
THE HOFER-ZEHNDER CAPACITY OF SYMPLECTIC TORIC MANIFOLDS

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ABSTRACT

The famous Gromov non-squeezing theorem says that we cannot symplectically embed a $2n$ -dimensional open ball into a $2n$ -dimensional cylinder with smaller radius. The notion of symplectic capacity plays an important role in understanding the proof. A special kind of symplectic capacity, the Hofer-Zehnder capacity seems closely related to the moment polytope of symplectic toric manifolds, which is a Delzant polytope by Delzant's theorem. In this paper, we will give an estimate about the Hofer-Zehnder capacity for 4-dimensional symplectic toric manifolds by just looking at their Delzant polytopes.

Keywords Hofer-Zehnder capacity · Symplectic toric manifolds · Delzant polytope

1 Introduction

Given a Delzant polytope, we can construct the unique (up to equivariance) symplectic toric manifold corresponding to this polytope [3]. By studying the admissible functions, we can always get the Hofer-Zehnder capacity of this symplectic manifold. The question for this project is whether we can get any information about this capacity directly from the Delzant polytope without explicitly constructing the corresponding manifold. In section 2, we review some basic definitions and theorems in symplectic geometry. In section 3, we will present a few propositions and discuss how they relate to the estimate of the Hofer-Zehnder capacity of 4-dimensional symplectic toric manifolds. In section 4, we will present some examples of our result and provide a further conjecture in higher dimension.

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2 Symplectic Geometry

In the first three subsections, we closely follow Silva's notes on symplectic toric manifolds [1].

2.1 Preliminaries

Definition 2.1.1. A *symplectic form* ω on a manifold M is a closed, non-degenerate, 2-form. A *symplectic manifold* is a pair (M, ω) where M is a manifold and ω is a symplectic form on this manifold.

By symplectic linear algebra, a symplectic manifold is necessarily of even dimension.

Example 2.1.2. The following are examples of symplectic manifolds:

1. Let $M = \mathbb{R}^{2n}$ and we write the coordinates as $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$. Then $\omega_{std} = \sum_{i=1}^n dx_i \wedge dy_i$ is the standard symplectic form on \mathbb{R}^{2n} and $(\mathbb{R}^{2n}, \omega_{std})$ is a symplectic manifold.
2. Equivalently, $M = \mathbb{C}^n$ and $\omega_{std} = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ is a symplectic manifold.
3. Let $M = S^2$ and the coordinate charts are given by (θ, h) , where $0 \leq \theta < 2\pi$ and $-1 < h < 1$. Let $\omega_{std} = d\theta \wedge dh$, then (S^2, ω_{std}) is a symplectic manifold. In fact, this form coincides with the area form.

Proposition 2.1.3 (properties of symplectic manifolds). Suppose (M, ω) is a $2n$ -dimensional symplectic manifold, then we have:

- M is orientable.
- If M is compact without boundary, then for any $1 \leq k \leq n$, the de Rham cohomology group $H_{dR}^{2k}(M) \neq 0$.

Proof.

- Let's consider the $2n$ -form $\alpha = \wedge^n \omega$. Since ω is a symplectic form, we know α is nowhere vanishing. So we have a volume form on M . M is orientable.
- For any k , let $\omega_k = \wedge^k \omega$. Since ω is closed, ω_k is closed. Suppose by contradiction, $H_{dR}^{2k}(M) = 0$, then ω_k is exact, meaning that $\omega_k = d\beta$ for some $\beta \in \Omega^{2k-1}(M)$. This implies

$$\alpha = d\beta \wedge \omega_{n-k} = d(\beta \wedge \omega_{n-k})$$

By Stoke's Theorem, we have $\int_M \alpha = \int_{\partial M} \beta \wedge \omega_{n-k} = 0$. A contradiction!

Corollary 2.1.4. S^n has a symplectic structure iff $n = 2$.

The following theorem implies that in symplectic geometry, the only local difference is the dimension. To introduce the theorem, we first introduce the symplectic analogue of diffeomorphism between manifolds.

Definition 2.1.5. Let (M_1, ω_1) and (M_2, ω_2) be two $2n$ -dimensional symplectic manifolds and $\phi : M_1 \rightarrow M_2$ be a diffeomorphism. ϕ is a *symplectomorphism* if $\phi^* \omega_2 = \omega_1$

Theorem 2.1.6. (Darboux) Let (M, ω) be a $2n$ -dimensional symplectic manifold. For any $p \in M$, there exists a chart $(\mathcal{U}, \phi = (x_1, \dots, x_n, y_1, \dots, y_n))$ centered at p , such that $\omega|_{\mathcal{U}} = \sum_{i=1}^n dx_i \wedge dy_i$. Such a chart is called a **Darboux chart**. See section 8 in [2] for a proof of the theorem.

2.2 Hamiltonian Actions

We start by introducing some related notions. In the following context, let (M, ω) be a $2n$ -dimensional symplectic manifold.

Definition 2.2.1. A vector field X on M is **symplectic**, if the contraction $\iota_X \omega$ is closed. X is **Hamiltonian**, if $\iota_X \omega$ is exact.

A 1-form being exact means it is the differential of some function. So we have the following definition corresponding to the Hamiltonian vector field.

Definition 2.2.2. A smooth function H on M is **Hamiltonian**, if there is a Hamiltonian vector field X such that $\iota_X \omega = dH$.

By non-degeneracy of ω , every smooth function is Hamiltonian and Hamiltonian functions are unique up to locally constant functions.

Example 2.2.3.

1. Consider $(S^2, \omega_{std} = d\theta \wedge dh)$. Let $X = \frac{\partial}{\partial \theta}$, then $\iota_X \omega = dh$. So X is a Hamiltonian vector field and the function $h : S^2 \rightarrow \mathbb{R}$ is a Hamiltonian function.

2. Let $M = \mathbb{C}\mathbb{P}^2$ and ω_{FS} be the Fubini-Study form induced by Fubini-Study metric, i.e. $\omega_{FS} = \frac{i}{2} \partial \bar{\partial} \log |Z|^2$, where $|Z|^2 = |z_0|^2 + |z_1|^2 + |z_2|^2$ and $[z_0 : z_1 : z_2]$ is the homogeneous coordinate on M . Then $H : M \rightarrow \mathbb{R}$ by

$$H([z_0 : z_1 : z_2]) = -\frac{1}{2} \frac{|z_1|^2}{|Z|^2}$$

is a Hamiltonian function with the Hamiltonian vector field given by $X_{[z_0 : z_1 : z_2]} = \frac{d}{dt} [z_0 : e^{it} z_1 : z_2] |_{t=0}$. See more about this example in **Example 2.2.6**.

3. **Non-example.** Let $M = T^2$, the 2-dimensional torus and $\omega = d\theta_1 \wedge d\theta_2$ be the symplectic form. Then $X = \frac{\partial}{\partial \theta_1}$ is a symplectic but not Hamiltonian vector field on T^2 , since θ_2 is not a smooth function on T^2 . Similarly $X = \frac{\partial}{\partial \theta_2}$ is symplectic but not Hamiltonian.

Now, we move on to the notion of Hamiltonian actions. In the following discussion, we will assume **the group action is a smooth left action by a Lie group**. We will denote the homomorphism between the Lie group G and $\text{Diff}(M)$ by ψ and denote the exponential mapping from Lie algebra \mathfrak{g} to G by \exp .

Definition 2.2.4. The action ψ is a **symplectic action** if it maps G into $\text{Sympl}(M, \omega) \subset \text{Diff}(M)$, where $\text{Sympl}(M, \omega)$ is the group of symplectomorphism on M .

Definition 2.2.5. The action $\psi : G \rightarrow \text{Sympl}(M, \omega)$ is a **Hamiltonian action** if there exists a map

$$\mu : M \rightarrow \mathfrak{g}^*$$

satisfying the following two conditions:

- For each $X \in \mathfrak{g}$, let $\mu^X : M \rightarrow \mathbb{R}$, $\mu^X(p) = \langle \mu(p), X \rangle$, be the component of μ along X , and let $X^\#$ be the vector field on M generated by the one-parameter subgroup $\{\exp tX | t \in \mathbb{R}\} \subset G$. Then $d\mu^X = \iota_{X^\#} \omega$, i.e., the function μ^X is a Hamiltonian function for the vector field X .
- The map μ is equivariant with respect to the given action ψ of G on M and the coadjoint action Ad^* of G on \mathfrak{g} : $\mu \circ \psi_g = \text{Ad}_g^* \circ \mu$, for all $g \in G$.

Then (M, ω, G, μ) is called a **Hamiltonian G -space** and μ is called a **moment map**.

Several remarks about this definition:

1. The flow of $X^\#$ is just $\Psi(p, t) = \exp(tX) \cdot p$
2. If $G = T^n$ is an n -dimensional torus, then the equivariant condition becomes: $\mu \circ \psi_g = \mu$, for all $g \in T^n$.
And the moment map is unique (up to constant) for a given Hamiltonian torus action.
3. If $H \subset G$ is a closed subgroup, and $i : H \rightarrow G$ is the inclusion, then we can restrict the G -action to H and get a Hamiltonian H -space, where the moment map is given by $i^* \circ \mu : M \rightarrow \mathfrak{h}^*$

Example 2.2.6.

1. Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ act on $(S^2, d\theta \wedge dh)$ by rotation: for $t \in \mathbb{R}$ and $(\theta, h) \in S^2$, we have $t \cdot (\theta, h) = (\theta + t, h)$. It's clearly a symplectic action. To see it's Hamiltonian, we consider the function $h : S^2 \rightarrow \mathbb{R}$, where we identify the dual space with itself. If we take $X = 1$, then $X^\# = \frac{\partial}{\partial \theta}$, so we have $\iota_{X^\#} \omega = dh$.
2. Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ act on $(T^2, d\theta_1 \wedge d\theta_2)$ by rotation: for $t \in \mathbb{R}$ and $(\theta_1, \theta_2) \in T^2$, we have $t \cdot (\theta_1, \theta_2) = (\theta_1 + t, \theta_2)$. This action is symplectic but not Hamiltonian, since $d\theta_1, d\theta_2$ are not exact.
3. Let $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ act on $(\mathbb{C}\mathbb{P}^2, \omega_{FS})$ in the following way: $(e^{it_1}, e^{it_2}) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$. Let $\mu : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{R}^2$ be given by:

$$\mu([z_0 : z_1 : z_2]) = -\frac{1}{2} \left(\frac{|z_1|^2}{|Z|^2}, \frac{|z_2|^2}{|Z|^2} \right)$$

where $|Z|^2 = |z_0|^2 + |z_1|^2 + |z_2|^2$. This is a Hamiltonian action.

From now on, we will assume G to be a torus.

Theorem 2.2.7. (Convexity Theorem) *Let (M, ω) be a compact connected symplectic manifold, and \mathbb{T}^m be an m -torus. Suppose $\psi : \mathbb{T}^m \rightarrow \text{Symp}(M, \omega)$ is a Hamiltonian action with moment map $\mu : M \rightarrow \mathbb{R}^m$. Then:*

- the levels of μ are connected;
- the image of μ is convex;
- the image of μ is the convex hull of the images of the fixed points of the action.

The image $\mu(M)$ is called the **moment polytope**. A proof of the theorem can be found in [6].

At the end, we need to introduce an important notion, symplectic toric manifolds, which is a special kind of symplectic manifolds with certain torus action.

Definition 2.2.8. *A **symplectic toric manifold** is a compact connected symplectic manifold (M, ω) equipped with an effective Hamiltonian torus \mathbb{T} -action, such that $\dim \mathbb{T} = \frac{1}{2} \dim M$ and with a choice of the moment map μ .*

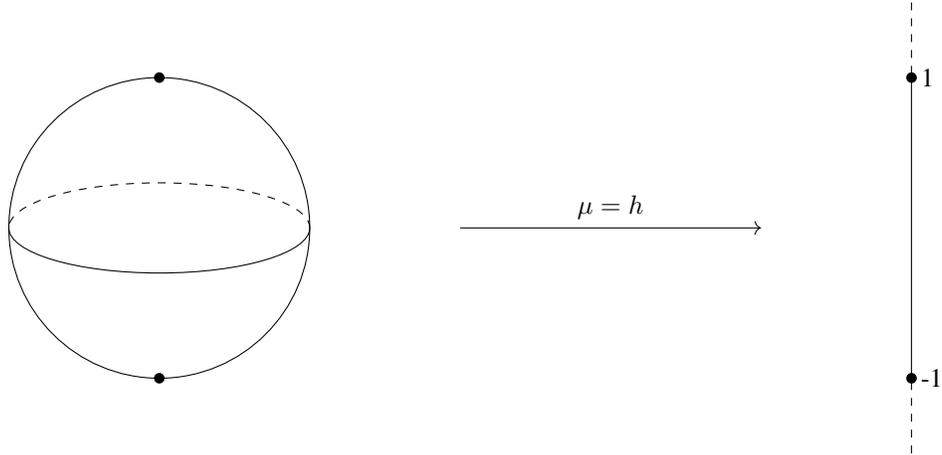
We say a group action by G on M is **effective** if the homomorphism $\psi : G \rightarrow \text{Diff}(M)$ is injective, i.e. every $g \neq e \in G$ will shift at least one point on M .

Example 2.2.9. Examples of symplectic toric manifolds:

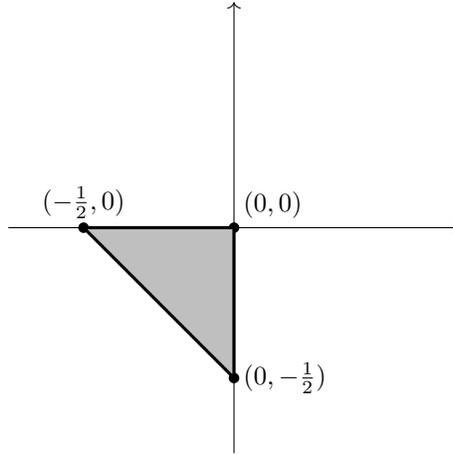
1. Consider the symplectic manifold $(S^2, d\theta \wedge dh)$. Let S^1 act on it by rotation:

$$e^{it}(\theta, h) = (\theta + t, h)$$

The moment map is $h : S^2 \rightarrow \mathbb{R}$ and the moment polytope is $[-1, 1]$.



2. Consider the example of $(\mathbb{C}\mathbb{P}^2, \omega_{FS})$ with the \mathbb{T}^2 -action described in **Example 2.2.6.2**. The moment polytope is an isosceles right-angled triangle.



Three vertices of this triangle correspond to the fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$ of this action. The preimage of a point on the edge is a copy of S^1 and the preimage of an interior point is a copy of \mathbb{T}^2 . Correspondingly, the stabilizer group of the preimage of a point on the edge is S^1 , and that of an interior point is the trivial group. Actually, this relationship is not a coincidence. See more in section 3.

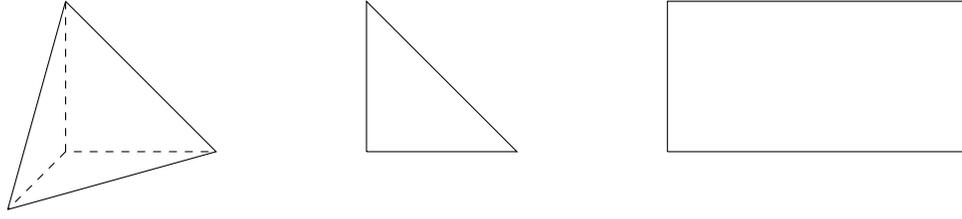
2.3 Delzant's Theorem: Classification of Symplectic Toric Manifolds

In this section, we introduce Delzant's Theorem which gives the one-to-one correspondence between symplectic toric manifolds and Delzant polytopes. We start by introducing the definition of Delzant polytope.

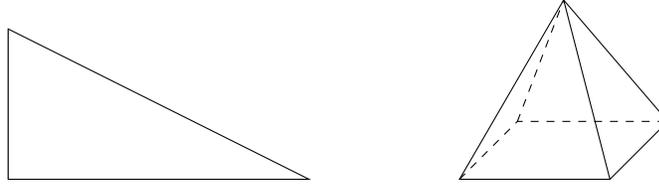
Definition 2.3.1. A *Delzant polytope* $\Delta \subset \mathbb{R}^n$ is a polytope satisfying:

- **simplicity:** there are n edges meeting at each vertex;
- **rationality:** the edges meeting at the vertex p are rational in the sense that each edge is of the form $p + tu_i$, $t \geq 0$, with $u_i \in \mathbb{Z}$;
- **smoothness:** for each vertex, the corresponding u_1, \dots, u_n can be chosen to be a \mathbb{Z} -basis of \mathbb{Z}^n .

Example 2.3.2. Examples of Delzant polytope:



Example 2.3.3. Non-examples of Delzant polytope:



The left one is not smooth, because the triangle is not isosceles. The right one is not simple because of the top vertex.

Delzant's Theorem gives classification of symplectic toric manifolds up to equivalence. To understand the theorem, we need first to understand what equivalence means in the category of symplectic toric manifolds.

Definition 2.3.4. Two symplectic toric manifolds $(M_1, \omega_1, \mathbb{T}_1, \mu_1)$, $(M_2, \omega_2, \mathbb{T}_2, \mu_2)$ are **equivalent** if there exists an isomorphism $\lambda : \mathbb{T}_1 \rightarrow \mathbb{T}_2$ and a λ -equivariant symplectomorphism $\phi : M_1 \rightarrow M_2$ such that $\mu_1 = \mu_2 \circ \phi$.

Theorem 2.3.5. (Delzant [3]) Symplectic toric manifolds are classified by Delzant polytopes. More specifically, the bijective correspondence between these two sets is given by the moment map:

$$\{\text{symplectic toric manifolds}\} \xleftrightarrow{1-1} \{\text{Delzant polytopes}\}$$

$$(M^{2n}, \omega, \mathbb{T}^n, \mu) \longmapsto \mu(M)$$

Proof. We'll give a sketch for the surjectivity. Readers may find a full proof of surjectivity in [1].

Suppose $\Delta \subset \mathbb{R}^n$ is a Delzant polytope with d facets, i.e.

$$\Delta = \bigcap_{i=1}^d \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \leq \lambda_i\}$$

where v_i are primitive outward-pointing normal vectors.

Step 1. Let $e_1 = (1, 0, \dots, 0), \dots, e_d = (0, \dots, 0, 1)$ be the standard basis of \mathbb{R}^d . Consider

$$\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^n$$

$$e_i \longmapsto v_i$$

Then π is onto and maps \mathbb{Z}^d onto \mathbb{Z}^n . Hence, it induces a map, still called $\pi : \mathbb{T}^d \rightarrow \mathbb{T}^n$. Let the kernel of π be N , with the natural inclusion map $i : N \hookrightarrow \mathbb{T}^d$.

Step 2. Let \mathbb{T}^d act on \mathbb{C}^d in the canonical way, i.e.

$$(e^{it_1}, \dots, e^{it_d}) \cdot (z_1, \dots, z_d) = (e^{it_1} z_1, \dots, e^{it_d} z_d)$$

We choose the moment map to be $\phi : \mathbb{C}^d \rightarrow (\mathbb{R}^d)^*$ to be

$$\phi(z_1, \dots, z_d) = (\lambda_1 - \frac{1}{2}|z_1|^2, \dots, \lambda_d - \frac{1}{2}|z_d|^2)$$

Consider the dual map of $i, i^* : (\mathbb{R}^d)^* \rightarrow \mathfrak{n}^*$. Then $i^* \circ \phi : \mathbb{C}^d \rightarrow \mathfrak{n}^*$. Take $Z = (i^* \circ \phi)^{-1}(0)$

Step 3. Z is compact and N acts freely on Z . So we can take the symplectic reduction and denote it by $(M_\Delta = Z/N, \omega_\Delta)$.

Step 4. Notice that we have the exact sequence:

$$1 \rightarrow N \xrightarrow{i} \mathbb{T}^d \xrightarrow{\pi} \mathbb{T}^n \rightarrow 1$$

And the sequence splits, so we have a map $\sigma : \mathbb{T}^n \rightarrow \mathbb{T}^d$ such that $\sigma(v_i) = e_i$. So we consider the following diagram:

$$\begin{array}{ccccccc} Z & \xleftarrow{j} & \mathbb{C}^d & \xrightarrow{\phi} & (\mathbb{R}^d)^* & \cong \mathfrak{n}^* \oplus (\mathbb{R}^n)^* & \xrightarrow{\sigma^*} & (\mathbb{R}^n)^* \\ & & \downarrow \pi_\Delta & & & & & \\ & & M_\Delta & & & & & \end{array}$$

The composition of horizontal maps is constant along N -orbits, it descends to $\mu_\Delta : M_\Delta \rightarrow (\mathbb{R}^n)^*$ such that $\mu_\Delta \circ \pi_\Delta = \sigma^* \circ \phi \circ j$.

Then $(M_\Delta, \omega_\Delta)$ is a Hamiltonian \mathbb{T}^n -space with a moment map μ_Δ such that $\mu_\Delta(M_\Delta) = \Delta$. \square

2.4 Hofer-Zehnder Capacity

The last thing we need to introduce is the notion of Hofer-Zehnder capacity, which is a symplectic capacity. We start from the notion of symplectic capacity.

Definition 2.4.1. A *symplectic capacity* c assigns to every symplectic manifold (M, ω) a nonnegative number $c(M, \omega)$ (possibly infinite) such that the following axioms hold:

- **monotonicity** If there is a symplectic embedding $(M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$ and $\dim M_1 = \dim M_2$, then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$.
- **conformality** $c(M, \lambda\omega) = \lambda c(M, \omega)$.
- **nontriviality** $c(B^{2n}(1), \omega_0) > 0$ and $c(Z^{2n}(1), \omega_0) < \infty$, where $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ is the standard symplectic form on \mathbb{R}^{2n} , $Z^{2n}(1) = \{(x_1, \dots, x_n, y_1, \dots, y_n) | x_1^2 + y_1^2 < 1\}$

Example 2.4.2. We define the **Gromov width** of a symplectic manifold (M, ω) by $w_G(M, \omega) = \sup\{\pi r^2 | B^{2n}(r) \text{ embeds symplectically in } M\}$. Monotonicity and conformality are clear. The nontriviality follows from the following important theorem.

Theorem 2.4.3. (Nonsqueezing Theorem) If there is a symplectic embedding $(B^{2n}(r), \omega_0) \hookrightarrow (Z^{2n}(R), \omega_0)$, then $r \leq R$.

One way to prove it is to use a special kind of symplectic capacity, the Hofer-Zehnder capacity. This capacity is based on properties of periodic orbits of Hamiltonian flows on (M, ω) and was introduced in [4]. Let $\mathcal{H}(M) = \{H \in C_0^\infty(\text{int}M) | H \geq 0, H|_U = \sup H \text{ for some open set } U\}$. For every function H , let X_H be its Hamiltonian vector field and $\phi_H^t \in \text{Symp}(M, \omega)$ be the time-independent Hamiltonian flow.

Definition 2.4.4. An orbit $x(t) = \phi_H^t(x_0)$ is called **T -periodic** if $x(t+T) = x(t)$ for every $t \in \mathbb{R}$.

Definition 2.4.5. We say $H \in \mathcal{H}(M)$ is **admissible** if the corresponding Hamiltonian flow has no nonconstant T -periodic orbits with period $T \leq 1$.

We denote the set of admissible Hamiltonian functions by $\mathcal{H}_{ad}(M, \omega) = \{H \in \mathcal{H}(M) \mid H \text{ is admissible}\}$. Admissible functions exist, since for every Hamiltonian function $H \in \mathcal{H}(M)$, ϵH is admissible for $\epsilon > 0$ sufficiently small.

Definition 2.4.6. The **Hofer-Zehnder capacity** of (M, ω) is defined by

$$c_{HZ}(M, \omega) = \sup_{H \in \mathcal{H}_{ad}(M, \omega)} \|H\|$$

where $\|H\| = \sup_{x \in M} H(x) - \inf_{x \in M} H(x)$.

Theorem 2.4.7. Hofer-Zehnder capacity is a symplectic capacity.

A proof of this theorem can be found in [6].

For computation purpose, we have an equivalent definition for the Hofer-Zehnder capacity.

Definition 2.4.8. Let $\tilde{\mathcal{H}}(M) := \{H \in C_0^\infty(M) \mid \text{Im} H = [0, 1], H \text{ attains } 0, 1 \text{ on open sets}\}$. Then the **Hofer-Zehnder capacity** can be defined as:

$$c_{HZ}(M, \omega) = \sup_{H \in \tilde{\mathcal{H}}(M)} \tau_H$$

where τ_H is the smallest positive period of the orbits of H .

Proof of the equivalence. For any $H \in \mathcal{H}_{ad}(M, \omega)$, we have

$$\tilde{H} := \frac{H}{\|H\|} \in \tilde{\mathcal{H}}(M)$$

Then the corresponding Hamiltonian vector field satisfies: $X_{\tilde{H}} = \frac{X_H}{\|H\|}$. So the flow for $X_{\tilde{H}}$, denoted by $\phi_{\tilde{H}}^t$ satisfies:

$$\phi_{\tilde{H}}^t = \phi_H^{\frac{t}{\|H\|}}$$

Therefore, for any periodic orbits of X_H with period T , the corresponding periodic orbits of $X_{\tilde{H}}$ will have period $\|H\|T$. Then $\tau_{\tilde{H}} = \inf \{\|H\|T \mid T\} \geq \|H\|$, since H is admissible. Taking supremum, we have

$$\sup_{H \in \tilde{\mathcal{H}}(M)} \tau_H \geq \sup_{H \in \mathcal{H}_{ad}(M, \omega)} \|H\|$$

Conversely, for any $\tilde{H} \in \tilde{\mathcal{H}}(M)$, we have $\tau_{\tilde{H}} > 0$, so $\exists \epsilon > 0$ such that $\tau_{\tilde{H}} - \epsilon > 0$. Let $H_\epsilon := (\tau_{\tilde{H}} - \epsilon)\tilde{H}$. Repeat the above discussion, we know all the periods

$$T = \frac{\tilde{T}}{\tau_{\tilde{H}} - \epsilon} \geq \frac{\tau_{\tilde{H}}}{\tau_{\tilde{H}} - \epsilon} > 1$$

Thus, $H_\epsilon \in \mathcal{H}_{ad}(M, \omega)$ and $\|H_\epsilon\| = \tau_{\tilde{H}} - \epsilon$. Then,

$$\tau_{\tilde{H}} \leq \sup_{H \in \mathcal{H}_{ad}(M, \omega)} \|H\|$$

Therefore,

$$\sup_{H \in \tilde{\mathcal{H}}(M)} \tau_H \leq \sup_{H \in \mathcal{H}_{ad}(M, \omega)} \|H\|$$

Then the equality holds, and we conclude that the two definitions are equivalent. \square

Given a Hamiltonian S^1 -action, its moment map might not be admissible. But if we can get information about all the stabilizer groups, we can find a minimal period of this action. Then by rescaling the moment map, we can get an admissible function and estimate the Hofer-Zehnder capacity. And symplectic toric manifolds provide a lot of such S^1 actions, so that's why we want to study this type of symplectic manifolds.

3 Main Results

In this section, we state the main propositions for this project and give some examples. From now on, if not otherwise specified, we will follow the following notation:

- $\Delta \subset \mathbb{R}^n$ will be a Delzant polytope with d facets, i.e.

$$\Delta = \bigcap_{i=1}^d \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \leq \lambda_i\}$$

where v_i are primitive outward-pointing normal vectors.

- $\Delta_i = \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle = \lambda_i\}$ will be the i^{th} facet of the polytope. Δ_i° will be the relative interior of this facet, meaning $\Delta_i^\circ = \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle = \lambda_i, \langle x, v_j \rangle < \lambda_j, \forall j \neq i.\}$
- $(M_\Delta, \omega_\Delta)$ will be the symplectic toric manifold corresponding to Δ .
- μ will be the moment map that sends M_Δ to Δ .
- $S_{v_i}^1 \subset \mathbb{T}^n$ will be the subgroup whose Lie algebra is generated by $v_i \in \mathbb{R}^n \cong \mathfrak{t}$.
- \mathbb{R}/\mathbb{Z} would be the model for S^1 .

3.1 Stabilizer

Stabilizer group plays an important role in determining the periods of the orbits. For example, if $p \in M$ has the stabilizer group (under S^1 action) $\mathbb{Z}/5\mathbb{Z}$, then the period of its orbit will be $\frac{1}{5}$. After scaling by multiplying $\frac{1}{5}$, the period of the new orbit will be 1. Hence, if we can find the minimal period, which will be the reciprocal of the cardinality of some stabilizer group, and scale by this, then the function will be admissible. To find the scalar that makes the function admissible, we need to consider the stabilizer group.

Proposition 3.1.1. *Given a Delzant polytope $\Delta \subset \mathbb{R}^n$. The subgroup $S_{v_i}^1 \subset \mathbb{T}^n$ acts Hamiltonianly on M_Δ . Then, for any $x \in \Delta_i^\circ$, for any $p \in \mu^{-1}(x)$, the stabilizer group of p , under the \mathbb{T}^n action, is $Stab_p = S_{v_i}^1$.*

Proof. Recall that in Delzant's construction, we have the following:

$$\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^n$$

$$e_i \longmapsto v_i$$

Let the kernel of π be N , with the natural inclusion map $i : N \hookrightarrow \mathbb{T}^d$.

\mathbb{T}^d acts on \mathbb{C}^d in the canonical way, i.e.

$$(e^{it_1}, \dots, e^{it_d}) \cdot (z_1, \dots, z_d) = (e^{it_1} z_1, \dots, e^{it_d} z_d)$$

with the moment map to be $\phi : \mathbb{C}^d \rightarrow (\mathbb{R}^d)^*$

$$\phi(z_1, \dots, z_d) = (\lambda_1 - \frac{1}{2}|z_1|^2, \dots, \lambda_d - \frac{1}{2}|z_d|^2)$$

Consider the dual map of i , $i^* : (\mathbb{R}^d)^* \rightarrow \mathfrak{n}^*$. $Z = (i^* \circ \phi)^{-1}(0)$ and $M_\Delta = Z/N$. We also have the following exact sequence:

$$1 \rightarrow N \xrightarrow{i} \mathbb{T}^d \xrightarrow{\pi} \mathbb{T}^n \rightarrow 1$$

And the sequence splits, so we have a map $\sigma : \mathbb{T}^n \rightarrow \mathbb{T}^d$ such that $\sigma(v_i) = e_i$. So $S_{v_i}^1 \subset \mathbb{T}^n$ acts on M_Δ via \mathbb{C}^d , i.e.

$$e^{itv_i} \cdot [(z_1, \dots, z_d)] = [(z_1, \dots, z_{i-1}, e^{it} z_i, z_{i+1}, \dots, z_d)] \quad (1)$$

Here, the bracket stands for an equivalence class in \mathbb{C}^d or a point in M_Δ . For any $p \in \mu^{-1}(x)$, there is a $z \in \mathbb{C}^d$ such that $\pi_\Delta(z) = p$. Then $\sigma^*(\phi(z)) = \mu(p) = x$. So

$$\lambda_i = \langle x, v_i \rangle = \langle x, \pi(e_i) \rangle = \langle \pi^*(x), e_i \rangle = \langle (\pi^* \circ \sigma^*)(\phi(z)), e_i \rangle = \langle \phi(z), e_i \rangle$$

Meaning that the i^{th} coordinate of z is zero, i.e. $z_i = 0$. Then (1) implies that $S_{v_i}^1$ fixes z . Passing to the quotient, we know $S_{v_i}^1$ fixes p . Therefore $S_{v_i}^1 \subset \text{Stab}_p$.

Now suppose there exists $v \in \mathbb{T}^n$ such that $v.p = p$, which means in the upstairs space Z , we have $v.z = n.z$ for some $n \in N$. However, the stabilizer of z under the \mathbb{T}^d action is exactly $S_{v_i}^1$ (similar argument as above shows that $z_j \neq 0$ for all $j \neq i$) and the splitting implies that $\mathbb{T}^n \cap N = \{1\}$, so the only possible case is $v = n = 1$. Therefore, we have another inclusion.

Hence, we conclude that $\text{Stab}_p = S_{v_i}^1$. □

Corollary 3.1.2. *Given a Delzant polytope $\Delta \subset \mathbb{R}^n$. Suppose*

$$\Delta^k = \bigcap_{i \in I \subset \{1, 2, \dots, d\}, |I|=n-k} \Delta_i$$

is a k -dimensional face of Δ , $(\Delta^k)^\circ$ is the relative interior. Then, for any $x \in (\Delta^k)^\circ$, for any $p \in \mu^{-1}(x)$, the stabilizer group of p , under the \mathbb{T}^n action, is $\text{Stab}_p = \prod_{i \in I} S_{v_i}^1 \cong \mathbb{T}^{n-k}$.

Proof. Notice that the Delzant polytope is simple and smooth. So v_i 's ($i \in I$) generate an $(n-k)$ -dimensional subspace of \mathbb{R}^n and apply proposition 3.1.1. □

The above two results tell us that to consider the stabilizer group under a S^1 action, we need to find the intersection of the given S^1 and the T^{n-k} 's in corollary 3.1.2. In dimension 2, we have a clear description of the stabilizer group. To state this result, we need the help of **Pick's Theorem**.

Theorem 3.1.3. (Pick's Theorem) *Suppose $\Delta \in \mathbb{R}^2$ is a simple polygon such that all the vertices are lattice points. Let A denote the area of Δ , i denote the number of lattice points in the interior of the polygon, b denote the number of lattice points on the boundary. Then we have the following identity:*

$$A = i + \frac{b}{2} - 1$$

Lemma 3.1.4. *Suppose $(a, b), (c, d) \in \mathbb{Z}^2$ are primitive vectors, i.e. $\gcd(a, b) = \gcd(c, d) = 1$, and $(a, b) \neq \pm(c, d)$. Consider the subgroups $S_{(a,b)}^1, S_{(c,d)}^1 \subset \mathbb{T}^2$. We have*

$$S_{(a,b)}^1 \cap S_{(c,d)}^1 \cong \mathbb{Z} / \det \begin{pmatrix} a & -c \\ b & -d \end{pmatrix} \mathbb{Z} = \mathbb{Z} / (bc - ad) \mathbb{Z}$$

Proof. We write points in $S_{(a,b)}^1$ as $(e^{2\pi i a t}, e^{2\pi i b t})$, and points in $S_{(c,d)}^1$ as $(e^{2\pi i c u}, e^{2\pi i d u})$. Then for the points in the intersection, the corresponding (t, u) satisfies:

$$\begin{cases} at - cu = k_1 \\ bt - du = k_2 \end{cases}$$

for some $k_1, k_2 \in \mathbb{Z}$ and $t, u \in [0, 1)$. This linear system with varying k_1, k_2 completely characterizes the intersection. To solve this system, we need to discuss two cases:

- $\det \begin{pmatrix} a & -c \\ b & -d \end{pmatrix} = 0$. Then $(a, b), (c, d)$ are linearly dependent, and primitive condition implies they're equal or $a = -c, b = -d$. These two cases are ruled out by the assumption. So we only need to consider the second case.
- $\det \begin{pmatrix} a & -c \\ b & -d \end{pmatrix} \neq 0$. In this case, for any given $k_1, k_2 \in \mathbb{Z}$, the system has a unique solution. However, not every k_1, k_2 will give a solution in $[0, 1)^2$. So the question now becomes, for which k_1, k_2 , we can get a solution in $[0, 1)^2$. Equivalently, we want to count the number of lattice points in the polytope $A[0, 1)^2$, where $A = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix}$. Notice this polytope satisfies the assumption in **Pick's Theorem**. Since $(a, b), (c, d)$ are primitive, the only lattice points on the edge are the vertices. So by **Pick's Theorem**, we know the number of lattice points in $A[0, 1)^2$ is $\text{area}(A[0, 1)^2) + 1 - \frac{4}{2} = \det A - 1$. Therefore, the number of lattice points in $A[0, 1)^2$ is just $\det A$. This means, the system has $\det A$ many different solutions in $[0, 1)^2$. Therefore, the intersection has cardinality $\det A$. Since the only subgroup of S^1 with cardinality $\det A$ is isomorphic to $\mathbb{Z}/(\det A)\mathbb{Z}$, we conclude that

$$S_{(a,b)}^1 \cap S_{(c,d)}^1 \cong \mathbb{Z}/\det \begin{pmatrix} a & -c \\ b & -d \end{pmatrix} \mathbb{Z} = \mathbb{Z}/(bc - ad)\mathbb{Z}$$

□

Remark: when $(a, b) = \pm(c, d)$, they generate the same group and the intersection is the whole circle, which is a triviality.

With this lemma, we can give the estimate for Delzant polytopes in \mathbb{R}^2 .

3.2 Estimate in \mathbb{R}^2

To state the main results, we need a few notations:

- Let $v = (v_x, v_y) \in \mathbb{Z}^2$ be a primitive vector.
- Define $k_v^i := |\det(v, -v_i)|$
- Define $l_v := \max_i k_v^i$
- Define $\tau_v := \frac{1}{l_v} (\max_{x \in \Delta} \langle x, v \rangle - \min_{x \in \Delta} \langle x, v \rangle)$

Theorem 3.2.1. *Suppose $\Delta \subset \mathbb{R}^2$ is a Delzant polytope. We have the following estimate:*

$$c_{HZ}(M_\Delta, \omega_\Delta) \geq \sup_v \tau_v \tag{2}$$

Proof.

For any given $v \in \mathbb{Z}^2$, the corresponding circle group S_v^1 acts on $(M_\Delta, \omega_\Delta)$ in a Hamiltonian way with the moment map $H = \langle \mu, v \rangle$. The corresponding Hamiltonian vector field, by definition of Hamiltonian action, is generated by

the action of S_v^1 and its flow is just $\phi_H^t(p) = \exp(tv).p$. So the orbits are of the form $p(t) = \exp(tv).p_0$. We want to figure out the stabilizer group under the S_v^1 -action of $p \in M_\Delta$, which determines the period of the orbits.

By lemma 3.1.4, for $p \in \mu^{-1}(\Delta_i^\circ)$, the stabilizer group is $S_v^1 \cap S_{v_i}^1 \cong \mathbb{Z}/(\det(v, -v_i))\mathbb{Z} = \mathbb{Z}/k_v^i\mathbb{Z}$.

For $p \in M_\Delta$ such that $\mu(p)$ is a vertex, the stabilizer group is the whole S_v^1 , since these p are fixed by the whole \mathbb{T}^2 .

For $p \in M_\Delta$ such that $\mu(p)$ is an interior point, the stabilizer group is the trivial group.

Thus, the orbits of points mapped to vertices and interior points will either be constant or have period 1. The orbits of $p \in \mu^{-1}(\Delta_i^\circ)$ will have period $\frac{1}{k_v^i}$.

After scaling, the function $\frac{H}{l_v + \epsilon}$ will be admissible, since all the periodic orbits will have period $\frac{l_v + \epsilon}{k_v^i} > 1$.

Since the Hofer-Zehnder capacity is the sup among all the admissible functions and $\frac{H}{l_v + \epsilon}$ approximate $\frac{H}{l_v}$, we have

$$c_{HZ}(M_\Delta, \omega_\Delta) \geq \tau_v$$

Since $v \in \mathbb{Z}^2$ is arbitrary, we get the estimate:

$$c_{HZ}(M_\Delta, \omega_\Delta) \geq \sup_v \tau_v$$

□

Definition 3.2.2. The *toric width* of Delzant polytope $\Delta \subset \mathbb{R}^2$ is defined to be $w_T(\Delta) := \sup_v \tau_v$.

Since the toric width gives an estimate for the Hofer-Zehnder capacity of the symplectic manifold, it should not vary as the torus action changes, nor as the moment map shifts. As expected, we have the following invariance about th toric width.

Proposition 3.2.3. The toric width of a Delzant polytope is invariant under translation and $SL_2(\mathbb{Z})$ transformation.

Proof. When we shift Δ by a vector $u \in \mathbb{R}^2$, l_v will not change for any $v \in \mathbb{Z}^2$ primitive.

So we only need to consider how $\max_{x \in \Delta} \langle x, v \rangle - \min_{x \in \Delta} \langle x, v \rangle$ changes. Since Δ is compact, there exist x_M, x_m such that $\langle x_M, v \rangle = \max_{x \in \Delta} \langle x, v \rangle$ and $\langle x_m, v \rangle = \min_{x \in \Delta} \langle x, v \rangle$

After shifting, the max will be attained at $x_M + u$ and min will be attained at $x_m + u$. So

$$\max_{x \in (\Delta + u)} \langle x, v \rangle - \min_{x \in (\Delta + u)} \langle x, v \rangle = \langle x_M + u, v \rangle - \langle x_m + u, v \rangle = \langle x_M, v \rangle - \langle x_m, v \rangle$$

is unchanged. Therefore, τ_v is invariant under translation.

Now, let's consider $SL_2(\mathbb{Z})$ transformation. Let $M \in SL_2(\mathbb{Z})$. We want to show $w_T(M\Delta) = w_T(\Delta)$. Notice for any $x \in \Delta$, we have $\langle x, v_i \rangle \leq \lambda_i$. Thus, for any $y \in M\Delta$, we have $\langle M^{-1}y, v_i \rangle \leq \lambda_i$, i.e. $\langle y, (M^{-1})^T v_i \rangle \leq \lambda_i$. This means

$$M\Delta = \bigcap_{i=1}^d \{x \in \mathbb{R}^n \mid \langle x, (M^{-1})^T v_i \rangle \leq \lambda_i\}$$

Then, for any $v \in \mathbb{Z}^2$ primitive, $\det(v, -(M^{-1})^T v_i) = \det((M^{-1})^T) \det(M^T v, -v_i) = \det(M^T v, -v_i)$. If we denote all the quantity for $M\Delta$ with a tilde, then $\tilde{k}_v^i = \det(M^T v, -v_i) = k_{M^T v}^i$ and $\tilde{l}_v = l_{M^T v}$.

As for $\tilde{\tau}_v$, notice that $\max_{x \in M\Delta} \langle x, v \rangle = \max_{y \in \Delta} \langle My, v \rangle = \max_{y \in \Delta} \langle y, M^T v \rangle$, so as the min. Hence, we have $\tilde{\tau}_v = \tau_{M^T v}$. The only thing left is to show M sends primitive vectors to primitive ones. Then the toric width is $SL_2(\mathbb{Z})$ -invariant. We have the following lemma:

Lemma. Suppose $(x, y)^T \in \mathbb{Z}^2$ is primitive and $M \in SL_2(\mathbb{Z})$, then $M(x, y)^T$ is primitive.

Proof of the lemma. (x, y) primitive $\implies px + qy = 1$ for some $p, q \in \mathbb{Z}$. Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

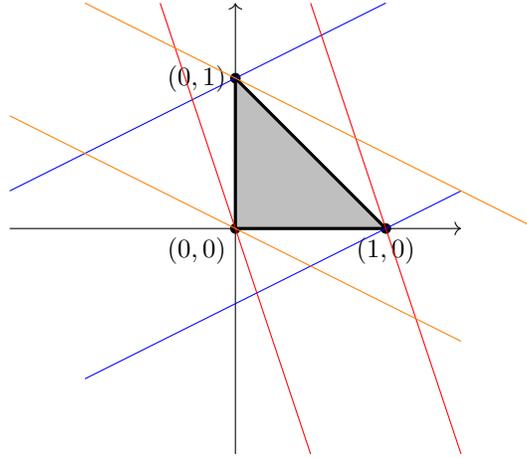
Then $M(x, y)^T = (ax+by, cx+dy)^T$, consider $pd-qc, aq-bp \in \mathbb{Z}$, we have $(pd-qc)(ax+by) + (aq-bp)(cx+dy) = (ad-bc)(px+qy) = 1$, since $\det M = ad-bc = 1$. Therefore, $M(x, y)^T$ is still primitive. And this completes the whole proof. \square

4 Discussion

4.1 Examples.

We want to first present a few example to see how good the estimate is.

Example 4.1.1. Consider the right-angled isosceles triangle in \mathbb{R}^2 , whose corresponding manifold is $(\mathbb{C}\mathbb{P}^2, \omega)$. Here we choose $\omega = -2\omega_{FS}$.



First, we give the notation $v_1 = (0, -1), v_2 = (1, 1), v_3 = (-1, 0)$. Let's consider the case when v is one of these three outward-pointing normal vector. We take $v = v_1$ for example. Then $k_v^1 = 0, k_v^2 = 1, k_v^3 = 1$, so $l_v = 1$. And $\tau_v = \max_{(x,y) \in \Delta} (-y) - \min_{(x,y) \in \Delta} (-y) = 1$. In fact all the three edges give the same number $\tau_{v_i} = 1$.

We now consider $v = (p, -q)$ primitive with $p, q > 0$.

Then $k_v^1 = p, k_v^2 = p+q, k_v^3 = q$, so $l_v = p+q, \tau_v = \frac{1}{p+q} (\max_{(x,y) \in \Delta} (px - qy) - \min_{(x,y) \in \Delta} (px - qy))$.

If we think of the geometric meaning of $px - qy$ and write $c = px - qy$, then c is maximized when the y-intercept of the line $y = \frac{p}{q}x + y_0$ is minimized.

As the blue line in the picture shows, c is maximized at $(1, 0)$ and minimized at $(0, 1)$. So $\tau_v = \frac{1}{p+q}(p - (-q)) = 1$.

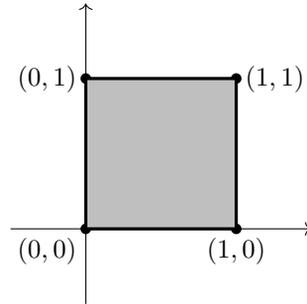
The last case is $v = (p, q)$ primitive with $p, q > 0$.

Then $k_v^1 = p, k_v^2 = |p - q|, k_v^3 = q$, so $l_v = \max(p, q)$.

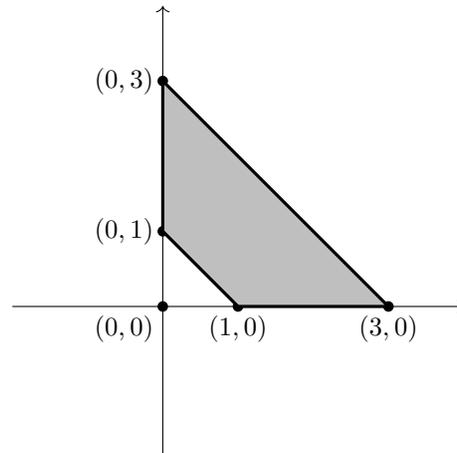
When $p > q$, we want to maximize and minimize $px + qy$ on the restricted region Δ . Similarly, as the red line shows, it's maximized at $(1, 0)$ and minimized at $(0, 0)$. So $\tau_v = \frac{1}{p}(p - 0) = 1$. The same argument works for $p < q$ (the orange line) and we still get $\tau_v = 1$.

With the above computation, we get the estimate $c_{HZ}(\mathbb{C}\mathbb{P}^2, \omega) \geq 1$. And actually, $c_{HZ}(\mathbb{C}\mathbb{P}^2, \omega) = 1$, as shown in [5].

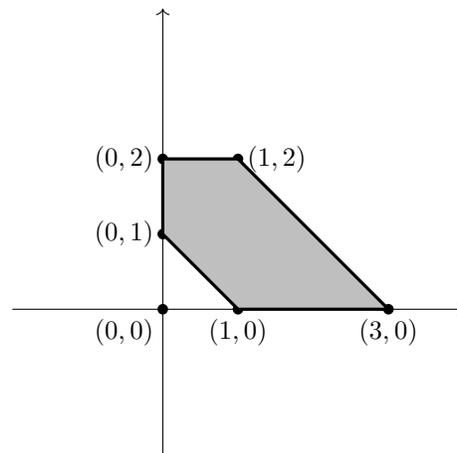
Example 4.1.2. We consider the unit square in \mathbb{R}^2 , whose corresponding manifold is $(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1, \omega)$ with a suitable ω . Computation shows that $c_{HZ}(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1, \omega) \geq 2$, which is the exact number of the Hofer-Zehnder capacity as shown in [5].



Example 4.1.3. We consider the one-point blow up of $\mathbb{C}\mathbb{P}^2$. The corresponding polytope can be the following picture. Apply our estimate, we get $c_{HZ}(\widehat{\mathbb{C}\mathbb{P}^2}, \omega) \geq 3$, which is the exact number of the Hofer-Zehnder capacity as shown in [5].



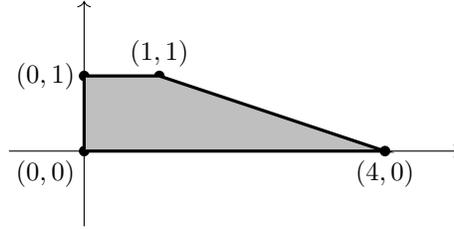
Example 4.1.4. We consider $\mathbb{C}\mathbb{P}^2$ with two points blown up. The corresponding polytope can be the following picture. Apply our estimate, we get $c_{HZ}(\widehat{\mathbb{C}\mathbb{P}^2}, \omega) \geq 3$, which is the exact number of the Hofer-Zehnder capacity as shown in [5].



Remark. All of the above three examples have one thing in common: the underlying manifold is **Fano**. According to [5], if we have a closed Fano symplectic manifold with a semi-free S^1 -action and the Hamiltonian function attains its max at a single point, then the corresponding Hamiltonian H will give the Hofer-Zehnder capacity $c_{HZ} = H_{max} - H_{min}$.

Now, let's turn to a random example.

Example 4.1.5. Consider the following polytope. $v_1 = (0, -1), v_2 = (1, 3), v_3 = (0, 1), v_4 = (-1, 0)$. We have the table for the data.



v	k_v^1	k_v^2	k_v^3	k_v^4	l_v	τ_v
$(0, \pm 1)$	0	1	0	1	1	1
$(1, 3)$	1	0	1	3	3	$\frac{4}{3}$
$(\pm 1, 0)$	1	3	1	0	3	$\frac{4}{3}$
$(p, -q)$	p	$3p+q$	p	q	$3p+q$	$\frac{4p+q}{3p+q}$
$(p, q), q \geq 3p$	p	$q-3p$	p	q	q	$\frac{p+q}{q}$
$(p, q), 3p > q \geq \frac{3}{2}p$	p	$3p-q$	p	q	q	$\frac{4p}{q}$
$(p, q), q < \frac{3}{2}p$	p	$3p-q$	p	q	$3p-q$	$\frac{4p}{3p-q}$

When $p, q > 0$,

$$\frac{4p+q}{3p+q} = 1 + \frac{p}{3p+q} < \frac{4}{3}$$

When $q \geq 3p > 0$,

$$\frac{p+q}{q} = 1 + \frac{p}{q} \leq \frac{4}{3}$$

When $3p > q \geq \frac{3}{2}p > 0$, $(p, q) = (2, 3)$ gives the equality in the following inequality:

$$\frac{4}{3} < \frac{4p}{q} \leq \frac{8}{3}$$

When $q < \frac{3}{2}p$, since $\frac{q}{p} < \frac{3}{2}$

$$\frac{4p}{3p-q} = \frac{4}{3-\frac{q}{p}} < \frac{8}{3}$$

Therefore, we have $c_{HZ}(M_\Delta, \omega_\Delta) \geq \frac{8}{3}$.

4.2 In Higher Dimension

In higher dimension, things become even more complicated. The stabilizer group will have higher dimension. There might be cases like $S_v^1 \cap S_u^1 = \{1\}$ but $S_v^1 \cap (S_u^1 \times S_w^1)$ is a nontrivial subgroup of S_v^1 . In other words, for a given $v \in \mathbb{Z}^n$, we need to consider $S_v^1 \cap (\prod_{i=1}^k S_{n_i}^1)$ for all $k = 1, 2, \dots, n-1$ and all suitable $n_i \in \{1, 2, \dots, d\}$ and determine the maximal cardinality for the nontrivial subgroup. If we can find that, then we can modify the definition for k_p^i in the estimate theorem to get the estimate in higher dimension.

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