ONE-DIMENSIONAL DOMAINS OF LARGE TYPE

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Abstract. We study one-dimensional domains, particularly those of the form $K[[t^a + t^b, t^c, t^d]] \subseteq K[[t]]$ (the formal power series ring in one variable over a field K) that contain all high powers of t. Here a, b, c, and d are positive integers. We seek to find families in which the minimal number of generators of the defining prime ideal can be arbitrarily large. We have results and conjectures which parallel the work of T.T. Moh as well as new conjectures for the rings studied by Moh. We will define and discuss the Cohen-Macaulay type of these rings. This work is related to the problem of determining whether there exist three $n \times n$ commuting matrices over a field K such that the algebra they generate has vector space dimension greater than n over K.

Contents

1. Motivation

Let K be a field of characteristic 0 and consider the map of formal power series rings $K[[x, y, z]] \to K[[t]]$ such that $f(x, y, z)$ is sent to $f(t^a + t^b, t^c, t^d)$. The image of this map is $K[[t^a + t^b, t^c, t^d]]$. T.T. Moh has shown in [7] that the kernel of this map is a prime that can require arbitrarily large numbers of generators for certain choices of a, b, c, and d. This implies that the type of these rings can be artbitrarily large. We will discuss this result and conjectures about the rings Moh studied in section 3.

Understanding rings that have this property may help understand an open quesiton about commuting matrices. Namely, let there be d $n \times n$ matrices with entries in K. Is the vector space dimension of the algebra generated by these matrices over K less than or equal to n? When $d = 1$, the answer is "yes". This follows from the Cayley-Hamilton theorem which says that a matrix satisfies its characteristic

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polynomial. The answer is also "yes" when $d = 2$. This was proved by Gerstenhaber in [3], and we refer the reader to [5] for further background on the topic. Counterexamples have been found which show that, in general, the answer to this question is "no" for $d \geq 4$. However, the question is still open for $d = 3$. The study of modules over these rings that have finite vector space dimension over K may lead to a counterexample.

2. Background and Key Definitions

We review key definitions and concepts needed to develop the notion of type.

Definition 2.1. A *domain* is a commutative ring A such that $ab = 0$ implies that $a = 0$ or $b = 0$ for all $a, b \in A$.

Definition 2.2. The *fraction field* of a domain A is defined as

$$\text{frac}(A) = \{(a, b) \in A \times A : b \neq 0\}/ \backsim$$

where $\backsim$ is an equivalence relation s.t. $(a, b) \backsim (a', b')$ if and only if $ab' = a'b$.

We define the multiplication and addition of two elements of $frac(A)$ as follows:

$$[(a, b)] \cdot [(a', b')] = [(aa', bb')]$$
$$[(a, b)] + [(a', b')] = [(ab' + a'b, bb')]$$

When $b \neq 0$, $[(a, b)] \cdot [(b, a)] = [(ab, ab)] = [(1, 1)]$, so $[(b, a)]$ is the inverse of $[(a, b)]$. These defintions agree with the usual operations on fractions in $\mathbb{Q}$.

Definition 2.3. A ring is *Noetherian* if each of its ideals is finitely generated. That is, for every ideal $I \subseteq R$, there exist $f_1, \ldots, f_k \in R$ such that $I = \Sigma_{i=1}^{k} R f_i$.

Definition 2.4. A ring is said be be *quasilocal* if it has a unique maximal ideal.

Some authors refer to quasilocal rings as *local*, but we will reserve the term for Noetherian quasilocal rings.

One can study the power series rings $K[[t^a + t^b, t^c, t^d]]$ or the localized polynomial rings $K[t^a + t^b, t^c, t^d]$ as type is defined for one-dimensional local domains.

2.1. Dimension.

Definition 2.5. The *Krull dimension* of a ring R is the supremum of the lengths of strictly ascending chains of prime ideals in R.

When K is algebraically closed, the dimension of a closed algebraic set X in K^n (including K^n itself) is the Krull dimension of the coordinate ring $K[X]$ of X. The coordinate ring of K^n is the polynomial ring $K[x_1, \ldots, x_n]$, which has Krull dimension n. In this sense, the notion of Krull dimension can be considered as a generalization of the notion of dimension of a vector space. A field K has dimension 0 since the zero ideal is its only prime ideal. The rings we are studying are one-dimensional since their only prime ideals are the zero ideal and the maximal ideal.

Definition 2.6. Let R be a local ring of Krull dimension d. Then there exist $x_1, \ldots, x_d \in m$ such if $I = (x_1, \ldots, x_d)R$ then $\text{Rad}(I) = m$. This sequence of elements is called a *system of parameters*.

2.2. **Type.** We first present the definition of type in terms of the socle of a ring and then in terms of free resolutions. An example follows which shows how to apply both defintions.

Definition 2.7. Let R be a ring and M an R-module. A set of elements $x_1, \ldots, x_n \in R$ is called a *regular sequence* in M if

(1) $(x_1, \ldots, x_n)M \neq M$. In the case where $M = R$, this says that $(x_1, \ldots, x_m)$ is a proper ideal.
(2) x_{i+1} is not a zerodivisor in $M/(x_1, \ldots x_i)M$ for $0 \leq i \leq n-1$.

For example, $x_1, \ldots, x_n$ is a regular sequence in $R = K[x_1, \ldots, x_n]$.

Definition 2.8. Let R be a local (Noetherian) ring. If R has Krull dimension n, and $x_1, \ldots, x_n$ is a system of parameters, then R is *Cohen-Macaulay* if $x_1, \ldots, x_n$ is a regular sequence. This is independent of the choice of system of parameters. If a system of parameters $x_1, \ldots, x_n$ generates m, R is called *regular*.

We refer the reader to [2] for a general reference for facts about Cohen-Macaulay rings.

Let R be a Cohen-Macaulay ring with $x_1, \ldots, x_n$ a system of parameters. Then $\text{socle}\left(\frac{R}{(x_1, \ldots, x_n)}\right) = \text{Ann}_{R/(x_1, \ldots, x_n)} m$ is an R/m-module. That is, it is a vector space over $R/m = K$.

Definition 2.9. The *type* of a ring R is the K-vector space dimension of $\text{socle}\left(\frac{R}{(x_1, \ldots, x_n)}\right)$ where $x_1, \ldots, x_n$ is any system of parameters. The type is independent of the specific choice.

Now we define type in terms of free resolutions. Direct summands of free modules are not always free; instead, they are said to be projective. However, if they are over a local ring and finitely generated, then they are free. The same holds over a polynomial ring. Let M be a finitely generated R-module with generators $m_1, \ldots, m_{b_0}$ where R is local. Consider the map $R^{b_0} \twoheadrightarrow M$ which maps $(r_1, \ldots, r_{b_0})$ to $\Sigma_{i=1}^{b_0} r_i m_i$. Let Z_0 the kernel of this map, which corresponds to elements $(r_1, \ldots, r_{b_0})$ such that $\Sigma_{i=1}^{b_0} r_i m_i = 0$. Then one can define a composite map $R^{b_1} \twoheadrightarrow Z_0 \to R^{b_0}$ and construct an exact sequence:

$$
\begin{array}{ccccccccc}
 & & & & & Z_0 & & & \\
 & & & & \nearrow\!\!\!\!\nearrow & & \searrow & & \\
\cdots \longrightarrow & R^{b_2} & \longrightarrow & & R^{b_1} & \longrightarrow & & R^{b_0} & \longrightarrow M \\
 & & \searrow\!\!\!\!\searrow & & \nearrow & & & & \\
 & & & Z_1 & & & & &
\end{array}
$$

This gives a free resolution since each of the modules is free.

Hilbert proved the following theorem for polynomial rings and finitely generated graded modules [6]. The regular local case is due to Auslander and Buchsbaum [1].

Theorem 2.10 (Hilbert syzygy theorem)**.** *If R is a regular local ring, every finitely generated module over R has a finite free resolution of length at most* $\dim R$.

Let R be a quasilocal ring and M a finitely generated R-module with generators. Then there is a "good" notion of minimal generators.

Lemma 2.11 (Nakayama's lemma)**.** *If $mM = M$ then $M = 0$.*

Proof. Let $u_1, \dots, u_n$ be generators of M of the smallest possible cardinality. One can express $u_n \in mM$ as

$$m_1 u_1 + \dots m_n u_n = r_1 u_1 + \dots r_n u_n$$

for $r_i \in m$. It follows that

$$(1 - r_n) u_n = r_1 u_1 + \dots + r_{n-1} u_{n-1}$$

Note that $1 - r_n$ is a unit in R since R is local. Hence $u_n \in Ru_1 + \dots + Ru_n$. This contradicts the minimality of the generators. □

Corollary 2.12. *Let M be a finitely generated module over R, a quasilocal ring. Then $u_1, \dots, u_n$ generate M if and only if their images generate $\frac{M}{mM}$.*

Proof. It sufficies to show that $\frac{M}{Ru_1 + \dots + Ru_n}$ is zero. Consider

$$\frac{M/Ru_1 + \dots + Ru_n}{m(M/Ru_1 + \dots + Ru_n)} \cong \frac{M/mM}{\text{Images of } (Ru_1 + \dots + Ru_n)}$$

But the second of these is equal to 0 since $u_1, \dots, u_n$ generate M. It follows that $\frac{M}{Ru_1 + \dots + Ru_n} = 0$. □

A set of elements $u_1, \dots, u_n$ are a minimal generating set of M if and only if $\bar{u_1}, \dots \bar{u_n} \in M/mM$ (a R/M-module) are a vector space basis over $R/M = K$. Let $\mu(M)$ denote the cardinality of the smallest generating set. This is equal to $\dim_K M/mM$.

Let R be a Cohen-Macaulay ring of dimension d and S be a regular and local ring of dimension n which is a homomorphic image of R. Then there is a minimal free resolution:

$$0 \to S^{b_{n-d}} \to \dots \to S^{b_1} \to S \to R$$

By a theorem in Section 3.3 of [2], b_{n-d} is the type of R. For example, if R is one dimensional and S is three dimensional, then one obtains the following exact sequence

$$0 \to S^{h-1} \to S^h \to S \to R$$

Example 2.13. Consider the example of the ring $K[[t^3, t^4, t^5]]$ and the K-algebra homomorphism $K[[x, y, z]] \to K[[t^3, t^4, t^5]]$ such that

$$x \mapsto t^3$$
$$y \mapsto t^4$$
$$z \mapsto t^5$$

We can impose a grading on the ring $K[[x, y, z]]$ such that x has weight 3, y has 4, and z has 5, corresponding to the power of t each variable is mapped to. Under this map a polynomial in x, y, z, $f(x, y, z)$ is mapped to a polynomial $f(t^3, t^4, t^5)$. This map is surjective but not injective. The kernel of the map is generated by the 2×2 minors

$$I_2 \begin{bmatrix} x & y & z \\ y & z & x^2 \end{bmatrix}$$

The resulting relations, which generate the kernel, are $xz-y^2$, yx^2-z^2, and x^3-yz. Let B denote

$$\frac{K[[x,y,z]]}{(xz-y^2, yx^2-z^2, x^3-yz)}$$

We have that

$$B \cong K[[t^3,t^4,t^5]]$$

Since B is one-dimensional, its type is defined as the dimension of socle$(\frac{B}{(g)})$ where g is a non-zero element of the maximal ideal $m=(x,y,z)$. In this example, let $g=x$. From the above, it follows that

$$\frac{B}{(x)} \cong \frac{K[[y,z]]}{(y^2,z^2,yz)}$$

The elements remaining in the ring are of the form $K+Ky+Kz$. The generators of the maximal ideal in $\frac{B}{(x)}$ are y and z, and each is killed by y and z. Thus the socle of B mod x has dimension two, so the type of the ring $K[t^3,t^4,t^5]$ is two. In this example, we obtain the following free resolution:

$$0 \to R^2 \to R^3 \to R \to K[[t^3,t^4,t^5]]$$

where $R=K[[x,y,z]]$, the map $R^2 \to R^3$ is given by

$$\begin{bmatrix} x & y \\ y & z \\ z & x^2 \end{bmatrix}$$

and the map $R^3 \to R$ is given by $\begin{bmatrix} F_{12} & -F_{13} & F_{23} \end{bmatrix}$ where $F_{12}=xz-y^2$, $F_{13}=x^3-yz$, and $F_{23}=x^2y-z^2$.

3. Results

In the following sections, let K be a field of characteristic 0. All calculations were done over the rational numbers in Macaulay2 [4].

3.1. **Moh Rings.**

T.T. Moh studied a class of rings of the form $K[[t^a+t^b,t^c,t^d]]$ where

$$\begin{aligned} a &= nm \\ b &= nm+\lambda \\ c &= (n+1)m \\ d &= (n+2)m \end{aligned}$$

for n odd, $m=(n+1)/2$, $\lambda > n(n+1)m$, and λ and m are coprime. He showed that the kernel P_n of the map $K[[x,y,z]] \to K[[t]]$ such that

$$\begin{aligned} x &\mapsto t^{nm}+t^{nm+\lambda} \\ y &\mapsto t^{(n+1)m} \\ z &\mapsto t^{(n+2)m} \end{aligned}$$

has at least n generators. This implies that the type is at least $n-1$. Let h be a positive integer such that $n=2h+1$ and $m=h+1$. Computation in Macaulay2 for $1 \le h \le 20$ supports the following conjectures:

Conjecture 3.1. *Let* $B = \frac{K[[x,y,z]]}{P_n} \cong K[[t^{nm} + t^{nm+\lambda}, t^{(n+1)m}, t^{(n+2)m}]]$ *be a Moh ring with* $\lambda = n(n+1)m + 1$*. Then the type of* B *is* n *and socle of* $\frac{B}{(z)}$ *is generated by*

$$y^n, xy^{n-1}, \ldots, x^{n-2}y^2, x^{n-1}y$$

One notes that socle in B/z is generated by all monomials in x and y of degree n. Further computation for $1 \leq h \leq 20$ yields the following conjecture:

Conjecture 3.2.

$$\frac{B}{(z)} \cong \frac{K[x,y]}{(y^{n+1}, xy^n, \ldots x^n y, x^{n+1} + (-1)^{m-1}\binom{n}{m-1}y^n)}$$

3.2. "Moh-like" Rings.

We introduce a new family of "Moh-like" rings of the form $K[[t^a + t^b, t^c, t^d]]$ where

$$\begin{aligned} a &= 2k^2 + 1 \\ b &= 2k^2 + 2k \\ c &= 2k^2 \\ d &= 2k^2 - k \end{aligned}$$

Since b, c, and d are all multiples of k while $a \equiv 1 \mod k$, these rings contain all t^n for n greater than some N. Let P_k denote the kernel of the map $K[[x,y,z]] \to K[[t]]$ such that

$$\begin{aligned} x &\mapsto t^{2k^2+1} + t^{2k^2+2k} \\ y &\mapsto t^{2k^2} \\ z &\mapsto t^{2k^2-k} \end{aligned}$$

and let $B = \frac{K[[x,y,z]]}{P_k} \cong K[[t^{2k^2} + t^{2k^2+2k}, t^{2k^2}, t^{2k^2-k}]]$. We present the following conjecture based on computation for $1 \leq k \leq 26$.

Conjecture 3.3. *When* $k \not\equiv 2 \mod 3$ *the socle of* $\frac{B}{(z)}$ *is generated by*

$$y^{2k-2}, xy^{2k-3}, \ldots, x^{2k-3}y, x^{2k-2}$$

and the type of the ring is $2k-1$*. When* $k \equiv 2 \mod 3$ *the socle of* $\frac{B}{(z)}$ *is generated by*

$$xy^{2k-2}, x^4y^{2k-5}, \ldots, x^{2k-3}y^2$$

and the type is $2h+1$ *when* $k = 3h+2$*.*

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