, we introduce another type of local move on bumpless pipe dreams called a *column move*. A **column move** moves an empty tile to an adjacent column. In the following diagram, the tile to be moved is indicated in blue.



Note that a column move is only valid when the southeast most tile is the *only* empty tile in the rectangle. Then the empty tile is moved to the northwest most tile and the SE elbow is bent down to a NW elbow. Any "kinks" (indicated in red) must be shifted left, as illustrated in the above diagram. We remark that a column move is a droop when no "kinks" are present in the rectangle, i.e. column moves are not necessarily droop moves. We call the transpose of a column move a **row move**, as it modifies a pipe dream in two adjacent rows.

Lemma 2.8 (Lemma 5.15 in [LLS18])**.** *Let w be a permutation, then the set of bumpless pipe dreams for w is closed under column moves and row moves.*

2.4. **Quivers and Representation Space.** We move on to discuss quivers, generally following the notation in [Bri12]. A **quiver** is a directed graph $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of *vertices*, and Q_1 is the set of *arrows*, and $s, t : Q_1 \to Q_0$ are maps assigning each arrow to its **source** and **target**. If we fix a field, $\Bbbk$, a **representation** $M = ((V_x)_{x \in Q_0}, (T_\alpha)_{\alpha \in Q_1})$ of Q is an assignment of a $\Bbbk$-vector space V_x to every vertex x and a linear transformation $T_\alpha : V_x \to V_y$ to each edge $\alpha : x \to y$. A map $((\phi_x)_{x \in Q_0})$ between two quiver representations $M = ((V_x)_{x\in Q_0}, (f_\alpha)_{\alpha\in Q_1})$ and $N = ((W_x)_{x\in Q_0}, (g_\alpha)_{\alpha\in Q_1})$ is called a **morphism** if for all $x \xrightarrow{\alpha} y$, the following diagram commutes:

$$\begin{array}{ccc} V_x & \xrightarrow{f_\alpha} & V_y \\ \downarrow{\scriptstyle \phi_x} & & \downarrow{\scriptstyle \phi_y} \\ W_x & \xrightarrow{g_\alpha} & W_y \end{array} \tag{2.8.1}$$

Naturally, we say a morphism is an isomorphism if in addition to the condition above, ϕ_x is an isomorphism for all $x \in Q_0$.

Fixing a dimension vector $\mathbf{d} = (\dim V_x) \in \mathbb{N}^{Q_0}$, recall from Section 1 that the associated representation space is defined as

$$\mathrm{Rep}_Q(\mathbf{d}) := \bigoplus_{x\xrightarrow{\alpha} y\in Q_1} \mathrm{Mat}(\dim V_x, \dim V_y).$$

By choosing an ordered basis at each $x \in Q_0$, we may express a quiver representation as tuple of matrices. Then a quiver representation with dimension vector $\mathbf{d}$ can be considered as an element of $\mathrm{Rep}_Q(\mathbf{d})$. Furthermore, recall that

$$\mathrm{GL}_Q(\mathbf{d}) := \prod_{x\in Q_0} \mathrm{GL}(\dim V_x),$$

with a natural action on $\mathrm{Rep}_Q(\mathbf{d})$ defined as

$$G \cdot M = (G_x \cdot M \cdot G_y^{-1})_{x\xrightarrow{\alpha} y\in Q_1},$$

for $G \in \mathrm{GL}_Q(\mathbf{d})$ and $M \in \mathrm{Rep}_Q(\mathbf{d})$. Observe that for $M, M' \in \mathrm{Rep}_Q(\mathbf{d})$, we have $M \cong M'$ if and only if they are in the same orbit under the action of $\mathrm{GL}_Q(\mathbf{d})$.

2.5. **Lace Arrays, Rank Arrays, and Zelevinsky Permutations.** In this subsection, we briefly recall some necessary information covered in Chapter 17 of [MS05]. From now on, Q denotes a **Type A_{m+1} equioriented quiver**, which is a directed path of length m. Then we fix a dimension vector $\mathbf{d} = (d_0, d_1, \dots, d_m)$. It is well-known that the orbit closures in $\mathrm{Rep}_Q(\mathbf{d})$ can be uniquely identified with the ranks of its successive matrix products, encoded in rank arrays. In this section, we will recall how to encode type A_{m+1} using lace arrays and rank arrays. Then we will give a brief explanation between the connection between quiver loci and matrix Schubert varieties.

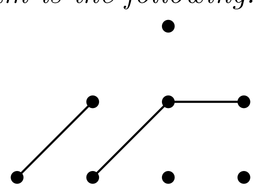
2.5.1. *Lacing Diagrams.* Note that in any orbit of $\mathrm{Rep}_Q(\mathbf{d})$, we can always find a representative element $\mathbf{M} = (M_1, \dots, M_m)$ such that M_i is a partial permutation matrix for all $1 \le i \le m$. Then we can encode its information using a **lacing diagram** – a graph arranged into columns such that column i contains d_i vertices; we draw an edge between the k-th vertex in column i and the ℓ-th vertex in column $i+1$ if $M_i(k,\ell) = 1$. (Note that within a column in a lacing diagram, we count the vertices from bottom to top.) We call a maximal path in a lacing diagram a **lace**.

2.5.2. *Lace Arrays and Rank Arrays.* For the lacing diagram of a quiver representation $\mathbf{M} = (M_1, \ldots, M_m)$ where M_i is a partial permutation matrix for all $1 \leq i \leq m$, its **lace array** $\mathbf{q} = (q_{ij})_{0\leq i\leq j\leq m}$ denotes the number of laces starting in column i and ending in column j; its **rank array** $\mathbf{r} = (r_{ij})_{0\leq i\leq j\leq m}$ denotes the number of laces *passing* through column i and column j. Observe that we can compute entries of a rank array from its corresponding lace arrays by summing all the entries weakly southeast of the desired position, namely $r_{ij} = \sum_{k=0}^{i}\sum_{\ell=j}^{m} q_{k\ell}$. Notice that the definition of a rank array coincides with $r_{ij} = \operatorname{rank}\prod_{k=i}^{j} M_k$, i.e. rank arrays encode the ranks of successive matrix products in quiver representations.

Example 2.9. *Let Q be a type A_4 equioriented quiver and consider the following representation of Q in $\operatorname{Rep}_Q(\mathbf{d})$ with $\mathbf{d} = (1, 2, 3, 2)$.*

$$\mathbf{M} = \left\{ \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \in \operatorname{Rep}_Q(\mathbf{d})$$

Then its corresponding lacing diagram is the following.



From the lacing diagram, we obtain its lace array and rank array to be the following.

$$\mathbf{q} = \begin{array}{cccc|c} 3 & 2 & 1 & 0 & \diagup \\ \hline & & & 0 & 0 \\ & & 0 & 1 & 1 \\ & 2 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 3 \end{array} \qquad \mathbf{r} = \begin{array}{cccc|c} 3 & 2 & 1 & 0 & \diagup \\ \hline & & & 1 & 0 \\ & & 2 & 1 & 1 \\ & 3 & 1 & 0 & 2 \\ 2 & 1 & 1 & 0 & 3 \end{array}$$

Now consider a generic quiver representation $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_m)$; we use the variables $f^i_{\alpha\beta}$ to label the (α, β) in the $r_{i-1} \times r_i$ matrices $\Phi_i : V_{i-1} \to V_i$. Given a rank array $\mathbf{r} = (r_{ij})_{0\leq i,j\leq m}$ of nonnegative integers, the **quiver locus** $\Omega_{\mathbf{r}}$ is the variety of the **quiver ideal** $I_{\mathbf{r}} \subset \Bbbk[f^i_{\alpha\beta}]$ defined as

$$I_{\mathbf{r}} = \langle \text{minors of size } (1 + r_{ij}) \text{ in } \Phi_{i+1} \cdots \Phi_j \text{ for } i < j \rangle.$$

In order to study quiver loci, we use matrix Schubert varieties. Let us briefly recall the definition of matrix Schubert varieties.

2.6. **Matrix Schubert Varieties.** Let M_n be the variety of $n \times n$ matrices over a field $\Bbbk$ with coordinate ring $\Bbbk[z_{ij}]$. Let $Z = (z_{ij})_{1\leq i,j\leq n}$ be the generic $n \times n$ matrix. For a permutation $w \in S_n$, let M_w denote its permutation matrix. Then we can define its **rank function** $\mathbf{r}_w$ as $r_w(i, j) = \operatorname{rank} M_w[i][j]$ where $M_w[i][j]$ denotes the submatrix of M_w consisting of the first i rows and the first j columns. From a rank function, we define the **Schubert determinantal ideal** in $\Bbbk[z_{ij}]$ as

$$I_w = \langle r_w(i, j) + 1 \text{ sized minors in } Z_{[i][j]} \ \forall\, i, j \rangle.$$

The **matrix Schubert variety** is defined as

$$\overline{X}_w = \{Z \in M_n \mid \operatorname{rank}(Z_{[i][j]}) \leq r_w(i, j)\}.$$

The connection between quiver loci and matrix Schubert varieties is given by Zelevinsky permutations and the Zelevinsky map.

2.7. **Zelevinsky Permutations.** Given a rank array $\mathbf{r}$, we can construct a unique **Zelevinsky permutation**, $v(\mathbf{r}) \in S_d$ as follows ([MS05]). First divide the matrix into blocks of size $d_i \times d_{m+1-i}$ for $1 \leq i \leq m$. Next, we put 1's in the blocks in position (i, j) of the permutation matrix as follows:

(1) For blocks in position $i \geq j+2$, we do not put any 1's.
(2) For blocks in positions $i = j+1$, we put exactly r_{ij} 1's.
(3) For blocks in positions $i \leq j$, we put exactly q_{ij} 1's.
(4) In every block, no nonzero entry is northeast of another.

For a fixed dimension vector $\mathbf{d}$, there is a unique Zelevinsky permutation corresponding to the quiver locus that is the entire representation space. We denote it $v(Hom)$, and explicitly, $v(Hom)$ is the Zelevinsky permutation which has no diagram boxes below the main super-antidiagonal in its Rothe diagram. Figure 3 illustrates the Zelevinsky permutation of the rank array in Example 2.9 and $v(Hom) = (6, 3, 4, 1, 2, 5, 7, 8)$ for $\mathbf{d} = (1, 2, 3, 2)$.

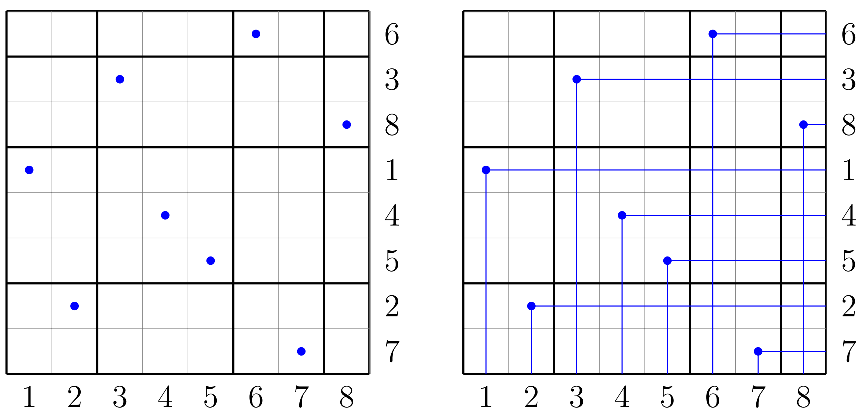


FIGURE 3. The Zelevinsky permutation $v(\mathbf{r}) = (6, 3, 8, 1, 4, 5, 2, 7)$ and its Rothe diagram of the rank array in Example 2.9

The **Zelevinsky map** $\mathcal{Z} : \mathrm{Rep}_Q(\mathbf{d}) \to M_d$ is defined as

$$\mathcal{Z} : (\phi_1, \ldots, \phi_m) \mapsto \begin{bmatrix} 0 & \cdots & 0 & \phi_1 & \mathbf{1} \\ 0 & & \phi_2 & \mathbf{1} & 0 \\ 0 & \iddots & \mathbf{1} & 0 & 0 \\ \phi_m & \iddots & 0 & 0 & 0 \\ \mathbf{1} & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Hence, $\mathcal{Z}(\phi)$ is a $d \times d$ block matrix and the Zelevinsky map gives us a way to embed the quiver representation space in a matrix Schubert variety. In particular, we have an induced surjective map $\Bbbk[x_{ij}] \to \Bbbk[f^i_{\alpha\beta}]$, where $\Bbbk[x_{ij}]$ is the coordinate ring of M_d and $\Bbbk[f^i_{\alpha\beta}]$ is the coordinate ring of $\mathrm{Rep}_Q(\mathbf{d})$. This notion is encapsulated in the following theorem.

Theorem 2.10 ([MS05])**.** *Let $\mathbf{r}$ be a rank array and $v(\mathbf{r})$ its Zelevinsky permutation. Then the image of the Schubert determinantal ideal $I_{v(\mathbf{r})}$ under the map $\Bbbk[x_{ij}] \to \Bbbk[f^i_{\alpha\beta}]$ equals the quiver ideal $I_{\mathbf{r}}$.*

3. Bumpless Pipe Formula and Factor Sequence Formula

Recall from the introduction that we presented two methods for computing quiver polynomials using bumpless pipe dreams in Proposition 1.4 and Proposition 1.6. In this section, we give the proofs of these propositions.

3.1. Bumpless Pipe Formula.

In this subsection, we give the detailed proof of Proposition 1.4.

Lemma 3.1. *Given a rank array* $\mathbf{r}$*, then for any bumpless pipe dream* $P \in \mathrm{BPD}(v(\mathbf{r}))$*, the set of empty tiles in* P *must contain the set of empty tiles in* $P(Hom)$*, where* $P(Hom)$ *denotes the unique bumpless pipe dream for* $v(Hom)$*.*

Proof. First, note that the blank tiles in $P(Hom)$ are exactly all the tiles above the main block antidiagonal. Observe that by definition, the empty tiles of the Rothe pipe dream of $v(\mathbf{r})$ consist of all the tiles above the main block antidiagonal and some additional empty tiles below the main block super-antidiagonal (none if $v(\mathbf{r}) = v(Hom)$). By Proposition 2.7, we obtain all the bumpless pipe dreams of $v(\mathbf{r})$ by drooping the empty tiles which are below the main block super-antidiagonal since the tiles above the main block antidiagonal are already top-left justified. Therefore, all the bumpless pipe dreams of $v(\mathbf{r})$ contain the empty tiles in $v(Hom)$. □

Now we are ready for the proof of Proposition 1.4.

Proof of Proposition 1.4. The proof follows directly from Theorem 2.4 and the ratio formula. It is easy to see that $\mathrm{BPD}(v(Hom)) = \{P(Hom)\}$ and by Lemma 3.1, the set of empty tiles for any $P \in \mathrm{BPD}(v(\mathbf{r})$ contains the set of empty tiles in $P(Hom)$. Now from the ratio formula, we have

$$\begin{aligned}
\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) &= \frac{\mathfrak{S}_{v(\mathbf{r})}(\mathbf{x} - \mathring{\mathbf{y}})}{\mathfrak{S}_{v(Hom)}(\mathbf{x} - \mathring{\mathbf{y}})} && \text{(Ratio formula)} \\
&= \sum_{P \in \mathrm{BPD}(v(\mathbf{r}))} \frac{\mathrm{wt}(P)}{\mathrm{wt}(P(Hom))} && \text{(Theorem 2.4)} \\
&= \sum_{P \in \mathrm{BPD}(v(\mathbf{r}))} (\mathbf{x} - \mathring{\mathbf{y}})^{P/P(Hom)}. && \text{(Lemma 3.1)}
\end{aligned}$$

Then we are done. □

This proposition is illustrated in Example 1.5.

3.2. Factor Sequence Formula.

In this subsection, we give the proof of Proposition 1.6 and present the factor sequence formula formulated in terms of bumpless pipe dreams.

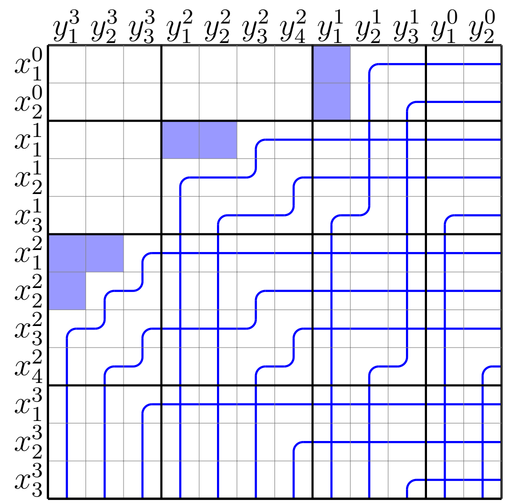
We combine the works of [Buc+06] and [FGS18] to show that quiver constants can be extracted from *Edelman-Greene bumpless pipe dreams*, which are bumpless pipe dreams with empty tiles top-left justified. In [Buc+06], the authors gave a bijection between $\mathbf{r}$-factor sequences and *decreasing tableaux* representing $v(\mathbf{r})$; later in [LLS18], it is shown that there exists a bijection between decreasing tableaux and EG-BPDs. Hence $\mathbf{r}$-factor sequences are in fact in bijection with EG-BPDS of $v(\mathbf{r})$. Therefore, $\mathbf{r}$-factor sequences are in bijection with EG-BPDs of $v(\mathbf{r})$.

To determine the quiver constants, first we extend the notion of underlying partition shapes to EG-BPDs. If P is an EG-BPD, let shape(P) denote the partition shape formed by the empty tiles in the top-left corner of P. Then, we can define a map

$$\tau_{\mathbf{d}} : \lambda \mapsto \boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$$

that truncates a partition λ to a sequence of partition $\boldsymbol{\lambda}$ according to the dimension vector $\mathbf{d} = (d_0, \dots, d_m)$. Intuitively, we superimpose a $d \times d$ grid divided into $(m+1) \times (m+1)$ blocks according to $\mathbf{d}$ in convention such that the partition is top-left justified in the grid. Then we obtain $\boldsymbol{\lambda}$ by reading off the partition shapes in the blocks on the main block super antidiagonal from top to bottom, i.e. λ_i is the partition shape in the $d_{i-1} \times d_{m-1}$ sized block in position $(i, i+1)$. The following example illustrates the map $\tau_{\mathbf{d}}$.

Example 3.2. *Let* $\mathbf{d} = (2, 3, 4, 3)$ *and consider the following EG-BPD, denote as* P*.*



Then by definition shape$(P) = (8, 8, 5, 3, 3, 2, 1)$ *and*

$$\tau_{\mathbf{d}}(\text{shape}(P)) = \boldsymbol{\lambda} = \left(\ydiagram{1,1}, \ydiagram{2}, \ydiagram{2,1}\right) = ((1,1),\ (2),\ (2,1)).$$

As we see in Example 1.7, we can obtain a sequence of partitions from an EG-BPD under the composition of the maps shape and $\tau_{\mathbf{d}}$. We now show formally that these sequences of partitions determine the quiver constants.

Proposition 3.3. *Given a rank array* $\mathbf{r}$*, the quiver constant* $c_{\boldsymbol{\lambda}}(\boldsymbol{r})$ *is the number of Edelman-Greene bumpless pipe dreams* P *for* $v(\mathbf{r})$ *such that* $\tau_{\mathbf{d}}(\text{shape}(P)) = \boldsymbol{\lambda}$*, i.e.*

$$c_{\boldsymbol{\lambda}}(\mathbf{r}) = |\{P \in \text{EGBPD}(v(\mathbf{r})) : \tau_{\mathbf{d}}(\text{shape}(P)) = \boldsymbol{\lambda}\}|.$$

Proof. By [KMS06], we know that $c_{\boldsymbol{\lambda}}$ counts the number of $\mathbf{r}$-factor sequences. Then from [Buc+06], it is shown that factor sequences of $\mathbf{r}$ are in bijection with the set of decreasing tableaux for $v(\mathbf{r})$. Specifically, the authors gave a bijection $\psi : \text{FS}(\mathbf{r}) \to \text{DT}(v(\mathbf{r}))$ where $\text{FS}(\mathbf{r})$ denotes the set of $\mathbf{r}$-factor sequences and $\text{DT}(v(\mathbf{r}))$ denotes the set of decreasing tableaux representing $v(\mathbf{r})$. It is also known from [LLS18] (see also [FGS18] and [Wei20]) that there exists a shape-preserving bijection between the set of decreasing tableaux for $v(\mathbf{r})$ and the set of EG-BPDs of $v(\mathbf{r})$, denote as $\chi : \text{DT}(v(\mathbf{r})) \to \text{EGBPD}(v(\mathbf{r}))$. In particular, suppose $T \in \text{DT}(v(\mathbf{r}))$, we have

$$\text{shape}(T) = \text{shape}(\chi(T)).$$

The composition $\psi \circ \chi : \mathrm{FS}(\mathbf{r}) \to \mathrm{EGBPD}(v(\mathbf{r}))$ is a bijection between $\mathbf{r}$-factor sequences and EG-BPDs of $v(\mathbf{r})$.

Recall that from Theorem 1.3, we have

$$c_{\boldsymbol{\lambda}}(\mathbf{r}) = |\{\mathbf{T} \in \mathrm{FS}(\mathbf{r}) : \mathrm{shape}(\mathbf{T}) = \boldsymbol{\lambda}\}|.$$

Applying the bijection ψ, we obtain

$$c_{\boldsymbol{\lambda}}(\mathbf{r}) = |\{T \in \mathrm{DT}(v(\mathbf{r})) : \mathrm{shape}(T) = \boldsymbol{\lambda}\}|.$$

Since $\tau_{\mathbf{d}}$ is a truncation of partition shapes, it is clear that for $T \in \mathrm{DT}(v(\mathbf{r}))$,

$$\tau_{\mathbf{d}}(\mathrm{shape}(T)) = \tau_{\mathbf{d}}(\mathrm{shape}(\chi(T))).$$

Now we can conclude that

$$c_{\boldsymbol{\lambda}}(\mathbf{r}) = |\{P \in \mathrm{EGBPD}(v(\mathbf{r})) : \tau_{\mathbf{d}}(\mathrm{shape}(P)) = \boldsymbol{\lambda}\}|$$

as desired. □

4. Partial Results

Through this project, our goal is to give a proof of the factor sequence formula using combinatorial properties of bumpless pipe dreams. In this section, we give some partial results in progress towards our goal and present some key ideas. First we give a toy lemma that demonstrates the connection between the diagonal moves on crossing diagrams and the column/row moves on bumpless pipe dreams.

Lemma 4.1. *Let Q be a type A_2 quiver. Given a rank array $\mathbf{r}$ with dimension vector $\mathbf{d} = (d_0, d_1)$, the quiver polynomial can be computed as*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) = s_{\lambda}(\mathbf{x}^0_{\mathbf{d}} - \mathbf{y}^1_{\mathbf{d}}),$$

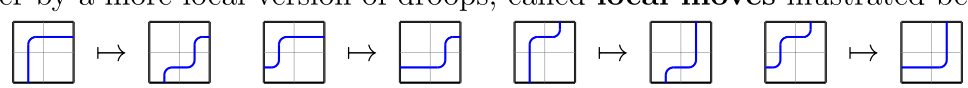
where $\lambda = \mathrm{shape}(P)$ and P denotes the unique Edelman-Greene bumpless pipe dream of $v(\mathbf{r})$.

A permutation $w = (w_1, \ldots, w_n) \in S_n$ is **Grassmannian** if it has a unique descent at position k such that $w_k > w_{k+1}$. The simplicity behind type A_2 equioriented quivers is based on the following observation.

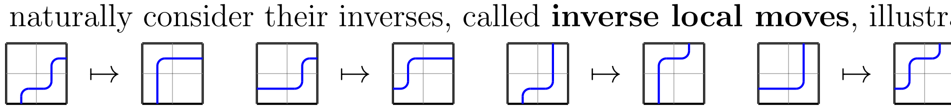
Lemma 4.2 ([MS05])**.** *Let Q be a type A_2 equioriented quiver and fix a dimension vector $\mathbf{d} = (d_0, d_1)$. Let $\mathbf{r}$ be the rank array of an element in $\mathrm{Rep}_Q(\mathbf{d})$, then its Zelevinsky permutation $v(\mathbf{r})$ is Grassmannian.*

Proof. Consider the block permutation matrix of $v(\mathbf{r})$. By definition, there is no nonzero entry that is northeast of another within a column of blocks. Therefore, there can be exactly one descent in $v(\mathbf{r})$ at exactly the d_0-th position if $v(\mathbf{r}) \neq v(Hom)$. □

In addition, the bumpless pipe dreams of Grassmannian permutations can be related to one another by a more local version of droops, called **local moves** illustrated below.



We may naturally consider their inverses, called **inverse local moves**, illustrated below.



Lemma 4.3 (Lemma 7.4 [Wei20])**.** *Let $w \in S_n$ be a Grassmannian permutation.*

(1) *Any bumpless pipe dream* $B \in \mathrm{BPD}(w)$ *can be uniquely determined by the positions of its empty tiles.*
(2) *There is a unique Edelman-Greene bumpless pipe dream* P *for* w *such that every bumpless pipe dream for* w *can be obtained from* P *through a sequence of inverse local moves.*

Now we are ready to prove Lemma 4.1.

Proof of Lemma 4.1. By Lemma 4.2 and Lemma 4.3, there is a unique Edelman-Greene bumpless pipe dream P for $v(\mathbf{r})$. In addition, note that $v(Hom)$ is just the identity permutation. Hence by Proposition 1.4, we have

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) = \sum_{B \in \mathrm{BPD}(v(\mathbf{r}))} (\mathbf{x} - \mathring{\mathbf{y}})^B.$$

Let $\lambda = \mathrm{shape}(P)$, consider the set of plus diagrams

$$\mathbf{D}(\lambda) = \{D \in \mathcal{D}(\lambda) : i \leq d_0, j \leq d_1, \forall (i,j) \in D\}.$$

Then we have the specialized Schur function

$$s_\lambda(\mathbf{x}^0_{\mathbf{d}} - \mathbf{y}^1_{\mathbf{d}}) = \sum_{D \in \mathbf{D}(\lambda)} (\mathbf{x}^0_{\mathbf{d}} - \mathbf{y}^1_{\mathbf{d}})^D,$$

since plus diagrams not in $\mathbf{D}(\lambda)$ specialize to weight zero in this specialized Schur function.

Observe that there is a direct correspondence between the local moves on plus diagrams and the inverse local moves on bumpless pipe dreams. An inverse local move on a bumpless pipe dream moves an empty tile one step along its diagonal, provided that there are no empty tiles directly to its right and below it. Notice the effect this has on the quiver polynomial is identical to the effect a local move on a plus diagram has on the Schur polynomial. This motivates us to define a map $f : \mathrm{BPD}(v(\mathbf{r})) \to \mathbf{D}(\lambda)$ between bumpless pipe dreams of $v(\mathbf{r})$ and plus diagrams of shape λ, where, for $B \in \mathrm{BPD}(v(\mathbf{r}))$, $f(B)$ is the plus diagram $\{(i,j) : (i,j) \text{ is an empty tile in } B\}$, and for $D \in \mathbf{D}(\lambda)$, $f^{-1}(D)$ is the bumpless pipe dream with empty tiles in the positions D. Now by Lemma 4.3, f is a weight-preserving bijection. Hence

$$\begin{aligned}\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{x}}) &= \sum_{B \in \mathrm{BPD}(v(\mathbf{r}))} (\mathbf{x} - \mathring{\mathbf{y}})^B = \sum_{f(B): B \in \mathrm{BPD}(v(\mathbf{r}))} (\mathbf{x}^0_{\mathbf{d}} - \mathbf{y}^1_{\mathbf{d}})^{f(B)} \\ &= \sum_{D \in \mathbf{D}(\lambda)} (\mathbf{x}^0_{\mathbf{d}} - \mathbf{y}^1_{\mathbf{d}})^D = s_\lambda(\mathbf{x}^0_{\mathbf{d}} - \mathbf{y}^1_{\mathbf{d}}),\end{aligned}$$

and the result follows. □

The above correspondence also holds in a more general case.

Proposition 4.4. *Let* Q *be a type* A_{m+1} *equioriented quiver. Let* $\mathbf{r}$ *be a rank array of* Q *with dimension vector* $\mathbf{d} = (d_0, \ldots, d_m)$ *such that the Rothe pipe dream of the Zelevinsky permutation* $v(\mathbf{r})$ *has no empty tiles below the main block super-antidiagonal, i.e. there are no diagram boxes below the main block super-antidiagonal in the Rothe diagram of* $v(\mathbf{r})$*. Then the quiver polynomial can be computed as*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) = s_{\tau_{\mathbf{d}}(\mathrm{shape}(P))}(\mathbf{x}_{\mathbf{d}} - \mathring{\mathbf{y}}_{\mathbf{d}}),$$

where P *denotes the unique Edelman-Greene bumpless pipe dream for* $v(\mathbf{r})$*.*

The proof of this lemma is based on the following observation similar to Lemma 4.3.

Lemma 4.5. *Let Q be a type A_{m+1} quiver and let $\mathbf{r}$ be a rank array of Q with dimension vector $\mathbf{d} = (d_0, \dots, d_m)$ such that the Rothe pipe dream of the Zelevinsky permutation $v(\mathbf{r})$ has no empty tiles below the main block super-antidiagonal. Then every bumpless pipe dream $P \in \mathrm{BPD}(v(\mathbf{r}))$ can be obtained through a sequence of local moves from the Rothe pipe dream $D(v(\mathbf{r}))$.*

Proof. Consider the block matrix of $v(\mathbf{r})$. By definition, we observe the 1's in the blocks on the main block super-antidiagonal are placed on a diagonal, starting from the top-leftmost position. Since the Rothe pipe dream $D(v(\mathbf{r}))$ has no empty tiles below the main block super-antidiagonal, then the empty tiles of $v(\mathbf{r})$ that are not in $v(Hom)$ are bottom-right justified rectangles in each block along the main block super-antidiagonal. Observe that any empty tile can only be drooped to a position occupied by a SE tile northwest of it. Therefore all empty tiles within block along the main super-antidiagonal may only be drooped to positions within its block. Furthermore, such droop moves are in fact local moves. Therefore, we can conclude that every bumpless pipe dream for $v(\mathbf{r})$ can be obtained through a sequence of local moves from $D(v(\mathbf{r}))$. ☐

Corollary 4.6. *Let Q be a type A_{m+1} quiver and let $\mathbf{r}$ be a rank array of Q with dimension vector $\mathbf{d} = (d_0, \dots, d_m)$ such that the Rothe pipe dream of the Zelevinsky permutation $v(\mathbf{r})$ has no empty tiles below the main block super-antidiagonal. Then every bumpless pipe dream $P \in \mathrm{BPD}(v(\mathbf{r}))$ can be uniquely determined by the positions of its empty tiles.*

Proof. Notice that local moves do not affect the positions of crossing tiles, the result follows directly from Lemma 4.5 and Lemma 3.6 in [Wei20]. ☐

Corollary 4.7. *Let Q be a type A_{m+1} equioriented quiver with dimension vector $\mathbf{d}$. Let $v(\mathbf{r})$ be the Zelevinsky permutation. Let P be the Rothe pipe dream of $v(\mathbf{r})$. Then valid droop moves on the empty tiles of the main block super-antidiagonal in P are local moves.*

Proof. The same proof for Lemma 4.5 suffices. ☐

Now we are ready for the proof of Proposition 4.4.

Proof. Recall that local moves preserve the number of empty tiles along each diagonal. By Lemma 4.5 and Corollary 4.6, it is clear that there is a unique Edelman-Greene bumpless pipe dream P for $v(\mathbf{r})$. Let $\tau_{\mathbf{d}}(\mathrm{shape}(P)) = \boldsymbol{\lambda}_P = (\lambda_P^{(1)}, \dots, \lambda_P^{(m)})$ and let $\mathbf{D}(\lambda)$ denote the set of plus diagrams with nonzero weight in the corresponding Schur function. We write out the Schur functions in terms of plus diagrams and we have

$$\begin{aligned} s_{\boldsymbol{\lambda}_P}(\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}}) &= s_{\lambda_P^{(1)}}(\mathbf{x}^0_{\mathbf{d}} - \mathbf{y}^1_{\mathbf{d}}) s_{\lambda_P^{(2)}}(\mathbf{x}^1_{\mathbf{d}} - \mathbf{y}^2_{\mathbf{d}}) \cdots s_{\lambda_P^{(m)}}(\mathbf{x}^{m-1}_{\mathbf{d}} - \mathbf{y}^m_{\mathbf{d}}) \\ &= \sum_{D_1 \in \mathbf{D}(\lambda_P^{(1)})} (\mathbf{x}^0_{\mathbf{d}} - \mathbf{y}^1_{\mathbf{d}})^{D_1} \cdots \sum_{D_m \in \mathbf{D}(\lambda_P^{(m)})} (\mathbf{x}^{m-1}_{\mathbf{d}} - \mathbf{y}^m_{\mathbf{d}})^{D_m} \\ &= \sum_{\underline{D} = (D_1, \dots, D_m) \in \prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)})} \left((\mathbf{x}^0_{\mathbf{d}} - \mathbf{y}^1_{\mathbf{d}})^{D_1} \cdots (\mathbf{x}^{m-1}_{\mathbf{d}} - \mathbf{y}^m_{\mathbf{d}})^{D_m} \right) \end{aligned}$$

Note that in this case, the set $\mathbf{D}(\lambda_P^{(k)})$ is equivalent to the set of plus diagrams with elements entirely contained in the corresponding $d_k \times d_{k-1}$ rectangle. Now we establish a bijective correspondence between ordered tuples of plus diagrams in $\prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)})$ and bumpless pipe dreams for $v(\mathbf{r})$. Notice that for a tuple of plus diagrams $\underline{D} = (D_1, \ldots, D_m) \in \prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)})$, we can place a plus sign for each element $(x, y) \in D_i$ at relative position (x, y) in the block at position $(i, i+1)$, i.e. we superimpose the plus diagrams in $\underline{D}$ on the main block super-antidiagonal. Then we obtain a plus diagram in a $d \times d$ grid. Let $\mathbf{S}(d)$ denote the set of plus diagrams contained in a $d \times d$ grid, i.e. for all $D \in \mathbf{S}(d)$, every $(i, j) \in D$ satisfies $i \le d$ and $j \le d$. We define an injective map $\alpha_{\mathbf{d}} : \prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)}) \to \mathbf{S}(d)$ as for $\underline{D} = (D_1, \ldots, D_m) \in \prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)}))$,

$$\alpha_{\mathbf{d}}(\underline{D}) = (D_1, \ldots, D_m) = \{(x + \sum_{k=0}^{i-1} d_k, y + \sum_{k=i+1}^{m} d_k) : \forall (x, y) \in D_i\}.$$

We observe that

$$(\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}})^{\alpha_{\mathbf{d}}(\underline{D})} = (\mathbf{x}_{\mathbf{d}}^0 - \mathbf{y}_{\mathbf{d}}^1)^{D_1} \cdots (\mathbf{x}_{\mathbf{d}}^{m-1} - \mathbf{y}_{\mathbf{d}}^m)^{D_m}.$$

Hence we have

$$s_{\boldsymbol{\lambda}_P}(\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}}) = \sum_{\underline{D} \in \prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)})} (\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}})^{\alpha_{\mathbf{d}}(\underline{D})}.$$

Similar to the proof of Lemma 4.1, observe that inverse local moves on bumpless pipe dreams correspond exactly to local moves on plus diagrams. Now, we define the map f : Im $\alpha_{\mathbf{d}} \to \mathrm{BPD}(v(\mathbf{r}))$ as $f(D)$ being the bumpless pipe dream with empty tiles in the positions $D \in$ Im $\alpha_{\mathbf{d}}$ in addition to the empty tiles in $P(Hom)$. For $B \in \mathrm{BPD}(v(\mathbf{r}))$, its inverse $f^{-1}(B)$ is the plus diagram $\{(i, j) : (i, j)$ is an empty tile in $B \setminus P(Hom)\}$. Clearly, f is well-defined since we can obtain $f(D)$ by applying the same inverse local moves on the EG-BPD P as the local moves on D and vice versa for f^{-1}. Now by Lemma 4.5 and Corollary 4.6, f is a bijection such that for $D \in$ Im $\alpha_{\mathbf{d}}$ we have

$$(\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}})^{\alpha_{\mathbf{d}}(\underline{D})} = (\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}})^{f(D) \setminus P(Hom)}.$$

Composing the maps $\alpha_{\mathbf{d}} \circ f : \prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)}) \to$ Im $\alpha_{\mathbf{d}} \to \mathrm{BPD}(v(\mathbf{r}))$, we obtain a bijection between the ordered tuples of plus diagrams in $\prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)})$ and bumpless pipe dreams for $v(\mathbf{r})$. Then we have

$$\begin{aligned} s_{\boldsymbol{\lambda}_P}(\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}}) &= \sum_{\underline{D} \in \prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)})} (\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}})^{\alpha_{\mathbf{d}}(\underline{D})} \\ &= \sum_{\underline{D} \in \prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)})} (\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}})^{\alpha_{\mathbf{d}} \circ f(\underline{D}) \setminus P(Hom)} \\ &= \sum_{B \in \mathrm{BPD}(v(\mathbf{r}))} (\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}})^{B \setminus P(Hom)} \\ &= \mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}). \end{aligned}$$

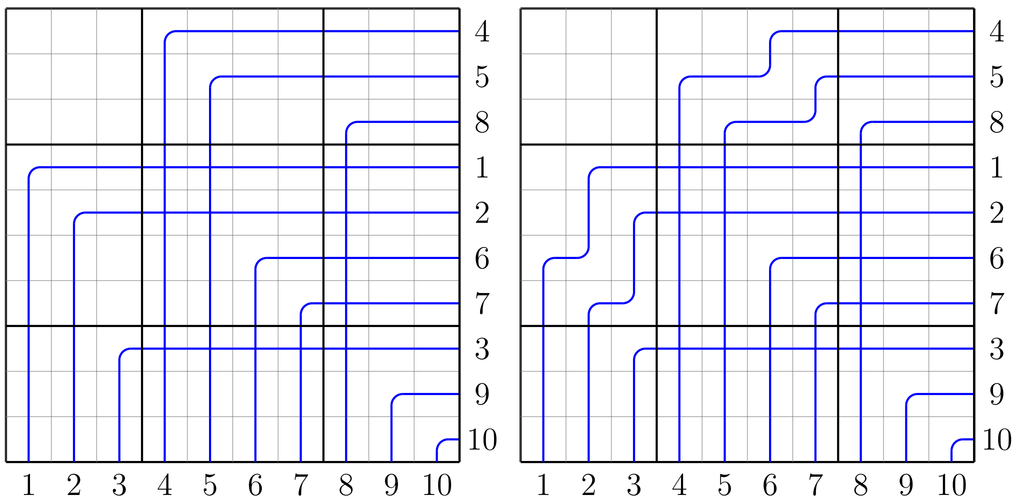
This concludes the proof. □

The following example illustrates the elements described in the proof of Proposition 4.4.

Example 4.8. *Let Q be a type A_3 equioriented quiver with $\mathbf{d} = (3,4,3)$ and consider the following rank array*

$$\mathbf{r} = \begin{array}{cccc|c} 2 & 1 & 0 & \\ \hline & & 3 & 0 \\ & 4 & 2 & 1 \\ 3 & 2 & 2 & 2 \end{array}.$$

Its Zelevinsky permutation is $v(\mathbf{r}) = (4,5,8,1,2,6,7,3,9,10)$. The corresponding Rothe pipe dream $D(v(\mathbf{r}))$ is on the left, and its unique Edelman-Greene pipe dream P on the right.



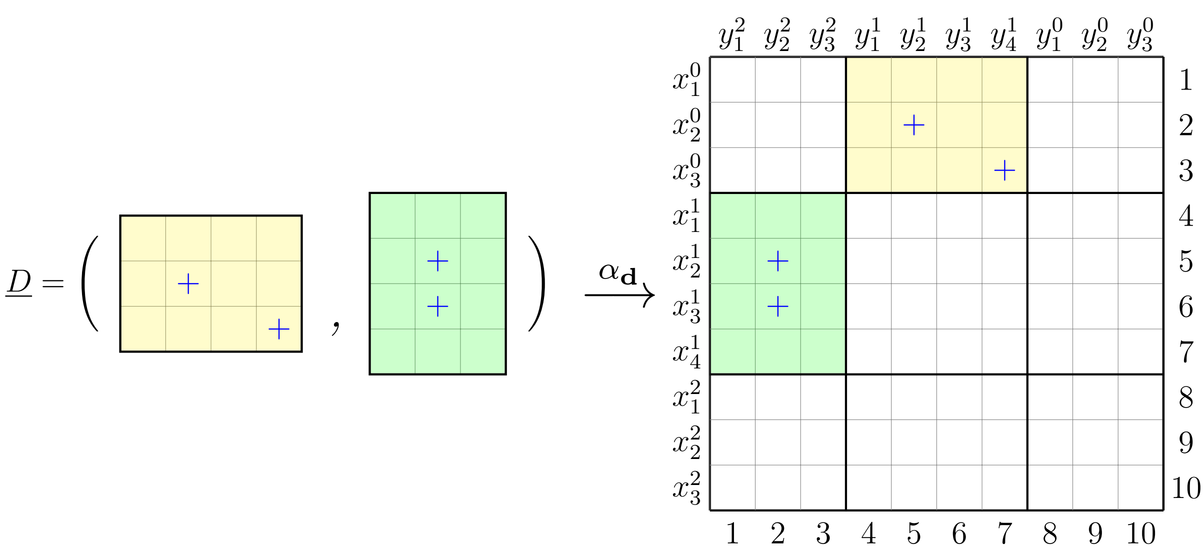
Using the same notation as the proof of Proposition 4.4, we have

$$\boldsymbol{\lambda}_P = (\lambda_P^{(1)}, \lambda_P^{(2)}) = (\square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}) = ((2),(1,1)).$$

Consider the tuple of plus diagrams

$$\underline{D} = (D_1, D_2) = (\{(2,2),(3,4)\}, \{(2,2),(3,2)\}).$$

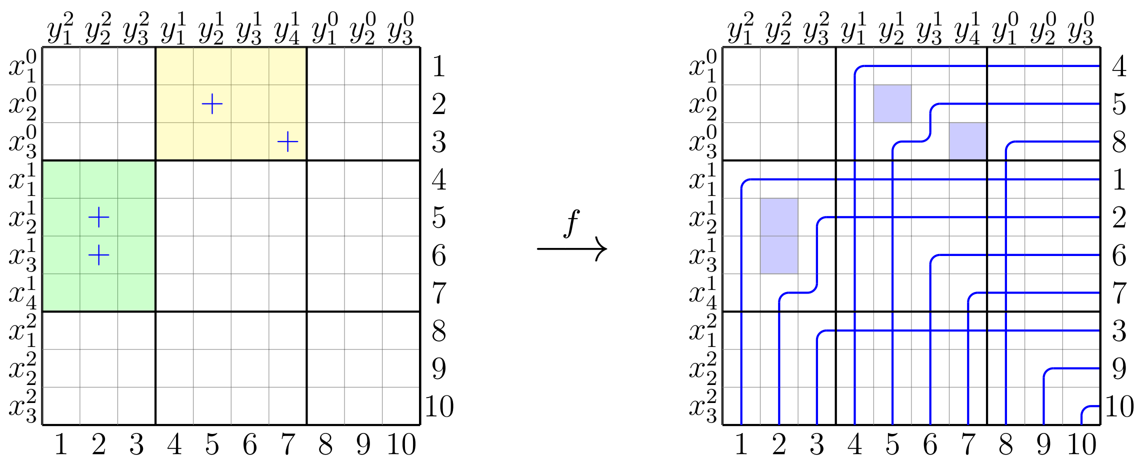
Then,



where the plus diagram shaded yellow is superposed onto the block shaded yellow on the main block super-antidiagonal, and the same is done for the plus diagram shaded green. In particular,

$$\begin{aligned}(\mathbf{x}^0 - \mathbf{x}^1)^{D_1} \cdot (\mathbf{x}^1 - \mathbf{x}^2)^{D_2} &= (x_2^0 - y_2^1)(x_3^0 - y_4^1)(x_2^1 - y_2^2)(x_3^1 - y_2^2) \\ &= (\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}})^{\alpha_{\mathbf{d}}(\underline{D})}.\end{aligned}$$

Then we can map the plus diagram $\alpha_{\mathbf{d}}(\underline{D})$ to a bumpless pipe dream as illustrated below.



Note that the positions of the plus signs in $\alpha_{\mathbf{d}}(\underline{D})$ correspond exactly to the positions of the empty tiles in $f(\alpha_{\mathbf{d}}(\underline{D})) \setminus P(Hom)$ and we have

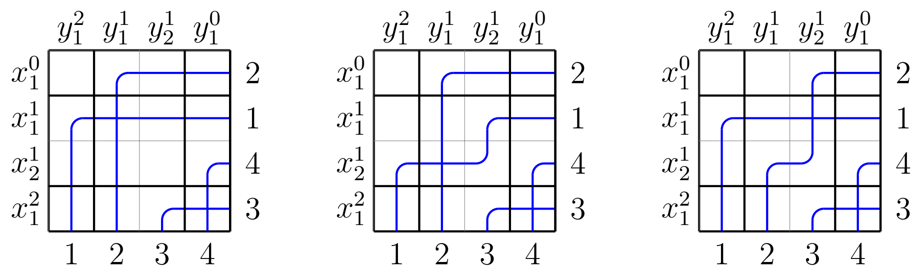
$$\begin{aligned}(\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}})^{\alpha_{\mathbf{d}}(\underline{D})} &= (x_2^0 - y_2^1)(x_3^0 - y_4^1)(x_2^1 - y_2^2)(x_3^1 - y_2^2) \\ &= (\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}})^{f(\alpha_{\mathbf{d}}(\underline{D}))\setminus P(Hom)}.\end{aligned}$$

Up to this point, we have only considered BPDs for which, in the Rothe pipe dream, any block which is not also an empty tile of $P(Hom)$ lies in the main block super-antidiagonal. Consequently, our previous proofs relied on the following observation: the factor that an element in a plus diagram that is contained in a block on the super-antidiagonal contributes to the product of specialized Schur functions is exactly the same as the factor that an empty tile on the main block super-antidiagonal contributes to the weight of the bumpless pipe dream. However, this does not apply to empty tiles below the main block super-antidiagonal. We illustrate this point with the following example.

Example 4.9. *Let Q be a type A_3 equioriented quiver with $\mathbf{d} = (1, 2, 1)$ and consider the following rank array*

$$\mathbf{r} = \begin{array}{ccc|c} 2 & 1 & 0 & \\ \hline & & 0 & 0 \\ & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{array}.$$

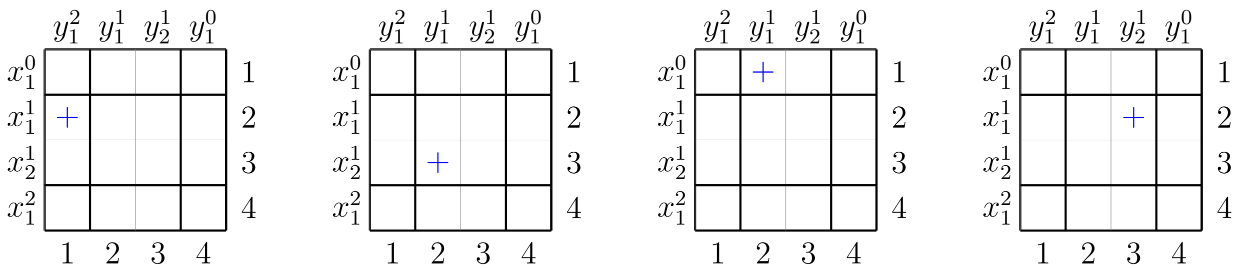
Then its Zelevinsky permutation is $v(\mathbf{r}) = (2, 1, 4, 3)$, with the following bumpless pipe dreams where the weight $(\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}})^{P\setminus P(Hom)}$ for each bumpless pipe dream P is labeled below each diagram.



$(x_2^1 - y_2^1)$ $\quad$ $(x_1^1 - y_1^2)$ $\quad$ $(x_1^0 - y_1^1)$

The two EG-BPDs above yield two sequences of partitions: $(\varnothing, \square)$ *and* $(\square, \varnothing)$*. For* $(\varnothing, \square)$*, there are two tuples plus diagrams which contribute nontrivially to the sum of products of specialized Schur functions:* $(\varnothing, \{(0,0)\})$ *and* $(\varnothing, \{(1,1)\})$*. Similarly, for* $(\square, \varnothing)$*, the tuples with nontrivial contribution are* $(\{(0,0)\}, \varnothing)$ *and* $(\{(1,1)\}, \varnothing)$*. In the convention of* $\alpha_{\mathbf{d}}$*, defined in the proof of Proposition 4.4, we achieve the following diagrams.*

$(\varnothing, \{(0,0)\})$ $\quad$ $(\varnothing, \{(1,1)\})$ $\quad$ $(\{(0,0)\}, \varnothing)$ $\quad$ $(\{(1,1)\}, \varnothing)$



$(x_1^1 - y_1^2)$ $\quad$ x_2^1 $\quad$ $(x_1^0 - y_1^1)$ $\quad$ $-y_2^1$

Notice, the second and third bumpless pipe dream correspond contribute the same factor to the formula as the first and third plus diagrams, respectively. However, observe that the contribution from the first bumpless pipe dream, which has an empty tile below the main block super-antidiagonal, is the sum of the contributions of the second and forth plus diagrams. The two plus diagrams corresponding to this empty tile are in the block directly rightwards and below on the main block super-antidiagonal.

Motivated by Example 4.9, we have the following definition.

Definition 4.10 (LU-labeled bumpless pipe dream)**.** *Let* w *be a Zelevinsky permutation of dimension vector* $\mathbf{d}$*. Let* P *be a bumpless pipe dream of* w *and let* $E(P)$ *denote the set of empty tiles strictly below the main block super-antidiagonal. Let* L, U *be symbols (indicating left and up respectively) and we say that a map* $\ell : E(P) \to \{L, U\}$ *is an LU-labeling of* P*. We say that a bumpless pipe dream together with an LU-labeling* (P, ℓ) *is an LU-labeled bumpless pipe dream. We use* $\mathrm{LUBPD}(w)$ *denote the set of LU-labeled bumpless pipe dreams for* $w \in S_n$*.*

Proposition 4.11. *Let* Q *be a type* A_{m+1} *quiver. Let* $\mathbf{r}$ *be a rank array of* Q *with dimension vector* $\mathbf{d} = (d_0, \ldots, d_m)$ *such that the Rothe pipe dream of the Zelevinsky permutation* $v(\mathbf{r})$ *has exactly one empty tile below the main block super-antidiagonal. For* $P \in \mathrm{EGBPD}(v(\mathbf{r}))$*, let* $\boldsymbol{\lambda}_P = (\lambda_P^{(1)}, \ldots, \lambda_P^{(m)}) = \tau_{\mathbf{d}}(\mathrm{shape}(P))$*. Then the quiver polynomial can be computed as*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{x} - \mathring{\mathbf{y}}) = \sum_{P \in \mathrm{EGBPD}(v(\mathbf{r}))} s_{\boldsymbol{\lambda}_P}(\mathbf{x}_{\mathbf{d}} - \mathring{\mathbf{y}}_{\mathbf{d}}).$$

Proof. Similar to the proof of Proposition 4.4, we construct a bijective map between tuples of plus diagrams and LU-labeled bumpless pipe dreams. In particular, note that since $v(\mathbf{r})$ only has one empty tile below the main super-antidiagonal, there is at most one empty tile that can be labeled for any $B \in \mathrm{BPD}(v(\mathbf{r}))$. For any $P \in \mathrm{EGBPD}(v(\mathbf{r}))$, we use $\boldsymbol{\lambda}_P = (\lambda_P^{(1)}, \ldots, \lambda_P^{(m)})$ to denote $\tau_{\mathbf{d}}(\mathrm{shape}(P))$. Let $\mathbf{D}(\lambda)$ denote the set of plus diagrams with nonzero weight in the corresponding specialized Schur function, and let $\mathbb{D}(v(\mathbf{r})) = \bigcup_{P \in \mathrm{EGBPD}(v(\mathbf{r}))} \prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)})$. Then we have

$$\begin{aligned}
\sum_{P \in \mathrm{EGBPD}(v(\mathbf{r}))} s_{\boldsymbol{\lambda}_P}(\mathbf{x_d} - \mathring{\mathbf{y}}_{\mathbf{d}}) &= \sum_{P \in \mathrm{EGBPD}(v(\mathbf{r}))} s_{\lambda_P^{(1)}}(\mathbf{x}_{\mathbf{d}}^0 - \mathbf{y}_{\mathbf{d}}^1) s_{\lambda_P^{(2)}}(\mathbf{x}_{\mathbf{d}}^1 - \mathbf{y}_{\mathbf{d}}^2) \cdots s_{\lambda_P^{(m)}}(\mathbf{x}_{\mathbf{d}}^{m-1} - \mathbf{y}_{\mathbf{d}}^m) \\
&= \sum_{P \in \mathrm{EGBPD}(v(\mathbf{r}))} \left(\prod_{i=1}^m \left(\sum_{D_i \in \mathbf{D}(\lambda_P^{(i)})} (\mathbf{x_d}^{i-1} - \mathbf{y_d}^i) \right) \right) \\
&= \sum_{P \in \mathrm{EGBPD}(v(\mathbf{r}))} \left(\sum_{\underline{D} = (D_1, \ldots, D_m) \in \prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)})} \left(\prod_{i=1}^m (\mathbf{x_d}^{i-1} - \mathbf{y_d}^i)^{D_i} \right) \right) \\
&= \sum_{\underline{D} = (D_1, \ldots, D_m) \in \mathbb{D}(v(\mathbf{r}))} \left(\prod_{i=1}^m (\mathbf{x_d}^{i-1} - \mathbf{y_d}^i)^{D_i} \right).
\end{aligned}$$

Note that since the Rothe pipe dream of $v(\mathbf{r})$ contains an empty tile below the main block super-antidiagonal, the set $\mathbf{D}(\lambda_P^{(k)})$ is **not** equivalent to the set of plus diagrams with elements entirely contained in the corresponding $d_k \times d_{k-1}$ rectangle. Again, let $\mathbf{S}(d)$ denote the set of plus diagrams entirely contained in a $d \times d$ grid and consider the injective map $\alpha_{\mathbf{d}} : \mathbb{D}(v(\mathbf{r})) \to \mathbf{S}(d)$ defined for $\underline{D} = (D_1, \ldots, D_m) \in \prod_{k=1}^m \mathbf{D}(\lambda_P^{(k)})$),

$$\alpha_{\mathbf{d}}(\underline{D}) = (D_1, \ldots, D_m) = \{(x + \sum_{k=0}^{i-1} d_k, y + \sum_{k=i+1}^{m} d_k) : \forall (x, y) \in D_i\}.$$

Its image set is set $\mathrm{Im}\ \alpha_{\mathbf{d}} = \{\alpha_{\mathbf{d}}(\underline{D}) : \underline{D} \in \mathbb{D}(v(\mathbf{r}))\} \subset \mathbf{S}(d)$, and we define a map $g : \mathrm{LUBPD}(v(\mathbf{r})) \to \mathrm{Im}\ \alpha_{\mathbf{d}}$ as follows: Given $(B, \ell) \in \mathrm{LUBPD}(v(\mathbf{r}))$, for each empty tile e in $B \setminus P(Hom)$, if e is in the main block super-antidiagonal, then by Corollary 4.7, use local moves to move e to the top left corner; if e is below the main block super-antidiagonal then consider $\ell(e)$. If $\ell(e) = U$, apply column moves to move e to the top left corner of the block on the super-antidiagonal to the north of the block containing e; otherwise if $\ell(e) = L$, apply row moves to move e to the top left corner of the block on the super-antidiagonal to the west of the block containing e. We do this operation on all the empty tiles in $B \setminus P(Hom)$ starting from the top-left most empty tile and working from left to right and top to bottom. In particular, we move all the empty tiles on the main block super-antidiagonal before moving any empty tiles below the main block super-antidiagonal (in this case at most one empty tile). This ordering is to ensure that, after this process, we obtain an Edelman Greene bumpless pipe dream. For each empty tile e in $B \setminus P(Hom)$, we record its final position e' and the number of moves c_e taken to move it to that position. Then we define

$$g(B, \ell) = \{(i + c_e, j + c_e) : \forall e' = (i, j)\}.$$

We now argue that g is well-defined. First, notice that the map g behaves like f^{-1} defined in the proof of Proposition 4.4 due the direct correspondence between local moves on bumpless pipe dreams and plus diagrams. If we ignore the element in $g(B, \ell)$ that came from an empty tile below the main block super-antidiagonal, the positions of all other elements of the plus diagram are well-defined. Hence if $B \in \mathrm{LUBPD}(v(\mathbf{r}))$ does not contain any empty tile below the main block super-antidiagonal, then we are done. Suppose there is a unique empty tile $e = (x, y)$ below the main block super anti-diagonal with final position $e' = (x', y')$ after using c_e moves. After having moved all empty tiles on the main block super-antidiagonal, but before moving e, we know that by Corollary 4.7, the empty tiles are arranged in rectangles in the top-left corner of each block on the main block super-antidiagonal. Therefore, if $\ell(e) = U$, we know that e' is in the bottom-most position among the empty tiles within the corresponding block. Since column moves modify bumpless pipe dreams within two consecutive columns, we know that $y' + c_e = y$, which by definition $y = \max\{j : (i, j) \in g(B, \ell)\}$ and therefore $g(B, \ell)$ is a valid plus diagram. Similarly, if $\ell(e) = L$, we know that e' is in the right-most position among the empty tiles within the corresponding block. Since row moves modify bumpless pipe dreams within two consecutive rows, we know that $x' + c_e = x$, which by definition $x = \max\{i : (i, j) \in g(B, \ell)\}$, meaning $g(B, \ell)$ is a valid plus diagram.

We now define the inverse g^{-1}. Given a plus diagram D, the empty tiles in $g^{-1}(D)$ on the main block super-antidiagonal are exactly in the position of the elements in D on the main block super-antidiagonal. If there is an element $p \in D$ below the main block super-antidiagonal, we know that it must map to an empty tile below the main block super-antidiagonal. Therefore we can recover the position of the empty tile by applying the same number of inverse column/row moves from its position in the corresponding Edelman-Greene bumpless pipe dream. Then we can apply inverse local moves to all the other empty tiles and obtain $g^{-1}(D)$. Hence we can conclude that g is a bijection. For $B \in \mathrm{BPD}(v(\mathbf{r}))$ with one empty tile e below the main block super-antidiagonal, let ℓ_U be the labeling such that $\ell_U(e) = U$ and ℓ_L be the labeling such that $\ell_L(e) = L$. Then we have

$$(\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{B \setminus P(Hom)} = (\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{g(B,\ell_U)} + (\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{g(B,\ell_L)}.$$

For any $B \in \mathrm{BPD}(v(\mathbf{r}))$ with no empty tile below the main block super-antidiagonal, we use $g(B)$ to denote the image of B under g since there is essentially no labeling for B. Then we have

$$(\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{B \setminus P(Hom)} = (\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{g(B)}.$$

Therefore the composition $\alpha_\mathbf{d} \circ g^{-1}$ is a bijection between $\mathbb{D}(v(\mathbf{r}))$ and $\mathrm{LUBPD}(v(\mathbf{r}))$.

Now let $\mathrm{BPD}_0(v(\mathbf{r}))$ denote the set of bumpless pipe dreams with no empty tiles below the main block super-antidiagonal and let $\mathrm{BPD}_1(v(\mathbf{r})) = \mathrm{BPD}(v(\mathbf{r})) \setminus \mathrm{BPD}_0(v(\mathbf{r}))$ and we have

$$\begin{aligned}
\sum_{B\in \mathrm{BPD}(v(\mathbf{r}))} (\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{B\setminus P(Hom)} &= \sum_{B\in \mathrm{BPD}_0(v(\mathbf{r}))} (\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{B\setminus P(Hom)} \\
&+ \sum_{B\in \mathrm{BPD}_1(v(\mathbf{r}))} (\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{B\setminus P(Hom)} \\
&= \sum_{B\in \mathrm{BPD}_0(v(\mathbf{r}))} \left((\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{g(B,\ell_U)} + (\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{g(B,\ell_L)}\right) \\
&+ \sum_{B\in \mathrm{BPD}_1(v(\mathbf{r}))} (\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{g(B)} \\
&= \sum_{D\in \mathrm{Im}\ \alpha_\mathbf{d}} (\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d})^{D} \\
&= \sum_{\underline{D}=(D_1,\ldots,D_m)\in \mathbb{D}(v(\mathbf{r}))} \left(\prod_{i=1}^{m} ({\mathbf{x_d}}^{i-1} - {\mathbf{y_d}}^{i})^{D_i}\right) \\
&= \sum_{P\in \mathrm{EGBPD}(v(\mathbf{r}))} s_{\boldsymbol{\lambda}_P}(\mathbf{x_d} - \mathring{\mathbf{y}}_\mathbf{d}).
\end{aligned}$$

Therefore the result follows.

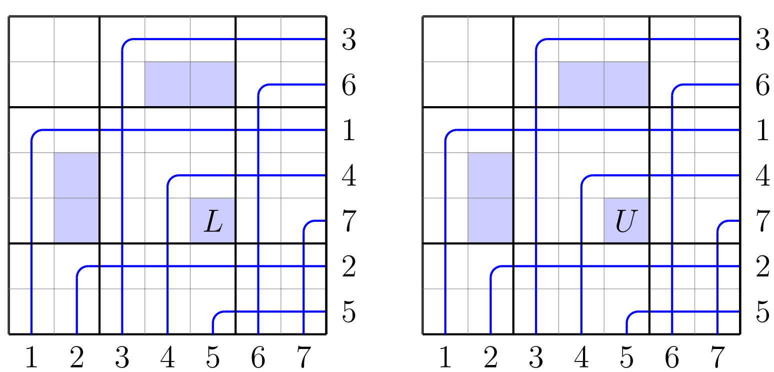
□

The following example illustrates the bijection described in the proof of Proposition 4.11.

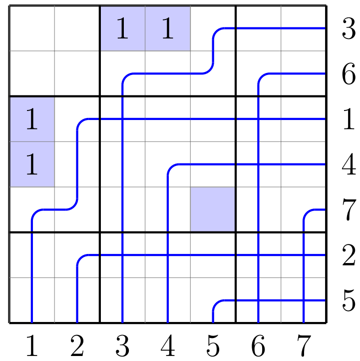
Example 4.12. *Let Q be a type A_3 equioriented quiver with $\mathbf{d} = (2, 3, 2)$ and consider the following rank array*

$$\mathbf{r} = \begin{array}{ccc|c} 2 & 1 & 0 & \\ \hline & & 2 & 0 \\ & 3 & 1 & 1 \\ 2 & 1 & 0 & 2 \end{array}.$$

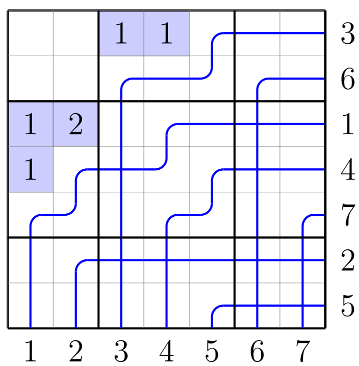
Its Zelevinsky permutation is $v(\mathbf{r}) = (3, 6, 1, 4, 7, 2, 5)$ with the following two LU-labelings for the Rothe pipe dream.

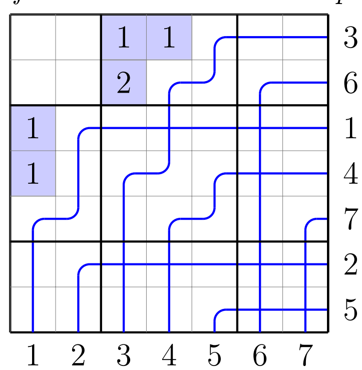


Note that the Rothe pipe dream only contains one empty tile below the main block super-antidiagonal. Now we show how to find the corresponding tuples of plus diagrams. First we top-left justify the empty tiles on the main block super-antidiagonal using local moves. Recording the number of moves taken for each tile we end up with the following bumpless pipe dream.

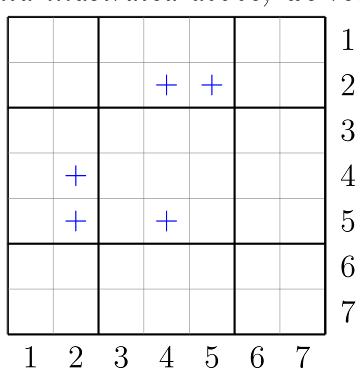


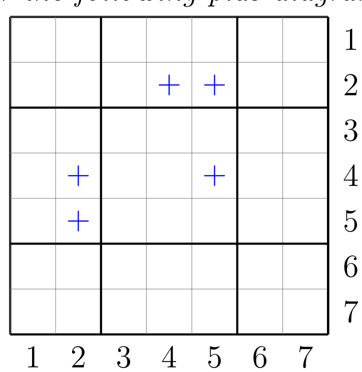
Then for the pipe dream labeled L, we apply row moves; for the pipe dream labeled U, we apply column moves. We obtain the following two Edelman-Greene bumpless pipe dreams respectively, illustrated below, with the number of moves used on the empty tiles labeled.





Now using the data illustrated above, we recover the following plus diagrams, respectively.





The above steps illustrated the map g defined in the proof of Proposition 4.11. Applying $\alpha_{\mathbf{d}}^{-1}$ we know that the above two plus diagrams corresponds to the tuples of plus diagrams $(\{(2,2),(2,3)\},\{(2,2),(3,2),(3,4)\})$ and $(\{(2,2),(2,3),(4,3)\},\{(2,2),(3,2)\})$ respectively. The reader can verify that the sum of the weights of the two plus diagrams is exactly the weight of the Rothe pipe dream.

5. Future Work

Motivated by the technique used in the proof of Proposition 4.11, we formulate the following conjecture.

Problem 5.1. *Let Q be a type A_{m+1} equioriented quiver with $\mathbf{d} = (d_0, \ldots, d_m)$ and let $\mathbf{r}$ be a rank array of Q. For any $P \in \mathrm{EGBPD}(v(\mathbf{r}))$, use $\boldsymbol{\lambda}_P = (\lambda_P^{(1)}, \ldots, \lambda_P^{(m)})$ to denote*

$\tau_{\mathbf{d}}(\text{shape}(P))$. *Let* $\mathbf{D}(\lambda)$ *denote the set of plus diagrams with nonzero weight in the corresponding specialized Schur function, and let*

$$\mathbb{D}(v(\mathbf{r})) = \bigcup_{P \in \text{EGBPD}(v(\mathbf{r}))} \prod_{k=1}^{m} \mathbf{D}(\lambda_P^{(k)}).$$

Find an explicit bijection between the set of LU-labeled bumpless pipe dreams for $v(\mathbf{r})$ *and* $\mathbb{D}(v(\mathbf{r}))$.

We would like to construct a proof using the same technique in the proof of Proposition 4.11, which is to move empty tile and track its positions and number of moves taken. However, we have yet to find an inverse when there is more than one empty tile below the main block super-antidiagonal. In fact, we have not yet found any natural ordering on moving the empty tiles that makes the desired map injective.

6. Acknowledgements

This project was part of the Research Experience for Undergraduate Program at the University of Michigan Department of Mathematics. We would like to thank Prof. David Speyer for organizing the program. We are grateful for our mentor Anna Weigandt for proposing this project and providing us guidance throughout this project.

References

[BF99] Anders Skovsted Buch and William Fulton. "Chern class formulas for quiver varieties". *Invent. Math.* 135.3 (1999), pp. 665–687.

[BFR05] Anders S. Buch, László M. Fehér, and Richárd Rimányi. "Positivity of quiver coefficients through Thom polynomials". *Adv. Math.* 197.1 (2005), pp. 306–320.

[BR07] Anders Skovsted Buch and Richárd Rimányi. "A formula for non-equioriented quiver orbits of type A". *J. Algebraic Geom.* 16.3 (2007), pp. 531–546.

[Bri12] Michel Brion. "Representations of quivers". In: *Geometric methods in representation theory. I.* Vol. 24. Sémin. Congr. Soc. Math. France, Paris, 2012, pp. 103–144.

[Buc+06] Anders S. Buch et al. "Stable Grothendieck polynomials and K-theoretic factor sequences". *Mathematische Annalen* 340 (2006), pp. 359–382.

[Buc05] Anders Skovsted Buch. "Alternating signs of quiver coefficients". *J. Amer. Math. Soc.* 18.1 (2005), pp. 217–237.

[FGS18] Neil J. Y. Fan, Peter L. Guo, and Sophie C. C. Sun. *Bumpless Pipedreams, Reduced Word Tableaux and Stanley Symmetric Functions.* 2018. eprint: `arXiv:1810.11916`.

[IN09] Takeshi Ikeda and Hiroshi Naruse. "Excited Young diagrams and equivariant Schubert calculus". *Trans. Amer. Math. Soc.* 361.10 (2009), pp. 5193–5221.

[KKR19] Ryan Kinser, Allen Knutson, and Jenna Rajchgot. "Three combinatorial formulas for type A quiver polynomials and K-polynomials". *Duke Math. J.* 168.4 (2019), pp. 505–551.

[KMS06] Allen Knutson, Ezra Miller, and Mark Shimozono. "Four positive formulae for type A quiver polynomials". *Invent. Math.* 166.2 (2006), pp. 229–325.

[KMY09] Allen Knutson, Ezra Miller, and Alexander Yong. “Gröbner geometry of vertex decompositions and of flagged tableaux”. *J. Reine Angew. Math.* 630 (2009), pp. 1–31.

[Kre06] V. Kreiman. “Schubert Classes in the Equivariant K-Theory and Equivariant Cohomology of the Grassmannian” (Jan. 2006).

[LLS18] Thomas Lam, Seung Jin Lee, and Mark Shimozono. *Back stable Schubert calculus.* 2018. eprint: arXiv:1806.11233.

[Man01] Laurent Manivel. *Symmetric functions, Schubert polynomials and degeneracy loci.* Vol. 6. SMF/AMS Texts and Monographs. Translated from the 1998 French original by John R. Swallow, Cours Spécialisés [Specialized Courses], 3. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001, pp. viii+167.

[MS05] Ezra Miller and Bernd Sturmfels. *Combinatorial commutative algebra.* Vol. 227. Graduate texts in mathematics. New York: Springer, 2005, pp. xiv, 417.

[Wei20] Anna Weigandt. *Bumpless pipe dreams and alternating sign matrices.* 2020. arXiv: 2003.07342 [math.CO].

(ACV) Dept. of Mathematics, University of Michigan, Ann Arbor MI 48109
Email address: avidinas@umich.edu

(ZX) Dept. of Mathematics, Massachusetts Institute of Technology, Cambridge MA 02139
Email address: zixuanxu@mit.edu