MEDIUM-SCALE RICCI CURVATURE FOR HYPERBOLIC GROUPS

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Abstract. We study the relationship between a notion of medium-scale Ricci curvature for finitely generated groups and that of hyperbolicity in the sense of Gromov. We give an example of a generating set that gives zero curvature with positive density for the free group of rank 2. We prove that, by making the radius used in computing the curvature sufficiently large, we can always have negative curvature outside of a ball in non-elementary hyperbolic groups. On the other hand, we give an example of a group which has negative curvature for all non-identity points but is not hyperbolic.

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1. Introduction

In Riemannian geometry, Ricci curvature measures the difference of the average distance between points on infinitesimal spheres and the distance between their centers. In [Oll09], Ollivier generalized this intuition to more general metric spaces

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equipped with Markov chains with his *transportation curvature*. Here we focus on the *comparison curvature* for finitely generated groups introduced by Bar-Natan, Duchin, and Kropholler in [BDK17]. They called this curvature *medium-scale* because, rather than being based on infinitesimal or global properties, it depends on some fixed comparison radius.

In this paper, following the notation introduced by Bar-Natan, Duchin, and Kropholler, we use $\kappa_k^{\mathcal{S}}(g)$ to denote the k-spherical comparison curvature at the group element g. As a special case, we write $\kappa(g)$ to denote the 1-spherical comparison curvature. It is well known that this type of curvature depends on the choice of generating set of the group. The first question that we seek to answer is whether the free group of rank 2 must have negative curvature for all but finitely many points with respect to any generating set. We show that this is false by proving the following theorem:

Theorem 1. *There exists a generating set of the free group $F_2 = \langle a, b \rangle$ such that F_2 has zero curvature with positive density. F_2 also has negative curvature with positive density with respect to this generating set.*

By positive density, we mean that there exist $\epsilon > 0$, $N \geq 0$ such that the fraction of points satisfying the desired property in the ball of radius n is bounded below by ϵ for all $n \geq N$.

We also explore the relationship between comparison curvature and hyperbolicity. We focus on non-elementary hyperbolic groups, which includes the group F_2. By [BDK17, Theorem 22], there exist non-elementary hyperbolic groups with sequences of elements $(g_n)_{n=1}^{\infty}$ such that $\kappa(g_n) > 0$ for all n. We prove the following theorem to show that, by using a sufficiently large comparison radius to compute the curvature, we can have at most finitely many points with non-negative curvature in any non-elementary hyperbolic group:

Theorem 2. *Let G be a non-elementary hyperbolic groups with some fixed generating set S. Then there exists $k \geq 1$ such that, for every $g \in G$ outside the ball of radius $10k$, we have $\kappa_k^{\mathcal{S}}(g) < 0$.*

On the other hand, we show that negative curvature for all non-identity points does not imply that a group is hyperbolic by proving the following counterexample:

Theorem 3. *The group $\mathbb{Z} * \mathbb{Z}^2 = \langle t, a, b | ab = ba \rangle$ is not hyperbolic, but there exists a generating set such that, for all $g \in \mathbb{Z} * \mathbb{Z}^2 - \{1\}$, we have $\kappa(g) < 0$.*

This paper is organized as follows: we begin by reviewing the definitions of comparison curvature and hyperbolicity in Section 2, going over some simple examples and proving lemmas that will help with the calculation of curvature. In Section 3, we give examples of generating sets giving negative curvature for F_2 before proving Theorem 1. We conclude in Section 4 by proving Theorems 2 and 3.

2. Background

In this section we review the definitions of comparison curvature and Gromov's δ-hyperbolicity, looking at some specfic examples. We then prove some preliminary results that will help us compute the curvature.

2.1. **Comparison Curvature.** Let G be a group that is generated by some finite set S, which we assume to be closed under inversion and not contain the group identity. Define the *word length* $|g|_S$ of an element $g \in G$ as the minimum number of generators $g_1, \ldots, g_n \in S$ such that $g = g_1 \ldots g_n$. We call $g_1 \ldots g_n$ a *word* of generators and a *spelling* of g. When $n = |g|_S$, we further call $g_1 \ldots g_n$ a *geodesic spelling* of g (such spellings need not be unique). In general, a *geodesic word* of generators is a word of generators such that no other word spelling the same group element has less generators.

The norm $|\cdot|_S : G \to \mathbb{Z}_{\geq 0}$ induces a left-invariant metric $d_S : G \times G \to \mathbb{Z}_{\geq 0}$ defined by $d_S(g,h) = |h^{-1}g|_S$, so that $d_S(g,1) = |g|_S$ where 1 denotes the group identity. Although the metric space (G, d_S) depends on the choice of generating set S, we will henceforth not include the subscript S because the generating set will be clear from context.

Let (X, d) be a metric space. Then we call (X, d) *geodesic* if any two points in X can be connected by a continuous curve $c : [a,b] \subset \mathbb{R} \to X$ such that $d(c(t_1), c(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [a,b]$.

Given a finitely generated group G and finite generating set S, define the *Cayley graph* of (G,S) as the graph whose vertices are the elements of G and whose edges connect the points $g, h \in G$ if $d(g,h) = 1$. This graph can be equipped with the natural graph metric such that the length of each edge is 1, and the distance between two points is the length of the shortest path connecting them. With this metric, the Cayley graph is a geodesic metric space such that the distance between two vertices is exactly the word distance between the corresponding group elements. An example of a Cayley graph is shown in Figure 1.

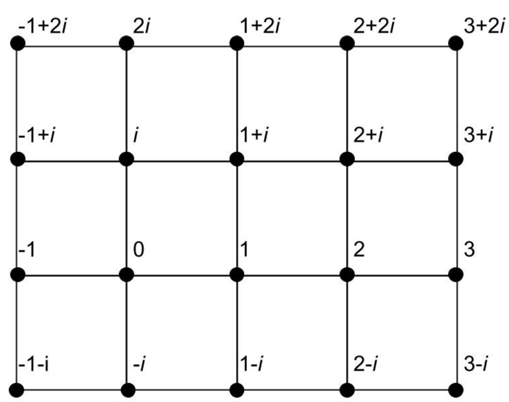

FIGURE 1. The Cayley graph of the group $\mathbb{Z}^2$ with respect to the generating set $S := \{1, -1, i, -i\}$.

By combining this geometric structure for a finitely generated group G with the algebraic structure of G, we can define a notion of curvature which shares many properties with Ricci curvature for Riemannian manifolds. For $k \geq 0$ and $g \in G$, define the ball and sphere of radius k centered at g to respectively be

$$B_k(g) := \{h \in G | d(g,h) \leq k\},$$
$$S_k(g) := \{h \in G | d(g,h) = k\}.$$

Usually we care about balls and spheres centered at the group identity 1, which we write as B_k and S_k respectively. In particular, note that we have $S_1 = S$. Given any distinct $g, h \in G$ and $k \geq 1$, define the *k-spherical comparison distance* between g and h to be

$$\mathcal{S}_k(g,h) := \frac{1}{|S_k|} \sum_{s \in S_k} d(gs, hs).$$

The quantity $\mathcal{S}_k(g,h)$ measures the average distance between corresponding points on the spheres of radius k about g and h, where the correspondence is by left-translation. Comparing this quantity to the distance between g and h, we can write the definition of the type of curvature that we are concerned with:

Definition 4. [BDK17] The *k-spherical comparison curvature* between $g, h \in G$ is

$$\kappa_k^{\mathcal{S}}(g,h) := \frac{d(g,h) - \mathcal{S}_k(g,h)}{d(g,h)} = \frac{d(g,h) - \frac{1}{|S_k|}\sum_{s \in S_k} d(gs, hs)}{d(g,h)}.$$

Notice that we get positive curvature when corresponding points on $S_k(g)$ and $S_k(h)$ are closer, on average, than g and h, a property that agrees with classical Ricci curvature. By left-invariance of our metric, it suffices to consider the comparison curvature between a point $g \in G - \{1\}$ and the group identity 1, which we write as $\kappa_k^{\mathcal{S}}(g) := \kappa_k^{\mathcal{S}}(g, 1)$. For the case of $k = 1$, we use the notation $\kappa(g) := \kappa_1^{\mathcal{S}}(g)$. Note that

$$\mathcal{S}_1(g,1) = \frac{1}{|S_1|} \sum_{s \in S_1} d(gs, 1s) = \frac{1}{|S|} \sum_{s \in S} |s^{-1}gs| := \mathrm{GenCon}(g),$$

where we use the notation $\mathrm{GenCon}(g)$ to emphasize the fact that this quantity is the average word length of the conjugation of g by all the generators in S. Therefore,

$$\kappa(g) = \frac{|g| - \mathrm{GenCon}(g)}{|g|},$$

and in order to find the sign of the curvature it suffices to evaluate whether conjugation by generators on average increases, decreases, or preserves the word length of g.

Example 5. Suppose that G is an abelian group. Then for all $g \in G - \{1\}$ we have

$$\begin{aligned}
\kappa(g) &= \frac{|g| - \mathrm{GenCon}(g)}{|g|} = \frac{|g| - \frac{1}{|S|}\sum_{s \in S} |s^{-1}gs|}{|g|} \\
&= \frac{|g| - \frac{1}{|S|}\sum_{s \in S} |gs^{-1}s|}{|g|} = \frac{|g| - \frac{1}{|S|}\sum_{s \in S} |g|}{|g|} \\
&= \frac{|g| - |g|}{|g|} = 0.
\end{aligned}$$

Therefore, abelian groups are everywhere "flat" with respect to the comparison curvature.

2.2. **Hyperbolic Groups.** The notion of a *hyperbolic group* was introduced by Gromov in [Gro87]. This is a group that, when equipped with a word metric, satisfies the following:

Definition 6. [Gro87] A metric space (X, d) is called δ-*hyperbolic* if, for all points $x, y, z, w \in X$, the *four-point condition*

$$(x, y)_w \geq \min\{(x, z)_w, (y, z)_w\} - \delta$$

is satisfied. Here the quantity $(p, q)_r$ denotes the *Gromov product* of $p, q, r \in X$ defined by

$$(p, q)_r := \frac{1}{2}(d(p, r) + d(r, q) - d(p, q)).$$

Gromov proved in [Gro87] that, if a group is δ-hyperbolic with respect to some finite generating set S, then with respect to another finite generating set S' it is δ'-hyperbolic for some $\delta' \geq 0$. Therefore, we can speak of groups being hyperbolic, but in order to speak of a group being δ-hyperbolic we must first fix a generating set.

Another useful formulation of hyperbolicity for geodesic metric spaces (such as a Cayley graph with a graph metric) is based on δ-*thin triangles*, which are defined as follows: Given a geodesic triangle Δ consisting of the vertices x, y, and z connected by geodesic segments, there exists a map τ from Δ to a tripod whose endpoints are $\tau(x)$, $\tau(y)$, and $\tau(z)$ such that $d(p_1, p_2) = d(\tau(p_1), \tau(p_2))$ whenever p_1 and p_2 lie on the same edge of Δ. The length of the branch of this tripod that ends in $\tau(z)$ is exactly the product $(x, y)_z$, and an analogous result holds for the other two branches. We call τ a *triangle-tripod transformation*, and its image is unique up to isometry. We call Δ a δ-thin triangle if, whenever $\tau(p) = \tau(q)$ for two points $p, q \in \Delta$, we have $d(p, q) \leq \delta$ (Figure 2).

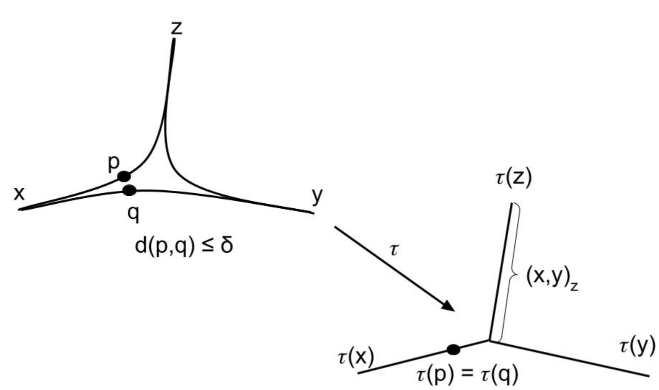

FIGURE 2. A δ-thin triangle Δ in a geodesic metric space, along with its image under a triangle-tripod transformation τ. Any points that map to the same point under τ must lie within a distance of δ in Δ.

According to [Gro87, Proposition 6.3.C], all geodesic triangles in a δ-hyperbolic space are 2δ-thin, and if all geodesic triangles in a geodesic metric space are δ-thin then the space is 2δ-hyperbolic. In this paper, when we write that a metric space is δ-hyperbolic, the constant δ is determined by the four-point condition.

Example 7. Consider a metric space (X, d) which is a tree. Any geodesic triangle in X will either be a tripod or a single geodesic segment containing all three of its vertices. Therefore, all geodesic triangles in X are 0-thin and X is a 0-hyperbolic metric space.

2.3. **Computing GenCon.** The following basic results will be used throughout this paper to transform the problem of computing the quantity $\mathrm{GenCon}(g)$ that appears in the formula for $\kappa(g)$ to that of finding spellings of g that begin and end with certain generators:

Lemma 8. *Let S be a generating set for the group G. Let $g \in G$ and $s \in S$. Then $|sg| = |g| - 1$ if and only if there exists a geodesic spelling $g_1 \ldots g_n$ of g with respect to S such that $g_1 = s^{-1}$. Similarly, $|gs| = |g| - 1$ if and only if there exists a geodesic spelling $g_1 \ldots g_n$ of g with respect to S such that $g_n = s^{-1}$.*

Proof. Assume that a geodesic spelling $s^{-1}g_2 \ldots g_n = g$ exists. Then $|sg| = |ss^{-1}g_2 \ldots g_n| = |g_2 \ldots g_n| \leq |g| - 1$, and from the triangle inequality we have $|sg| \geq |g| - 1$, so $|sg| = |g| - 1$.

Conversely, assume that $|sg| = |g| - 1$. Then there exists a geodesic spelling $h_1 \ldots h_{n-1}$ of sg, where $|g| = n$. Consider the word $s^{-1}h_1 \ldots h_{n-1}$, which is equal to $s^{-1}sg = g$. Because this word consists of n generators, it is a geodesic spelling of g, and it begins with s^{-1} as desired.

Note that we have $|g| = |g^{-1}|$ for all $g \in G$ because any geodesic spelling $g_1 \ldots g_n$ of g induces a spelling $g_n^{-1} \ldots g_1^{-1}$ of g^{-1} with the same number of generators. The proof for $|gs| = |g| - 1$ therefore follows from noting that $|gs| = |(gs)^{-1}| = |s^{-1}g^{-1}|$ and applying the result for when $|s^{-1}g^{-1}| = |g^{-1}| - 1 = |g| - 1$. □

Lemma 9. *Let S, G, s, and g be as in the statement of Lemma 8 with $|g| = n$. Then $|sg| = |g|$ if and only if there exists a spelling $g_1 \ldots g_{n+1}$ of g with respect to S such that $g_1 = s^{-1}$ and $g_2 \ldots g_{n+1}$ is a geodesic word. Similarly, $|gs| = |g|$ if and only if there exists a spelling $g_1 \ldots g_{n+1}$ of g with respect to S such that $g_{n+1} = s^{-1}$ and $g_1 \ldots g_n$ is a geodesic word.*

Proof. Assume that a spelling $s^{-1}g_2 \ldots g_{n+1} = g$ exists with $g_2 \ldots g_{n+1}$ a geodesic word. Then $|sg| = |ss^{-1}g_2 \ldots g_{n+1}| = |g_2 \ldots g_{n+1}| = n = |g|$.

Conversely, assume that $|sg| = |g|$. Then there exists a geodesic spelling $h_1 \ldots h_n$ of sg, so $g = s^{-1}sg = s^{-1}h_1 \ldots h_n$. This is indeed a spelling of g that is one letter longer than a geodesic spelling and that begins with s^{-1} as desired.

As in the proof of Lemma 8, the proof for $|gs| = |g|$ follows from applying the result for $|sg| = |g|$ to the inverse of gs. □

Remark 10. By the triangle inequality for the word metric, $|g| - |s| \leq |sg| \leq |g| + |s|$, so $|sg| = |g| - 1$, $|g|$, or $|g| + 1$ because $|s| = 1$. Lemmas 8 and 9 give necessary and sufficient conditions for the first two of these three possibilities, so if the hypotheses of neither of these lemmas hold then $|sg| = |g| + 1$. Similarly, if the necessary and sufficient conditions for $|gs| = |g| - 1$ and $|gs| = |g|$ both fail to hold, then $|gs| = |g| + 1$.

3. Curvature of Free Group

Throughout this section, let F_2 denote the free group of rank 2 with the presentation $\langle a, b\rangle$. We begin by giving examples of generating sets of F_2 for which all non-identity points have negative curvature. We then prove Theorem 1.

3.1. Negative Curvature Examples in Free Group.

Example 11. Consider the generating set $S_\alpha := \{a, a^{-1}, b, b^{-1}\}$ of F_2. Because the only way to shorten a word in these generators is by applying free reductions, any two words can only represent the same group element $g \in F_2 - \{1\}$ if they differ in length by an even number of generators. Therefore, by Lemma 9, there exist no generators $s \in S_\alpha$ or group elements $g \in F_2 - \{1\}$ such that $|sg| = |g|$ or $|gs| = |g|$. Moreover, because this group is free, every group element has only one geodesic spelling with respect to S_α. For each $g \in F_2 - \{1\}$, we therefore have by Lemma 8 that there exists only one $s \in S_\alpha$ such that $|sg| = |g| - 1$ and only one $s \in S_\alpha$ such that $|gs| = |g| - 1$. For all other $s \in S_\alpha$ we have $|sg| = |g| + 1$ and $|gs| = |g| + 1$, so we can conclude that

$$\kappa(g) = \frac{|g| - \mathrm{GenCon}(g)}{|g|} = \frac{|g| - \frac{1}{4}\sum_{s\in S_\alpha} |s^{-1}gs|}{|g|} = \frac{|g| - \frac{1}{4}(4|g| + 4)}{|g|} = -\frac{1}{|g|}.$$

The Cayley graph of (F_2, S_α) is shown in Figure 3. Note that this graph is a tree, proving that F_2 is a hyperbolic group.

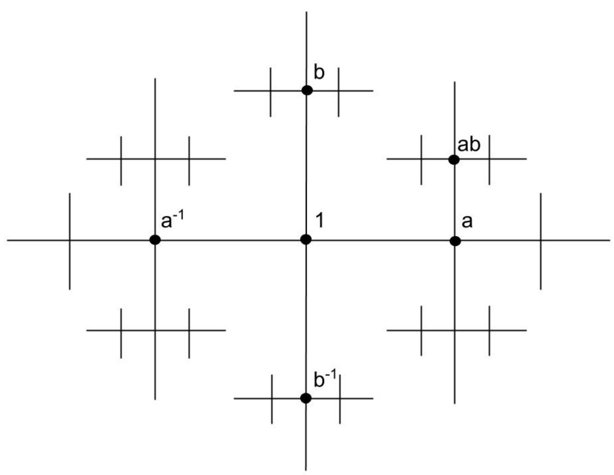

Figure 3. The Cayley graph of the group $F_2 = \langle a, b\rangle$ with respect to the generating set $S_\alpha := \{a, a^{-1}, b, b^{-1}\}$

Now consider the generating set $S_\beta := \{a, a^{-1}, b, b^{-1}, aba, a^{-1}b^{-1}a^{-1}\}$. We have the following result:

Proposition 12. *With respect to the generating set S_β, we have $\kappa(g) < 0$ for all $g \in F_2 - \{1\}$.*

Proof. Let $w = aba$ so that $w^{-1} = a^{-1}b^{-1}a^{-1}$. With respect to the generators in S_β, F_2 has the presentation $\langle a, b, w | abaw^{-1} = 1\rangle$. Suppose that we have

$g_1 \dots g_n = h_1 \dots h_{n+1}$ for some generators $g_1, \dots, g_n, h_1, \dots, h_{n+1} \in S_\beta$. Then $h_{n+1}^{-1} \dots h_1^{-1} g_1 \dots g_n = 1$, so this word with $2n+1$ letters can be reduced to the identity by applying free reductions and removing all occurrences of any inverse or cyclic permutation of $abaw^{-1}$. Because all of these operations change the number of letters by an even number and $2n+1$ is odd, we have a contradiction. Therefore, given any word of generators in S_β, there does not exist a word with exactly one more generator that represents the same group element in F_2. By Lemma 9, this implies that there do not exist generators $s \in S_\beta$ or group elements $g \in F_2 - \{1\}$ such that $|sg| = |g|$ or $|gs| = |g|$.

For $s \in S_\alpha = \{a, a^{-1}, b, b^{-1}\}$, let $A_s \subset F_2$ be the set of all $g \in F_2$ such that the first letter of g begins with s in its geodesic spelling with respect to S_α. Every non-identity element of F_2 belongs to exactly one of the four sets A_a, $A_{a^{-1}}$, A_b, or $A_{b^{-1}}$, which each correspond with one of the four primary branches stemming from the point 1 in Figure 3. We claim that, for all $g \in A_a$, every geodesic spelling of g with respect to S_β must begin with either a or w. Note that a and w are the only two generators in S_β that begin with a when spelled with respect to S_α and thus lie in A_a. Suppose that there exists $g \in A_a$ with a geodesic spelling $g = g_1 \dots g_n$ such that $g_1 \neq a$ and $g_1 \neq w$. Then for some $1 \leq k < n$ we have $g_1 \dots g_k \notin A_a \cup \{1\}$ and $(g_1 \dots g_k) g_{k+1} \in A_a$. The only non-identity point that is not in A_a for which right multiplication by a generator gives a point in A_a is $b^{-1}a^{-1}$, for which we have $(b^{-1}a^{-1})w = a$. Therefore, we must have $(g_1 \dots g_k) g_{k+1} = a$, contradicting our claim that the spelling $g_1 \dots g_n$ was geodesic. Hence, the first letter of a geodesic spelling of any element of A_a must be either a or w as desired. An identical argument shows that a geodesic spelling of an element of $A_{a^{-1}}$ must begin with either a^{-1} or w^{-1}.

We now claim that every geodesic spelling of an element of A_b must begin with either b or a^{-1}. Suppose that $g \in A_b$ has a geodesic spelling $g = g_1 \dots g_n$ such that $g_1 \neq b$. Then again there exists $1 \leq k < n$ such that $g_1 \dots g_k \notin A_b \cup \{1\}$ and $(g_1 \dots g_k) g_{k+1} \in A_b$. The only non-identity point that is not in A_b for which right multiplication by a generator gives a point in A_b is a^{-1}, for which we have $a^{-1}w = ba$. Therefore, $g_1 \dots g_k = a^{-1}$. Because the spelling $g_1 \dots g_n$ is geodesic, we must have $k = 1$ and $g_1 = a^{-1}$. Hence, the first letter of a geodesic spelling of an element of A_b must be either b or a^{-1}. An identical argument shows that a geodesic spelling of an element of $A_{b^{-1}}$ must begin with either b^{-1} or a.

We thus conclude by Lemma 8 that, for all $g \in F_2 - \{1\}$, there exist at most two distinct $s \in S_\beta$ such that $|sg| = |g| - 1$. By Remark 10 we must have $|sg| = |g| + 1$ for all other s. By applying the results of the previous paragraphs to the inverses of group elements, we also get that there exist at most two distinct $s \in S_\beta$ such that $|gs| = |g| - 1$.

Direct computation shows that $\kappa(g) < 0$ for $g = ba$ and $g = b^{-1}a^{-1}$, and whenever g is a generator in S_β. For all other $g \in F_2 - \{1\}$, note that g is in A_{s_0} for some $s_0 \in S_\alpha = \{a, a^{-1}, b, b^{-1}\}$ if and only if gs' is also in A_{s_0} for all generators $s' \in S_\beta$. Hence, the two distinct $s \in S_\beta$ for which it is possible to have $|sg| = |g| - 1$ are the same as the two distinct $s \in S_\beta$ for which it is possible to have $|sgs'| = |gs'| - 1$. Therefore,

$$\begin{aligned}\kappa(g) &= \frac{|g| - \mathrm{GenCon}(g)}{|g|} = \frac{|g| - \frac{1}{6}\sum_{s\in S_\beta}|s^{-1}gs|}{|g|} \\ &= \frac{\frac{1}{6}\sum_{s\in S_\beta}(|g|-|gs|) + \frac{1}{6}\sum_{s\in S_\beta}(|gs| - |s^{-1}gs|)}{|g|} \\ &\leq \frac{\frac{1}{6}(-2)+\frac{1}{6}(-2)}{|g|} = -\frac{2}{3|g|} < 0.\end{aligned}$$

□

3.2. **Zero Curvature with Positive Density in Free Group.** The generating set S_β from Proposition 12 did not allow for points of non-negative curvature because different geodesic words for the same group element could have no more than two different starting letters. In order to prove Theorem 1, we therefore seek a six-element generating set which allows for different geodesic words for the same group element to begin with three different generators. Let $S_\gamma := \{a, a^{-1}, b, b^{-1}, ababa, a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}\}$. We define the word $w = ababa$ so that $w^{-1} = a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}$, and we split S_γ into the two subsets $S^1_\gamma := \{a, b^{-1}, w\}$ and $S^2_\gamma := \{a^{-1}, b, w^{-1}\}$. Then we have the following useful results:

Lemma 13. *Let $h_1 \dots h_n$ and $h'_1 \dots h'_n$ be geodesic words spelling the same element of F_2. Then $h_1 \in S^1_\gamma$ if and only if $h'_1 \in S^1_\gamma$. Similarly, $h_n \in S^1_\gamma$ if and only if $h'_n \in S^1_\gamma$.*

Proof. Let the sets A_a, $A_{a^{-1}}$, A_b, and $A_{b^{-1}}$ be as in the proof of Proposition 12, and recall that every non-identity element of F_2 must lie in exactly one of these four sets. Let $g \in A_a$, and we claim that every geodesic spelling of g begins with an element of S^1_γ. Let $g_1 \dots g_n$ be a geodesic spelling of g such that $g_1 \neq a$ and $g_1 \neq w$. Then there must exist $1 \leq k < n$ such that $g_1 \dots g_k \notin A_a \cup \{1\}$ and $(g_1 \dots g_k)g_{k+1} \in A_a$. The only non-identity elements that are not in A_a for which right multiplication by a generator gives elements of A_a are $b^{-1}a^{-1}$ and $b^{-1}a^{-1}b^{-1}a^{-1}$. For $b^{-1}a^{-1}$, we have $(b^{-1}a^{-1})w = aba$, while for $b^{-1}a^{-1}b^{-1}a^{-1}$, we have $(b^{-1}a^{-1}b^{-1}a^{-1})w = a$. In this second case, though, note that if $g_1 \dots g_k = b^{-1}a^{-1}b^{-1}a^{-1} = aw^{-1}$ and $g_{k+1} = w$, then the spelling $g_1 \dots g_n$ would not have been geodesic and we would have a contradiction. Therefore, the only way we could have $g_1 \dots g_n \in A_a$ and $g_1 \notin A_a$ is if $g_1 \dots g_k = b^{-1}a^{-1}$ and $g_{k+1} = w$. The word $b^{-1}a^{-1}$ is the only geodesic spelling of itself, so we must in particular have $g_1 = b^{-1}$. We can conclude that either $g_1 \in A_a$, in which case it is either a or w, or $g_1 = b^{-1}$. Therefore, $g_1 \in S^1_\gamma$ as desired. An identical argument shows that, if $g \in A_{a^{-1}}$, then every geodesic spelling of g must begin with an element of S^2_γ.

We now claim that, if $g \in A_b$, then every geodesic spelling of g begins with an element S^2_γ. Let $g_1 \dots g_n$ be a geodesic spelling of g such that $g_1 \neq b$. Then there must exist $1 \leq k < n$ such that $g_1 \dots g_k \notin A_b \cup \{1\}$ and $(g_1 \dots g_k)g_{k+1} \in A_b$. The only non-identity elements not in A_b for which right multiplication by a generator gives elements of A_b are a^{-1} and $a^{-1}b^{-1}a^{-1}$. For a^{-1}, we have $a^{-1}w = baba$, while for $a^{-1}b^{-1}a^{-1}$, we have $(a^{-1}b^{-1}a^{-1})w = ba$. In this second case, we would have $g_1 \dots g_k = a^{-1}b^{-1}a^{-1} = baw^{-1}$ is a geodesic spelling, and $g_{k+1} = w$, so the spelling $g_1 \dots g_n$ is not geodesic and we would have a contradiction. This leaves the case where $g_1 \dots g_k = a^{-1}$, and because this spelling is geodesic we must have $k = 1$ and

$g_1 = a^{-1}$. Therefore, a geodesic spelling of $g \in A_b$ must begin with either a^{-1} or b, both of which are elements of S^2_γ. An identical argument shows that, if $g \in A_{b^{-1}}$, then every geodesic spelling of g must begin with an element of S^1_γ, either b^{-1} or a.

Now suppose we have two geodesic words $h_1 \dots h_n$ and $h'_1 \dots h'_n$ such that $h_1 \dots h_n = h'_1 \dots h'_n \coloneqq h$. If $h_1 \in S^1_\gamma$, then the results of the previous paragraphs tell us that $h \in A_a$ or $A_{b^{-1}}$. Therefore, we must also have $h'_1 \in S^1_\gamma$. If $h_1 \notin S^1_\gamma$, then $h_1 \in S^2_\gamma$, so $h \in A_{a^{-1}}$ or A_b and $h'_1 \in S^2_\gamma$. Finally, applying these results to h^{-1} gives the desired result for h_n and h'_n. □

Lemma 14. *With respect to the generating set S_γ, there do not exist generators $s \in S_\gamma$ or group elements $g \in F_2 - \{1\}$ such that $|sg| = |g|$ or $|gs| = |g|$.*

Proof. With respect to S_γ, F_2 has the presentation $\langle a, b, w | ababaw^{-1} = 1\rangle$. By Lemma 9, it suffices to show that no word of these generators has exactly one more letter than a geodesic word. Suppose, on the other hand, that we have $g_1 \dots g_n = h_1 \dots h_{n+1}$ for some generators $g_1, \dots, g_n, h_1, \dots, h_{n+1} \in S_\gamma$, where the word $g_1 \dots g_n$ is geodesic. Then $h^{-1}_{n+1} \dots h^{-1}_1 g_1 \dots g_n = 1$, so this word with $2n+1$ letters can be reduced to the identity by applying free reductions and removing all occurrences of any inverse or cyclic permutation of $ababaw^{-1}$. Because all of these operations change the number of letters by an even number and $2n + 1$ is odd, we have a contradiction. □

Lemma 14 immediately implies the following when considered with Lemma 8 and Remark 10.

Corollary 15. *With respect to the generating set S_γ, we must have $|sg| = |g| - 1$ or $|sg| = |g| + 1$ for all $g \in F_2 - \{1\}$ and $s \in S_\gamma$. Moreover, we have $|sg| = |g| - 1$ if and only if there exists a geodesic spelling of g beginning with s^{-1}. Similarly, we must have $|gs| = |g| - 1$ or $|gs| = |g| + 1$, and $|gs| = |g| - 1$ if and only if there exists a geodesic spelling of g ending with s^{-1}.* □

We have the following observations that use similar ideas to the proof of Lemma 13:

Lemma 16. *Suppose that, for some $g \in F_2 - \{1\}$, there exists a geodesic spelling of g of the form $aag_1 \dots g_n$ with $g_1, \dots, g_n \in S_\gamma$. Then there exists no geodesic spelling of g of the form $b^{-1}h_1 \dots h_{n+1}$ with $h_1, \dots, h_{n+1} \in S_\gamma$. Similarly, if a geodesic spelling of some $g' \in F_2 - \{1\}$ is of the form $a^{-1}a^{-1}g'_1 \dots g'_n$, then there exists no geodesic spelling of g' of the form $bh'_1 \dots h'_{n+1}$.*

Proof. In the proof of Lemma 13, we saw that a geodesic word can only begin with the letter a if it spells a group element that lies in either A_a or $A_{b^{-1}}$. We cannot have $g \in A_{b^{-1}}$ in this case because this would imply that the product of the first two letters of a geodesic spelling of g must lie in $A_{b^{-1}}$, but we have $aa \in A_a$. Thus, $g \in A_a$. If a geodesic spelling $b^{-1}h_1 \dots h_{n+1}$ of g exists, then we must have $h_1 = a^{-1}$ and $h_2 = w$ because we saw in the proof of Lemma 13 that all geodesic words in A_a that begin with b^{-1} must begin with the prefix $b^{-1}a^{-1}w$. Therefore, because $b^{-1}a^{-1}w = aba$, we have

$$aag_1 \dots g_n = g = b^{-1}a^{-1}wh_3 \dots h_{n+1} = abah_3 \dots h_{n+1} \implies$$
$$ag_1 \dots g_n = bah_3 \dots h_{n+1},$$

which contradicts Lemma 13 because we have equal geodesic words beginning with $a \in S^1_\gamma$ and $b \notin S^1_\gamma$. The argument for g' is identical. □

Lemma 17. *Let $g \in F_2 - \{1\}$ such that there exists a geodesic spelling of g of the form $baag_1 \dots g_n$ with $g_1, \dots, g_n \in S_\gamma$. Then there exists no geodesic spelling of g of the form $a^{-1}h_1 \dots h_{n+2}$ with $h_1, \dots, h_{n+2} \in S_\gamma$. Similarly, if $g' \in F_2 - \{1\}$ has a geodesic spelling of the form $b^{-1}a^{-1}a^{-1}g'_1 \dots g'_n$, then there exists no geodesic spelling of g' of the form $ah'_1 \dots h'_{n+2}$.*

Proof. Because $aag_1 \dots g_n = b^{-1}g$ is in A_a, we must have that $baag_1 \dots g_n = g$ is in A_b. Therefore, if a geodesic spelling $a^{-1}h_1 \dots h_{n+2}$ of g exists, then we must have $h_1 = w$ because we saw in the proof of Lemma 13 that all geodesic words in A_b that begin with a^{-1} must begin with the prefix $a^{-1}w$. Because $aba = wa^{-1}b^{-1}$, we have

$$\begin{aligned} & baag_1 \dots g_n = g = a^{-1}wh_2 \dots h_{n+2} \\ \implies & abaag_1 \dots g_n = wh_2 \dots h_{n+2} \\ \implies & wa^{-1}b^{-1}ag_1 \dots g_n = wh_2 \dots h_{n+2} \\ \implies & b^{-1}ag_1 \dots g_n = ah_2 \dots h_{n+2}. \end{aligned}$$

The word $ag_1 \dots g_n$ is geodesic because it is a subword of a geodesic word for g. Because this word begins with a, there exists no other geodesic spelling representing the same group element that begins with b by Lemma 13. Thus, by Corollary 15, $b^{-1}ag_1 \dots g_n$ is a geodesic word, so $ah_2 \dots h_{n+2}$ is also geodesic because it consists of the same number of generators. Note that $b^{-1}ag_1 \dots g_{n-1}$ must either lie in A_a or $A_{b^{-1}}$ because it begins with b^{-1}. If a geodesic word beginning with b^{-1} lies in A_a, then it must begin with the prefix $b^{-1}a^{-1}w$, so $b^{-1}ag_1 \dots g_{n-1}$ instead lies in $A_{b^{-1}}$. Therefore, $ah_2 \dots h_{n+2}$ must also lie in $A_{b^{-1}}$, which implies that $h_2 = w^{-1}$ because all geodesic words in $A_{b^{-1}}$ that begin with a must begin with the prefix aw^{-1}. We then have

$$\begin{aligned} & b^{-1}ag_1 \dots g_n = aw^{-1}h_3 \dots h_{n+2} \\ \implies & wa^{-1}b^{-1}ag_1 \dots g_n = h_3 \dots h_{n+2} \\ \implies & abaag_1 \dots g_n = h_3 \dots h_{n+2} \\ \implies & aag_1 \dots g_n = b^{-1}a^{-1}h_3 \dots h_{n+2}, \end{aligned}$$

which is a contradiction by Lemma 16. The argument for g' is identical. □

The following proposition gives a sufficient condition for a point in F_2 to have zero curvature with respect to S_γ. In our proof of Theorem 1, we show that points satisfying this condition not only exist, but exist with positive density.

Proposition 18. *With respect to S_γ, $\kappa(g) = 0$ for all $g \in F_2$ such that g has a geodesic spelling of the form $g = (aba)^{\pm 1}(h)(aba)^{\pm 1}$ or $g = (aba)^{\pm 1}(h)(aba)^{\mp 1}$, where h is a geodesic subword such that $|h| = |g| - 6$.*

Proof. Suppose first that $g = (aba)(h)(aba)$. Then

$$ga^{-1} = (aba)(h)(aba)a^{-1} = (aba)(h)(ab),$$

which must be a geodesic spelling of $|ga^{-1}|$ because it consists of $|g| - 1$ letters, the minimum length of a spelling of ga^{-1} by the triangle inequality. By Lemma 13, every geodesic spelling of ga^{-1} must begin with an element of S^1_γ so no geodesic spelling begins with a^{-1}. Combining this fact with Corollary 15, we have

$$|aga^{-1}| = |a(aba)(h)(ab)| = |(aba)(h)(ab)| + 1 = (|g| - 1) + 1 = |g|.$$

Likewise, we have

$$\begin{aligned} |a^{-1}ga| &= |a^{-1}(aba)(h)(aba)a| = |(ba)(h)(aba)a| \\ &= |(ba)(h)(aba)| + 1 = (|g| - 1) + 1 = |g|. \end{aligned}$$

To compute the length of conjugations by the other generators, we use the relation $aba = wa^{-1}b^{-1} = b^{-1}a^{-1}w$ to show that

$$\begin{aligned} |bgb^{-1}| &= |b(b^{-1}a^{-1}w)(h)(aba)b^{-1}| = |(a^{-1}w)(h)(aba)b^{-1}| \\ &= |(a^{-1}w)(h)(aba)| + 1 = (|g| - 1) + 1 = |g|, \\ |b^{-1}gb| &= |b^{-1}(aba)(h)(wa^{-1}b^{-1})b| = |b^{-1}(aba)(h)(wa^{-1})| \\ &= |(aba)(h)(wa^{-1})| + 1 = (|g| - 1) + 1 = |g|, \\ |wgw^{-1}| &= |w(aba)(h)(b^{-1}a^{-1}w)w^{-1}| = |w(aba)(h)(b^{-1}a^{-1})| \\ &= |(aba)(h)(b^{-1}a^{-1})| + 1 = (|g| - 1) + 1 = |g|, \\ |w^{-1}gw| &= |w^{-1}(wa^{-1}b^{-1})(h)(aba)w| = |(a^{-1}b^{-1})(h)(aba)w| \\ &= |(a^{-1}b^{-1})(h)(aba)| + 1 = (|g| - 1) + 1 = |g|. \end{aligned}$$

Therefore, $\mathrm{GenCon}(g) = \frac{1}{|S_\gamma|}\sum_{s\in S_\gamma} |s^{-1}gs| = |g|$, so $\kappa(g) = 0$.

To prove the case of $g = (aba)^{-1}(h)(aba)^{-1}$, note that g^{-1} is a group element of the type considered in the previous case. Therefore, because the word length of a group element is equal to the length of its inverse, we have $|s^{-1}gs| = |s^{-1}g^{-1}s| = |g^{-1}| = |g|$ for all $s \in S_\gamma$. Therefore, $\mathrm{GenCon}(g) = \frac{1}{|S_\gamma|}\sum_{s\in S_\gamma} |s^{-1}gs| = |g|$, so $\kappa(g) = 0$.

Now suppose that we have $g = (aba)(h)(aba)^{-1}$. Then, using the relations $aba = b^{-1}a^{-1}w = wa^{-1}b^{-1}$ and $a^{-1}b^{-1}a^{-1} = w^{-1}ab = baw^{-1}$, we have

$$\begin{aligned} |a^{-1}ga| &= |a^{-1}(aba)(h)(a^{-1}b^{-1}a^{-1})a| = |(ba)(h)(a^{-1}b^{-1})| \le |g| - 2, \\ |bgb^{-1}| &= |b(b^{-1}a^{-1}w)(h)(w^{-1}ab)b^{-1}| = |(a^{-1}w)(h)(w^{-1}a)| \le |g| - 2, \\ |w^{-1}gw| &= |w^{-1}(wa^{-1}b^{-1})(h)(baw^{-1})w| = |(a^{-1}b^{-1})(h)(ba)| \le |g| - 2. \end{aligned}$$

By the triangle inequality, $|s^{-1}gs| \ge |g| - 2$ for any generator $s \in S_\gamma$. Therefore, each of the above conjugations has a length of exactly $|g| - 2$.

Because we have a geodesic spelling of g beginning with a, no geodesic spelling of g can begin with a^{-1} by Lemma 13. Combining this fact with Corollary 15, we have that the spelling

$$ag = a(aba)(h)(aba)^{-1}$$

is geodesic. This spelling ends with a^{-1}, so we can once again apply Lemma 13 to show that no geodesic spelling of ag ends with a. Thus, the spelling

$$aga^{-1} = a(aba)(h)(aba)^{-1}a^{-1}$$

is geodesic, so $|aga^{-1}| = |g| + 2$. Likewise, no geodsic spelling of g starts with b or w^{-1} or ends with b^{-1} or w. Thus,

$$|b^{-1}gb| = |b^{-1}(aba)(h)(aba)^{-1}b| = |g| + 2,$$
$$|wgw^{-1}| = |w(aba)(h)(aba)^{-1}w^{-1}| = |g| + 2.$$

Therefore, $\mathrm{GenCon}(g) = \frac{1}{|S_\gamma|}\sum_{s\in S_\gamma}|s^{-1}gs| = \frac{1}{6}(3(|g|+2) + 3(|g|-2)) = |g|$, so $\kappa(g) = 0$. The proof for the case of $g = (aba)^{-1}(h)(aba)$ is identical, with $|s^{-1}gs| = |g| - 2$ for $s = a^{-1}$, b, or w^{-1}, and $|s^{-1}gs| = |g| + 2$ for $s = a$, b^{-1}, or w. □

The following proposition gives a sufficient condition for points in F_2 to have negative curvature with respect to S_γ, which we also show to be true with positive density in our proof of Theorem 1.

Proposition 19. *With respect to S_γ, we have $\kappa(g) < 0$ for all $g \in F_2$ such that a geodesic spelling of g has the form $g = (aa)^{\pm1}(h)(aa)^{\pm1}$ or $g = (aa)^{\pm1}(h)(aa)^{\mp1}$, where h is a geodesic subword such that $|h| = |g| - 4$.*

Proof. Suppose first that $g = (aa)(h)(a^{-1}a^{-1})$. Then no geodesic spelling of g begins with a^{-1}, b, or w^{-1} by Lemma 13, and no geodesic spelling begins with b^{-1} by Lemma 16. If $s \in \{a, b^{-1}, w, b\}$, then the word $s(aa)(h)(a^{-1}a^{-1}) = sg$ is geodesic by Corollary 15. Applying Lemmas 13 and 16 again, we have that no geodesic spelling of sg ends with a, b^{-1}, w, or b. Therefore, if $s \in \{a, b^{-1}, w, b\}$, then the spelling $s(aa)(h)(a^{-1}a^{-1})s^{-1} = sgs^{-1}$ is geodesic by Corollary 15, so $|sgs^{-1}| = |g| + 2$. Because this holds for four out of the six choices for generator $s \in S_\gamma$, and the minimum possible value for $|sgs^{-1}|$ is $|g| - 2$ by the triangle inequality, this implies that $\mathrm{GenCon}(g) = \frac{1}{|S_\gamma|}\sum_{s\in S_\gamma}|s^{-1}gs| > |g|$ so $\kappa(g) < 0$ for all such g. The proof for $g = (a^{-1}a^{-1})(h)(aa)$ is identical, with $|sgs^{-1}| = |g| + 2$ for $s \in \{a^{-1}, b^{-1}, w^{-1}, b\}$.

Now suppose that $g = (aa)(h)(aa)$. Then again by Lemmas 13 and 16, no geodesic spellings of g start or end with b or b^{-1}, so $|b^{-1}gb| = |bgb^{-1}| = |g| + 2$. No geodesic spelling of g begins or ends with a^{-1} or w^{-1} by Lemma 13, so $|s^{-1}gs| \geq |g|$ for $s \in \{a, a^{-1}, w, w^{-1}\}$. Therefore, we again have $\mathrm{GenCon}(g) = \frac{1}{|S_\gamma|}\sum_{s\in S_\gamma}|s^{-1}gs| > |g|$ so $\kappa(g) < 0$.

Finally, the proof for $g = (a^{-1}a^{-1})(h)(a^{-1}a^{-1})$ follows by noting that $|sgs^{-1}| = |sg^{-1}s^{-1}| = |g^{-1}| + 2 = |g| + 2$ for $s = b$ or b^{-1} because g^{-1} is a group element of the type considered in the previous case. Likewise, we have $|sgs^{-1}| = |sg^{-1}s^{-1}| \geq |g^{-1}| = |g|$ for $s \in \{a, a^{-1}, w, w^{-1}\}$. Therefore, $\mathrm{GenCon}(g) = \frac{1}{|S_\gamma|}\sum_{s\in S_\gamma}|s^{-1}gs| > |g|$ so $\kappa(g) < 0$. □

In order to prove Theorem 1, we recall the following useful inequality:

Lemma 20. *Let $a_1, \ldots, a_k, b_1, \ldots, b_k$ be positive integers such that $\frac{a_1}{b_1} \leq \ldots \leq \frac{a_k}{b_k}$. Then $\frac{a_1}{b_1} \leq \frac{\sum_{i=1}^k a_i}{\sum_{j=1}^k b_j}$.*

Proof. This follows from induction on k. For $k = 2$, we have

$$\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \implies a_1b_2 \leq a_2b_1 \implies$$
$$a_1b_2 + a_1b_1 \leq a_2b_1 + a_1b_1 \implies \frac{a_1}{b_1} \leq \frac{a_1 + a_2}{b_1 + b_2}.$$

Suppose that the desired inequality holds for $k \leq l-1$. Then we have $\frac{a_2}{b_2} \leq \frac{\sum_{i=2}^{l} a_i}{\sum_{j=2}^{l} b_j}$. By assumption, $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$. Therefore, we can apply the result for $k = 2$ to conclude that

$$\frac{a_1}{b_1} \leq \frac{\sum_{i=2}^{l} a_i}{\sum_{j=2}^{l} b_j} \implies \frac{a_1}{b_1} \leq \frac{a_1 + \sum_{i=2}^{l} a_i}{b_1 + \sum_{j=2}^{l} b_j} = \frac{\sum_{i=1}^{l} a_i}{\sum_{j=1}^{l} b_j}.$$

□

Proof of Theorem 1. Define the sets

$$P_n \coloneqq \{g \in S_n | \text{ there exists a geodesic spelling } g = g_1 \dots g_n \text{ with } g_1, g_n \in \{a, a^{-1}\}\}.$$

We claim that the ratio $\frac{|P_n|}{|S_n|}$ is bounded below by some positive number for all $n > 0$. For $n > 2$ and $g \in S_{n-2}$, let $g_1 \dots g_{n-2}$ be a geodesic spelling of g. If $g_1 \in S_\gamma^1$, then by Lemma 13 no geodesic spelling of g begins with a^{-1}. Hence, by Corollary 15 the word $ag_1 \dots g_{n-2} \in S_{n-1}$ is geodesic. Similarly, if $g_1 \in S_\gamma^2$, then the word $a^{-1}g_1 \dots g_{n-2} \in S_{n-1}$ is geodesic. Let $h_1 g_1 \dots g_{n-2}$ be a geodesic word where $h_1 \in \{a, a^{-1}\}$. If $g_{n-2} \in S_\gamma^1$, then no geodesic spelling of $h_1 g$ ends with a^{-1} by Lemma 13, so by Corollary 15 the word $h_1 g_1 \dots g_{n-2} a$ is geodesic. Similarly, if $g_{n-2} \in S_\gamma^2$, then $h_1 g_1 \dots g_{n-2} a^{-1}$ is geodesic. Therefore, for all $g \in S_{n-2}$, there exists at least one element of P_n of the form $h_1 g h_2$ where $h_1, h_2 \in \{a, a^{-1}\}$.

Note that every element of S_n for $n > 2$ can be written as $h_1 g h_2$ for some $h_1, h_2 \in S_\gamma$ and $g \in S_{n-2}$. Because there are six generators in S_γ, there are no more than $6^2 = 36$ elements of S_n of the form $h_1 g h_2$ for each $g \in S_{n-2}$. Therefore, for all $n > 2$,

$$\frac{|P_n|}{|S_n|} \geq \frac{1}{36}.$$

P_1 contains the element a, and P_2 contains the element aa, so both are nonempty. Therefore, letting $\eta = \min\left\{\frac{1}{|B_2|}, \frac{1}{36}\right\}$, we have

$$\frac{|P_n|}{|S_n|} \geq \eta$$

for all $n > 0$ as desired.

Let $g \in P_n$ be such that a geodesic spelling of g is $ag_1 \dots g_{n-1}$ for some generators $g_1, \dots, g_{n-1} \in S_\gamma$. Then no geodesic spelling of g begins with a^{-1} by Lemma 13, so the word $a(ag_1 \dots g_{n-1}) = ag$ is geodesic by Corollary 15. By Lemma 16, no geodesic spelling of ag begins with b^{-1}, so the word $b(aag_1 \dots g_{n-1}) = bag$ is geodesic by Corollary 15. Finally, no geodesic spelling of bag begins with a^{-1} by Lemma 17, so the word $a(baag_1 \dots g_{n-1}) = abag$ is geodesic by Corollary 15. Therefore, we have $|(aba)g| = n + 3$.

An identical argument shows that, if $g \in P_n$ is such that a geodesic spelling of g is $a^{-1}g_1 \dots g_{n-1}$ for generators $g_1, \dots, g_{n-1} \in S_\gamma$, then the word $(aba)^{-1}a^{-1}g_1 \dots g_{n-1}$ is geodesic with length $n + 3$. Applying these claims to the inverses of elements of P_n, we get that we either have $|g(aba)| = n + 3$ or $|g(aba)^{-1}| = n + 3$ for all $g \in P_n$. We can thus conclude that, if a geodesic spelling of some $g \in P_n$ is $a^{\pm 1}g_1 \dots g_{n-2}a^{\pm 1}$, then the word $(aba)^{\pm 1}(a^{\pm 1}g_1 \dots g_{n-2}a^{\pm 1})(aba)^{\pm 1}$ is geodesic with length $n+6$, and if a geodesic spelling of $g \in P_n$ is $a^{\pm 1}g_1 \dots g_{n-2}a^{\mp 1}$, then the word $(aba)^{\pm 1}(a^{\pm 1}g_1 \dots g_{n-2}a^{\mp 1})(aba)^{\mp 1}$ is geodesic with length $n+6$. This fact implies a natural bijection between the sets P_{n-6} and

$$Q_n \coloneqq \{g \in S_n \mid g = (aba)^{\pm 1}(h)(aba)^{\pm 1} \text{ or } g = (aba)^{\pm 1}(h)(aba)^{\mp 1} \text{ is a geodesic spelling for some } h \in P_{n-6}\}.$$

for all $n > 6$. Therefore, $|P_{n-6}| = |Q_n|$. Recall that elements of Q_n have zero curvature by Proposition 18.

Every element of the sphere S_n can be written as an element of the sphere S_{n-1} that has been left-multiplied by one of the six generators in S. Therefore, $|S_n| \leq 6|S_{n-1}|$. For $C > 6^6$ and $n > 6$, this implies that $|S_n| < C|S_{n-6}|$. We therefore have for all $n > 6$ that

$$\frac{|Q_n|}{|S_n|} > \frac{|Q_n|}{C|S_{n-6}|} = \frac{|P_{n-6}|}{C|S_{n-6}|} \geq \frac{\eta}{C}. \tag{3.1}$$

Let $\epsilon_1 = \min\left\{\frac{1}{|B_7|}, \frac{\eta}{C}\right\}$ and $N_1 = 7$. Applying Lemma 20, equation 3.1, and the fact that $|Q_7| > 1$ (it contains the points $(aba)(a)(aba)$ and $(aba)^{-1}(a^{-1})(aba)^{-1}$), we have for all $n \geq N_1$ that

$$\begin{aligned}\frac{\sum_{i=7}^{n}|Q_i|}{|B_n|} &> \frac{1+\sum_{i=8}^{n}|Q_i|}{|B_7|+\sum_{j=8}^{n}|S_j|} \\ &\geq \min\left\{\frac{1}{|B_7|}, \frac{|Q_8|}{|S_8|}, \dots, \frac{|Q_n|}{|S_n|}\right\} \\ &\geq \min\left\{\frac{1}{|B_7|}, \frac{\eta}{C}\right\} = \epsilon_1.\end{aligned}$$

Therefore, the set of points in the union of all the Q_n has positive density in F_2. Because $\kappa(g) = 0$ for all such points g, this proves that F_2 has zero curvature with positive density with respect to S_γ.

We now show that points with negative curvature also have positive density. For $n \geq 5$, define the sets

$$R_n \coloneqq \{g \in S_n \mid g \text{ has a geodesic spelling of the form } g_1(h)g_2 \text{ with } g_1, g_2 \in \{a, a^{-1}\} \text{ and } h \in P_{n-2}\}.$$

Note that $a^{\pm 1}(h)a^{\pm 1}$ is geodesic if and only if $h \in P_{n-2}$ is such that a geodesic spelling of h is $a^{\pm 1}h_1 \dots h_{n-4}a^{\pm 1}$, and $a^{\pm 1}(h)a^{\mp 1}$ is geodesic if and only if a geodesic spelling of h is $a^{\pm 1}h_1 \dots h_{n-4}a^{\mp 1}$. By definition, every element of P_{n-2} satisfies exactly one of these conditions, so there is a bijection between R_n and P_{n-2}. Therefore, $|R_n| = |P_{n-2}|$. Recall that elements of R_n have negative curvature for $n \geq 5$ by Proposition 19.

Let $D > 6^2 = 36$ so that $|S_n| < D|S_{n-2}|$ for all $n > 2$. Then we have

$$\frac{|R_n|}{|S_n|} > \frac{|R_n|}{D|S_{n-2}|} = \frac{|P_{n-2}|}{D|S_{n-2}|} \geq \frac{\eta}{D}. \tag{3.2}$$

Let $\epsilon_2 = \min\left\{\frac{1}{|B_5|}, \frac{\eta}{D}\right\}$ and $N_2 = 5$. Then by Lemma 20, equation 3.2, and the fact that $a^5, a^{-5} \in R_5$ so $|R_5| > 1$, we have for all $n \geq N_2$ that

$$\begin{aligned}\frac{\sum_{i=5}^{n}|R_i|}{|B_n|} &> \frac{1+\sum_{i=6}^{n}|R_i|}{|B_5|+\sum_{j=6}^{n}|S_j|}\\ &\geq \min\left\{\frac{1}{|B_5|}, \frac{|R_6|}{|S_6|}, \ldots, \frac{|R_n|}{|S_n|}\right\}\\ &\geq \min\left\{\frac{1}{|B_5|}, \frac{\eta}{D}\right\} = \epsilon_2.\end{aligned}$$

Therefore, the set of points in the union of all the R_n has positive density in F_2. Because $\kappa(g) < 0$ for all such points g, this proves that F_2 has negative curvature with positive density with respect to S_γ. □

Given the result of Theorem 1, it is natural to ask whether there exists a generating set for F_2 that gives positive, not just zero, curvature for infinitely many points. We do not have the answer to this question, but an approach to find such an example seems to be to construct a large generating set such that more than half of the generators can begin and end different geodesic spellings of the same group element.

4. Relationship with δ-Hyperbolicity

The group F_2 from Section 3 is an example of a *non-elementary hyperbolic group*, which is a hyperbolic group that is not virtually cyclic. We have thus proven that non-elementary hyperbolic groups can have a positive density of points with zero curvature. In this section we prove Theorem 2 to conclude that we get negative curvature for all but finitely many points in non-elementary hyperbolic groups if we compute the k-spherical comparison curvature for sufficiently large k. On the other hand, we show that there exist non-hyperbolic groups that have negative curvature for all non-identity points.

4.1. **Negative Curvature at Larger Radii for Hyperbolic Groups.** For the rest of this section, let G be a non-elementary δ-hyperbolic group that has a word metric $d(\cdot,\cdot)$ and Gromov product $(\cdot,\cdot).$ determined by some finite generating set S. We begin by proving a series of lemmas that give us constants to determine the desired value of k in Theorem 2. This theorem is analogous to and motivated by [Oll09, Example 15] for Ollivier's transportation curvature.

Lemma 21. *There exists a constant $C_1 \geq 0$ such that, for all k, $R > 10k$, $g \in G - B_R$, and $s \in S_k$ we have*

$$|(1, gs)_g - (s, gs)_g| \leq C_1.$$

Proof. By [Gro87, Section 6.2], there exists a tree T in the Cayley graph of (G, S) consisting of at most three geodesic segments T_1, T_2, and T_3 such that the extremal points (vertices of degree 1) of T are elements of $\{1, s, g, gs\}$, and which approximates distances between elements of $\{1, s, g, gs\}$ within some error for which we can compute an upper bound. We use $d_T(\cdot,\cdot)$ and $_T(\cdot,\cdot).$ to denote respectively the length metric and Gromov product induced on T by the graph metric of the

Cayley graph of (G, S). Note that, because T is a subset of this Cayley graph, we have $d_T(x, y) \geq d(x, y)$ for all $x, y \in T$. By the triangle inequality,

$$\begin{aligned}(s, gs)_1 + (s, gs)_g =& \frac{1}{2}(d(s, 1) + d(1, gs) - d(s, gs) + d(s, g) + d(g, gs) - d(s, gs)) \\ \leq& \frac{1}{2}(k + 11k - 8k + 11k + k - 8k) = 4k < |g|.\end{aligned}$$

Therefore, we have the following inequality:

$$(s, gs)_1 + (s, gs)_g < |g| + 12\delta = d(1, g) + 4(\log_2(4) + 1)\delta.$$

By Step 2 in [Gro87, Section 6.2], this inequality implies that we can let T_1 be a geodesic segment connecting 1 and g, and let T_2 and T_3 be two of the shortest geodesic segments connecting s and gs respectively to points on T_1 if s and gs do not already lie on T_1 (if they do then T will consist of only one or two geodesic segments). By Property 3 in [Gro87, Section 6.2], we have for all $x, y \in \{1, s, g, gs\}$ that

$$d_T(x, y) \leq d(x, y) + C\delta(\log_2(4))^2 = d(x, y) + 4C\delta$$

for some $C \leq 100$. Because $d_T(x, y) \geq d(x, y)$, we can rewrite this statement as

$$|d_T(x, y) - d(x, y)| \leq 4C\delta. \tag{4.1}$$

Because $d(1, s) = d(g, gs) = k$ and $d(1, g) > 10k$, T_2 and T_3 do not intersect and the intersection of T_3 and T_1 is closer to g than that of T_2 and T_1. Therefore, ${}_T(1, gs)_g = {}_T(s, gs)_g$ (if $gs \in T_1$, then both of these products are equal to the distance from g to gs). The tree T is shown in Figure 4.

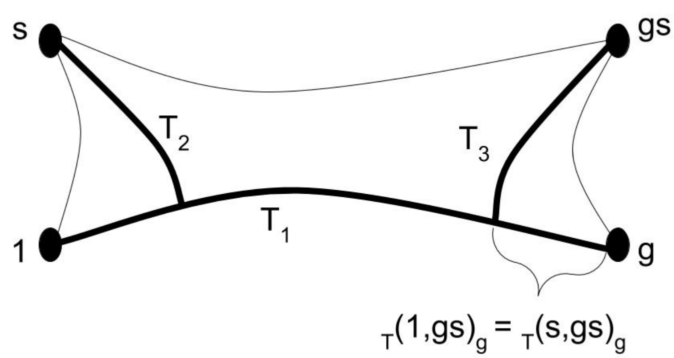

FIGURE 4. A geodesic tree (bold segments) connecting 1, s, g, and gs such that the induced Gromov products ${}_T(1, gs)_g$ and ${}_T(s, gs)_g$ are equal.

By the equation ${}_T(1, gs)_g = {}_T(s, gs)_g$, inequality 4.1, and the triangle inequality,

$$
\begin{aligned}
&|(1, gs)_g - (s, gs)_g| \\
=&|[\ {}_T(1, gs)_g - {}_T(s, gs)_g] + [(1, gs)_g - {}_T(1, gs)_g] + [\ {}_T(s, gs)_g - (s, gs)_g]| \\
\leq&|(1, gs)_g - {}_T(1, gs)_g| + |\ {}_T(s, gs)_g - (s, gs)_g| \\
\leq&\frac{1}{2}(|d(1, g) - d_T(1, g)| + |d(g, gs) - d_T(g, gs)| + |d_T(1, gs) - d(1, gs)| \\
&\quad + |d(s, g) - d_T(s, g)| + |d(g, gs) - d_T(g, gs)| + |d_T(s, gs) - d(s, gs)|) \\
\leq&\frac{1}{2}6(4C\delta) \leq 1200\delta.
\end{aligned}
$$

Therefore, letting $C_1 = 1200\delta$ gives the desired result. □

The following is a well-known fact for non-elementary hyperbolic groups. See, for example, [Cha12, Corollary IV.1.3.] for a proof.

Lemma 22. *There exists a constant $C_2 > 1$ such that, for all $k \geq 1$, we have*

$$|B_k| \geq C_2|B_{k-1}|.$$

□

We have the following lemma that shows that a majority of geodesic curves connecting points in a ball to points far away will pass close to the center of the ball. The proof is analogous to [Oll04, Proposition 21].

Lemma 23. *There exists a constant $C_3 > 0$ such that, for all $g \in G$, $k \geq 1$, and integers $L \leq k$ we have*

$$\frac{|A_k(g, L)|}{|B_k|} \leq C_3 C_2^{-L},$$

where

$$A_k(g, L) := \{s \in B_k | (g, s)_1 \geq L\},$$

and C_2 is the constant from Lemma 22.

Proof. Fix any such g, k, and L, and let $s \in B_k$. Suppose that $(g, s)_1 \geq L$. Then by the triangle inequality, $(g, s)_1 = \frac{1}{2}(|g| + |s| - d(g, s)) \leq \frac{1}{2}(|g| + |s| - (|g| - |s|)) = |s|$, so $L \leq |s|$, and we similarly have $L \leq |g|$. Let $h, h' \in G$ respectively be points on a geodesic segment connecting 1 and g and a geodesic segment connecting 1 and s such that $d(1, h) = d(1, h') = L$. Applying the 2δ-thin condition to a geodesic triangle with vertices 1, g, and s containing h and h', we have that $d(h, s) \leq d(h', s) + d(h, h') \leq (k - L) + 2\delta$ (see Figure 5 on page 19). Therefore, $A_k(g, L) \subset B_{k-L+2\delta}(h)$.

By left-invariance of the word metric, it suffices to show that $\frac{|B_{k-L+2\delta}|}{|B_k|}$ is bounded above by the desired function of L. Indeed, letting C_2 be the constant from Lemma 22, we have

$$
\begin{gathered}
C_2|B_{k-1}| \leq |B_k| \implies C_2^{L-2\delta}|B_{k-L+2\delta}| \leq |B_k| \implies \\
\frac{|B_{k-L+2\delta}|}{|B_k|} \leq C_2^{-L+2\delta} \implies \frac{|B_{k-L+2\delta}|}{|B_k|} \leq C_3 C_2^{-L}.
\end{gathered}
$$

where $C_3 := C_2^{2\delta}$. The fact that C_3 does not depend on g, k, or L implies the desired result. □

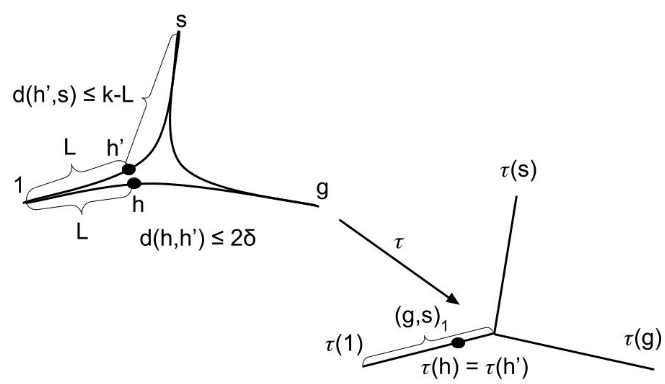

FIGURE 5. Using the 2δ-thin condition and triangle inequality to show that, because $(g,s)_1 \geq L$, we must have $d(h,s) \leq k - L + 2\delta$.

The following lemma uses the previous result to deduce an upper bound on the average value of the Gromov product between all the points on a sphere and any arbitrary point:

Lemma 24. *There exists a constant $C_4 > 0$ such that for all $g \in G$ and $k \geq 1$ we have*

$$\frac{1}{|S_k|}\sum_{s\in S_k}(g,s)_1 \leq C_4.$$

Proof. Define the set

$$D_k(g,L) := \{s \in S_k | \lfloor (g,s)_1 \rfloor = L\}.$$

Then we have

$$\frac{1}{|S_k|}\sum_{s\in S_k}(g,s)_1 = \sum_{L=0}^{k} L\frac{|D_k(g,L)|}{|S_k|}$$

because each $(g,s)_1 \leq |s| = k$. Let C_2 be the constant from Lemma 22. Then

$$\frac{|S_k|}{|B_k|} = \frac{|B_k| - |B_{k-1}|}{|B_k|} = 1 - \frac{|B_{k-1}|}{|B_k|} \geq 1 - \frac{1}{C_2} := K > 0.$$

Therefore, because $D_k(g,L) \subset A_k(g,L)$, we can apply Lemma 23 to conclude that

$$\sum_{L=0}^{k} L\frac{|D_k(g,L)|}{|S_k|} \leq \sum_{L=0}^{k} L\frac{|A_k(g,L)|}{|S_k|} \leq \sum_{L=0}^{k} L\frac{|A_k(g,L)|}{K|B_k|} \leq \frac{C_3}{K}\sum_{L=0}^{k} LC_2^{-L},$$

where C_3 is the constant from Lemma 23. For $L \geq \frac{1}{\ln(C_2)}$, the sequence $(LC_2^{-L})_{L=0}^{\infty}$ is monotonically decreasing. Letting

$$f := \left\lfloor \frac{1}{\ln(C_2)} \right\rfloor,$$
$$c := \left\lceil \frac{1}{\ln(C_2)} \right\rceil,$$
$$I := \int_c^{\infty} xC_2^{-x} dx < \infty,$$

we have by the integral test that

$$\frac{C_3}{K} \sum_{L=0}^{k} LC_2^{-L} < \frac{C_3}{K} \sum_{L=0}^{\infty} LC_2^{-L} \leq \frac{C_3}{K} \left(\sum_{L=0}^{f} LC_2^{-L} + cC_2^{-c} + I \right) := C_4,$$

where C_4 does not depend on g or k. □

The constants C_1 and C_4 can now be applied to prove our main result:

Proof of Theorem 2. Fix $k \geq 1$ and let $g \in G - B_{10k}$. For all $s \in S_k$,

$$\begin{aligned} d(s, gs) =& d(1, g) + d(1, s) + d(g, gs) \\ & - [d(g, 1) + d(1, s) - d(g, s)] - [d(s, g) + d(g, gs) - d(s, gs)] \\ =& |g| + 2k - 2(g, s)_1 - 2(s, gs)_g \\ =& |g| + 2k - 2(g, s)_1 - 2(1, gs)_g + 2[(1, gs)_g - (s, gs)_g]. \end{aligned}$$

The curvature is then given by

$$\begin{aligned} \kappa_k^{\mathcal{S}}(g) =& \frac{|g| - \frac{1}{|S_k|} \sum_{s \in S_k} d(s, gs)}{|g|} \\ =& \frac{|g| - \frac{1}{|S_k|} \sum_{s \in S_k} |g| + 2k - 2(g, s)_1 - 2(1, gs)_g + 2[(1, gs)_g - (s, gs)_g]}{|g|} \\ =& \frac{-2k + \left(\frac{1}{|S_k|} \sum_{s \in S_k} 2(g, s)_1 + 2(1, gs)_g - 2[(1, gs)_g - (s, gs)_g] \right)}{|g|}, \end{aligned}$$

so

$$\begin{aligned} |g|\kappa_k^{\mathcal{S}}(g) + 2k =& 2 \left(\sum_{s \in S_k} \frac{(g, s)_1}{|S_k|} + \sum_{s \in S_k} \frac{(1, gs)_g}{|S_k|} - \frac{1}{|S_k|} \sum_{s \in S_k} [(1, gs)_g - (s, gs)_g] \right) \\ \leq& 2(C_4 + C_4 + C_1) := 2C_5, \end{aligned}$$

where C_1 and C_4 are the constants from Lemmas 21 and 24 respectively (the inequality for the $(1, gs)_g$ term follows from using left-invariance to rewrite it as $(g^{-1}, s)_1$ and applying Lemma 24). We recall that the constants C_1 and C_4 are independent of k. Therefore, for all $k > C_5$ and $g \in G - B_{10k}$ we have

$$\kappa_k^{\mathcal{S}}(g) \leq \frac{2C_5 - 2k}{|g|} < 0.$$

□

4.2. **Negatively Curved Non-Hyperbolic Groups.** With our proof of Theorem 2, we have shown that hyperbolicity implies negative comparison curvature for sufficiently large comparison radius. It is natural to ask whether negative curvature everywhere conversely implies that a group is hyperbolic. We conclude this paper by proving that this is not the case.

Proof of Theorem 3. Gromov proved in [Gro87] that no hyperbolic group contains $\mathbb{Z}^2$ as a subgroup. Therefore, $\mathbb{Z} * \mathbb{Z}^2$ is not a hyperbolic group.

Using the presentation $\mathbb{Z}*\mathbb{Z}^2 = \langle a, b, t | ab = ba\rangle$, let $S_{\text{neg}} := \{a, a^{-1}, b, b^{-1}, t, t^{-1}\}$, and we will show that we always have negative curvature with respect to S_{neg}. Suppose that $g_1 \ldots g_n = h_1 \ldots h_{n+1}$ for some generators $g_1, \ldots, g_n, h_1, \ldots, h_{n+1} \in S_{\text{neg}}$. Then $h_{n+1}^{-1} \ldots h_1^{-1} g_1 \ldots g_n = 1$, so this word with $2n+1$ letters can be reduced to the identity by applying free reductions and commuting the letters a, a^{-1}, b, and b^{-1} with each other. Because free reductions reduce the number of letters by an even number and $2n + 1$ is odd, we have a contradiction. Therefore, given any geodesic word of generators in S_{neg}, there does not exist a word with exactly one more generator that represents the same group element in $\mathbb{Z}*\mathbb{Z}^2$. Thus, by Lemma 9 we do not have $|sg| = |g|$ or $|gs| = |g|$ for any $g \in \mathbb{Z} * \mathbb{Z}^2$ or $s \in S_{\text{neg}}$.

Let $g \in \mathbb{Z} * \mathbb{Z}^2 - \{1\}$. From the definition of $\mathbb{Z} * \mathbb{Z}^2 - \{1\}$, we see that a certain geodesic spelling of g contains the letter $t^{\pm 1}$ in the ith position if and only if every geodesic spelling of g does. Likewise, a geodesic spelling of g contains an element of $\{a, a^{-1}, b, b^{-1}\}$ in the ith position if and only if every geodesic spelling of g contains an element of this set in the ith position, and no geodesic spelling of g can contain s in the ith position for $s \in \{a, a^{-1}, b, b^{-1}\}$ if another geodesic spelling of g contains s^{-1} in the ith position.

Suppose first that every geodesic spelling of g does not include the letters t or t^{-1}. Then no geodesic spelling of tg or $t^{-1}g$ will end with t or t^{-1}, so $|tgt^{-1}| = |t^{-1}gt| = |g| + 2$ by applying Remark 10 twice. On the other hand, all the letters of a geodesic spelling of g commute with the letters in $\{a, a^{-1}, b, b^{-1}\}$, so for all $s \in \{a, a^{-1}, b, b^{-1}\}$ we have $|s^{-1}gs| = |gs^{-1}s| = |g|$. Adding these together, we get that $\text{GenCon}(g) = \frac{1}{6}\sum_{s\in S_{\text{neg}}} |s^{-1}gs| = |g| + \frac{4}{6}$ and $\kappa(g) = -\frac{2}{3|g|} < 0$.

Next suppose that every geodesic spelling of g starts and ends with elements of $\{a, a^{-1}, b, b^{-1}\}$ and also contains at least one element of $\{t, t^{-1}\}$. Then we still have $|tgt^{-1}| = |t^{-1}gt| = |g| + 2$. There exist at most two generators $s \in \{a, a^{-1}, b, b^{-1}\}$ such that $|sg| = |g| - 1$, and at most two such that $|gs| = |g| - 1$. Because every geodesic spelling of g contains a letter t or t^{-1} that does not commute with the letters in $\{a, a^{-1}, b, b^{-1}\}$, we have $|s^{-1}gs| = |gs| - 1$ if and only if $|s^{-1}g| = |g| - 1$. Therefore,

$$\begin{aligned}
\kappa(g) = \frac{|g| - \text{GenCon}(g)}{|g|} &= \frac{|g| - \frac{1}{6}\sum_{s\in S_{\text{neg}}} |s^{-1}gs|}{|g|} \\
&= \frac{\frac{1}{6}\sum_{s\in S_{\text{neg}}}(|g| - |gs|) + \frac{1}{6}\sum_{s\in S_{\text{neg}}}(|gs| - |s^{-1}gs|)}{|g|} \\
&\le \frac{\frac{1}{6}(-2) + \frac{1}{6}(-2)}{|g|} = -\frac{2}{3|g|} < 0.
\end{aligned}$$

If every geodesic spelling of g starts and ends with elements of $\{t, t^{-1}\}$, then for all $s \in \{a, a^{-1}, b, b^{-1}\}$ we have $|s^{-1}gs| = |g| + 2$. Therefore, because the minimum possible value of $|s^{-1}gs|$ for any $s \in S_{\text{neg}}$ is $|g|-2$ by the triangle inequality, we have $\text{GenCon}(g) = \frac{1}{6}\sum_{s\in S_{\text{neg}}} |s^{-1}gs| \geq |g| + \frac{4}{6}$ and $\kappa(g) \leq -\frac{2}{3|g|} < 0$.

Finally, suppose that every geodesic spelling of g either starts or ends with $t^{\pm 1}$ and ends or starts with an element of $\{a, a^{-1}, b, b^{-1}\}$. Without loss of generality, say that every geodesic spelling of g starts with t, and that at least one geodesic spelling of g ends with a. Then we must have

$$|tgt^{-1}| = |g| + 2,$$
$$|a^{-1}ga| = |g| + 2,$$

and $|s^{-1}gs| \geq |g|$ for all $s \in \{t, a^{-1}, b, b^{-1}\}$. Therefore, we have $\text{GenCon}(g) = \frac{1}{6}\sum_{s\in S_{\text{neg}}} |s^{-1}gs| \geq |g| + \frac{4}{6}$ and $\kappa(g) \leq -\frac{2}{3|g|} < 0$.

We can thus conclude that $\kappa(g) \leq -\frac{2}{3|g|} < 0$ for all $g \in \mathbb{Z}_2 * \mathbb{Z}_3 - \{1\}$. Therefore, $\mathbb{Z}_2 * \mathbb{Z}_3$ is a non-hyperbolic group that has negative curvature for all non-identity points. □

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