

DUPLICITOUS PERMUTATIONS AND BUMPLESS PIPE DREAMS

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1. INTRODUCTION

The goal of this project is to establish a geometric interpretation of a new formula for the Schubert polynomials of Lascoux and Schützenberger. The formula is in terms of combinatorial objects called *bumpless pipe dreams* [LLS18]. We will study a conjecture of Hamaker, Pechenik, and Weigandt [HPW19] about the diagonal Gröbner geometry of matrix Schubert varieties.

The original formula for Schubert polynomials was an algebraic formula in terms of divided difference operators [LS82]. The fact that the coefficients are non-negative integers hinted that there may be combinatorial formulas for the coefficients. Many such formulas have been provided: see for example [Koh91], [FGRS97]. Knutson and Miller provided a geometric explanation for a formula in terms of a combinatorial object called *pipe dreams* [KM05] [BB93] [FK96]. Knutson and Miller showed that the pipe dreams label coordinate subspaces in the Gröbner degeneration of a matrix Schubert variety with respect to an antidiagonal term order.

Hamaker, Pechenik, and Weigandt have conjectured an analogous result for the diagonal Gröbner geometry of matrix Schubert varieties [HPW19]. Their conjecture relates to the bumpless pipe dream formula. Often, the degenerations of matrix Schubert varieties with respect to diagonal term orders are not reduced. My project this summer has centered on trying to understand a class of permutations for which the conjecture predicts that the limit is reduced.

In Section 2, I will give some background on bumpless pipe dreams, the Schubert polynomial formula, and the geometry of matrix Schubert varieties. In Section 3, I will state the conjecture and explain the research question in more detail. In Section 4, I will discuss a phenomenon called pattern containment and evidence that it governs the permutations of interest, as well as discussing pattern containment results for related classes of permutations. In Section 5, I will show how we can find new examples of permutations of interest. Finally, in Section 6, I will discuss our current computational and theoretical results about the permutations of interest.

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2. BACKGROUND ON BUMPLESS PIPE DREAMS, SCHUBERT POLYNOMIALS AND MATRIX SCHUBERT VARIETIES

Lam, Lee and Shimozono defined a combinatorial object called a *bumpless pipe dream*. We follow [LLS18] as a reference.

Definition 1. A *bumpless pipe dream* (BPD) is a tiling of an $n \times n$ grid with the following tiles



which obeys these rules:

- pipes start at the right edge of the grid,
- pipes end at the bottom edge of the grid, and
- pipes cross pairwise at most once.

The set of positions of the blank tiles in a BPD is its *diagram*.

Since each pipe traces out a path from the right edge of the grid to the bottom edge, we can associate to each BPD a permutation. Given a BPD with n pipes, we can associate a permutation w in the symmetric group of n elements S_n in the following way. We label the pipes based on the column in which they originate and then trace that label to the row in which they terminate. We then obtain w in one line notation by reading the labels on the right side from top to bottom. We denote the set of bumpless pipe dreams corresponding to a given permutation w by $\text{BPD}(w)$.

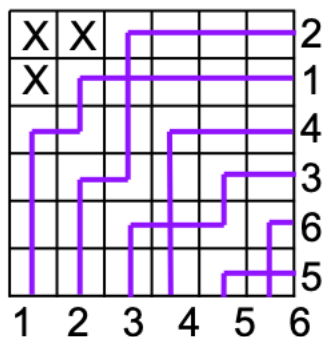


FIGURE 1. The permutation associated to this BPD is 214365, which we can read in one line notation on the right side of the diagram.

The bumpless pipe dream in which each pipe bends exactly once is called the *Rothe bumpless pipe dream* for the given permutation, and its diagram is called the *Rothe diagram* of its permutation.

We can associate a monomial weight to each bumpless pipe dream:

Theorem 1 ([LLS18]). *Given a permutation $w \in S_n$, the formula for the associated Schubert polynomial is given by*

$$(1) \quad \mathfrak{S}_w = \sum_{b \in \text{BPD}(w)} \text{wt}(b),$$

where

$$(2) \quad \text{wt}(b) = \prod_{i=1}^n x_i^{(\# \text{ blank tiles in row } i)}.$$

We take this theorem as our definition.

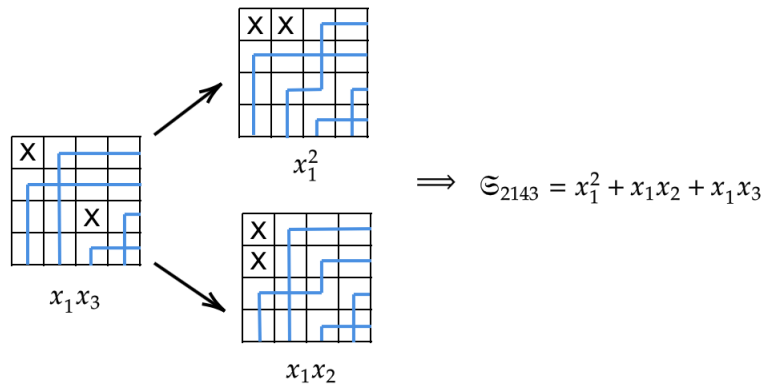


FIGURE 2. There are three BPDs associated to 2143. We assign the monomial weights as shown in (2) and then add them as in (1) to obtain \mathfrak{S}_{2143} .

2.1. The Gröbner geometry of matrix Schubert varieties. In this section, we follow [KM05] as a reference. Take $\text{Mat}(n)$ to be the set of $n \times n$ matrices and let $\mathbf{z} = (z_{ij})_{i,j=1}^n$ be a generic matrix. Let $\mathbb{C}[\mathbf{z}] := \mathbb{C}[z_{11}, z_{12}, \dots, z_{nn}]$.

Definition 2. A *term order* on monomials in $\mathbb{C}[\mathbf{z}]$ is a total order $>$ that satisfies the following properties given any m, m_0, \tilde{m} monomials in $\mathbb{C}[\mathbf{z}]$:

- $1 \leq m$, and
- If $m_0 < \tilde{m}$ then $m \cdot m_0 < m \cdot \tilde{m}$.

A *Gröbner degeneration* takes an ideal I to $\text{init}(I)$.

Definition 3. The *lead term* of a polynomial $f \in \mathbb{C}[\mathbf{z}]$ is the largest monomial of f with a non-zero coefficient with respect to a set term order. We denote the lead term by $\text{lt}(f)$ and define the initial ideal of an ideal I as $\text{init}(I) := \langle \text{lt}(f) : f \in I \rangle$.

Definition 4. Given a matrix minor, its *antidiagonal term* is given by the product of entries along its main antidiagonal. The *diagonal term* of a minor is given by the product of entries along its main diagonal. An *antidiagonal term order* on $\mathbb{C}[\mathbf{z}]$ satisfies that the lead term of any minor is its antidiagonal term, and a *diagonal term order* satisfies that the lead term of any minor is its diagonal term.

Definition 5. Fix an ideal I and a term order. Then a set $\{g_1, \dots, g_k\}$ is a Gröbner basis for I if and only if $I = \langle g_1, \dots, g_k \rangle$ and $\text{init}(I) = \langle \text{lt}(g_1), \dots, \text{lt}(g_k) \rangle$.

Take $T \subset \text{GL}(n) \subset \text{Mat}(n)$ as the torus of diagonal matrices. Then T acts via left multiplication on $\text{Mat}(n)$. We say that $X \subseteq \text{Mat}(n)$ is T -stable if $T \cdot X \subseteq X$. It is a fact that if X is T -stable, then X defines a class $[X]_T$ in $H_T^*(\text{Mat}(n)) = \mathbb{Z}[x_1, \dots, x_n]$, the T -equivariant cohomology of $\text{Mat}(n)$.

We can use the following strategy to compute $[X]_T$. If $X \rightsquigarrow X'$ is a “nice enough” degeneration, then $[X]_T = [X']_T$ [KM05]. If $X = \bigcup K$, where K are of equal dimension and reduced, then

$$(3) \quad [X]_T = \sum [K]_T.$$

Let $[n] := \{1, \dots, n\}$. Consider the following case, where $S \subseteq [n] \times [n]$. Then

$$(4) \quad L_s := V(\langle z_{i,j} : (i,j) \in S \rangle)$$

and

$$(5) \quad [L_s]_T = \prod_{(i,j) \in S} x_i \in \mathbb{Z}[x_1, \dots, x_n].$$

We consider the following example:

Example 1. If $S = (1, 1), (2, 2)$ then $L_s = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\} = V(\langle z_{11}, z_{22} \rangle)$. As such, $[L_{\{(1,1), (2,2)\}}]_T = x_1 x_2$.

Our research question concerns a type of variety called a matrix Schubert variety (see [Ful92]). These varieties are indexed by permutations $w \in S_n$, and the variety X_w is defined by determinantal conditions. We work with the example X_{2143} for simplicity and do not discuss the exact conditions on the determinants here.

It is a fact that

$$(6) \quad X_{2143} = \{[m_{ij}] \in \text{Mat}(n) : \text{rk}([m_{11}]) = 0, \text{rk} \left(\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \right) \leq 2\}.$$

It can also be shown that

$$(7) \quad X_{2143} = V(\langle z_{11}, \det \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} \rangle).$$

Knutson and Miller proved that for any antidiagonal term order, these generators form a Gröbner basis, and that the coordinate subspaces in the Gröbner degeneration of the ideal are naturally labeled by combinatorial objects called pipe dreams [KM05].

Our research question addresses instead a diagonal term order. Looking at the formula for X_{2143} in (6), we see that if we simply take the leading terms of each determinant, we will not get a Gröbner basis because both terms contain z_{11} . It can be shown that we obtain a Gröbner basis by ignoring the terms in the second determinant with z_{11} :

$$(8) \quad \cancel{z_{11}z_{22}z_{33}} - \cancel{z_{11}z_{23}z_{32}} - z_{12}z_{21}z_{33} + z_{12}z_{23}z_{31} + z_{13}z_{21}z_{32} - z_{13}z_{22}z_{31}$$

Then:

$$\begin{aligned} \mathbf{init}(X_{2143}) &= V(\langle z_{11}, z_{12}z_{21}z_{33} \rangle) \\ &= V(\langle z_{11}, z_{12} \rangle \cap \langle z_{11}, z_{21} \rangle \cap \langle z_{11}, z_{33} \rangle) \\ &= L_{\{(1,1),(1,2)\}} \cup L_{\{(1,1),(2,1)\}} \cup L_{\{(1,1),(3,3)\}}. \end{aligned}$$

This implies

$$[X_{2143}]_T = x_1^2 + x_1x_2 + x_1x_3,$$

which is exactly the Schubert polynomial for $w = 2143$. Now if we return to bumpless pipe dreams, we see that the coordinate subspaces in the Gröbner degeneration seem to be labelled by bumpless pipe dreams for w , as we can see in Figure 2.

3. RESEARCH QUESTION

My research question addresses the combinatorics related to the following conjecture:

Conjecture 1 ([HPW19]). *If a permutation $w \in S_n$ has no repeated diagrams, then the BPDs for w label coordinate subspaces in a diagonal term order Gröbner degeneration of the matrix Schubert variety. In particular, $\mathbf{init}(X_w)$ is reduced.*

My work this summer has centered around determining which permutations have bumpless pipe dreams with repeated diagrams from a combinatorial perspective.

In Figure 3, we show two examples of permutations in S_6 with duplicate diagrams in their bumpless pipe dreams.

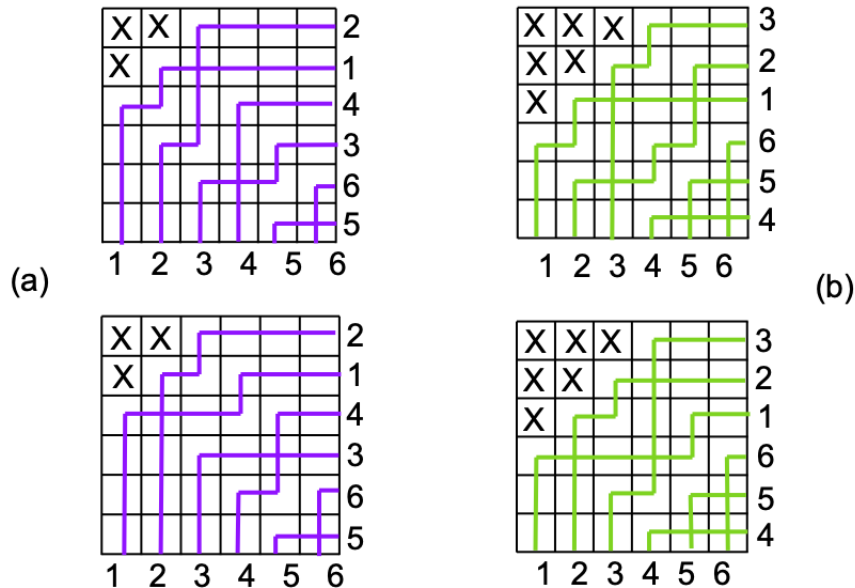


FIGURE 3. (a) Duplicate diagrams for 321654. (b) Duplicate diagrams for 214365.

Using a computer check, we determined that 321654 and 214365 are the only permutations in S_6 with duplicate diagrams and that there are no such permutations in S_n for $n < 6$. However, there are already 85 examples of these type of permutations in S_7 .

We call the set of permutations with repeated diagrams among their bumpless pipe dreams *duplicitous*. Such permutations have two distinct associated bumpless pipe dreams that leave the same set of tiles blank. Write D for the set of duplicitous permutations.

Returning for a moment to the geometric context, we note that if w is duplicitous, we expect at least one coordinate subspace in $\text{init}(X_w)$ to appear with multiplicity. This did not occur when degenerating with respect to antidiagonal term orders in [KM05].

4. EXPLORATIONS IN PATTERN CONTAINMENT

4.1. Evidence for pattern containment. One can study permutations in terms of the notion of pattern containment.

Definition 6. Given $v \in S_m$ and $w \in S_n$, we say that w *contains* v as a pattern if there is some subsequence $(w_{i_1}, \dots, w_{i_m})$, $1 \leq i_1 < i_2 < \dots < i_m \leq n$, in w such that if we let $r, s \in \{1, \dots, m\}$, then $w_{i_r} < w_{i_s}$ if and only if $v_r < v_s$. We say that w *avoids* v if w does not contain v as a pattern.

Example 2. If $v = 214365$ and $w = 432615978$ then w contains v . We have highlighted the subsequence in w for emphasis.

We say that w *avoids* v if w does not contain v as a pattern. Our main goal in this project was to try to characterize the set of duplicitous patterns in terms of pattern containment.

Other related classes of permutations have been understood in terms of pattern containment, which gives hope that D may be governed by the same type of rule.

Our approach is guided by the following questions:

Question: Fix $v \in D$ and consider w containing v . Is w duplicitous?

- If yes: what are the minimal duplicitous patterns?
- Is the list of minimal patterns finite?

Other researchers have answered these questions for several related classes of permutations, and we briefly highlight relevant results here.

4.1.1. *Permutations with Zero-One Schubert Polynomials.* Fink, Meszaros, and St. Dizier considered the set of permutations with associated Schubert polynomials with only zero and one as coefficients, denoted by `ZeroSchub` [FMD19].

The following theorem tells us that this class is governed by pattern containment.

Theorem 2 ([FMD19]). *A permutation w has a zero-one Schubert polynomial if and only if it avoids a set of 12 patterns in S_5 and S_6 .*

Lemma 1. *If a permutation is duplicitous then it does not have a zero-one Schubert polynomial, i.e. $D \subseteq \overline{\text{ZeroSchub}}$.*

Proof. If a permutation has repeated bumpless pipe dream diagrams, then it will have repeated monomial weights in its Schubert polynomial, and thus the Schubert polynomial will have a coefficient greater than 1 in its monomial expansion. \square

We also note that the containment is proper; there are permutations that have bumpless pipe dreams with the same monomial weight but different diagrams. The example in Figure 4 illustrates such a case.

4.1.2. *Multiplicity-Free Permutations.*

Definition 7. [LLS18] A BPD is *Edelman-Greene* if all diagram boxes are upper-left justified.

For example, the BPDs for $w = 321654$ shown in Figure 3(a) are Edelman-Greene because all of the boxes are in the upper left corner.

There is a bijection between Edelman-Greene bumpless pipe dreams and a combinatorial object called Edelman-Greene tableaux [LLS18].

Definition 8. An *Edelman-Greene tableau* for $w \in S_n$ is an increasing tableau such that its reading word (with the order right to left, then top to bottom) is a reduced word for w .

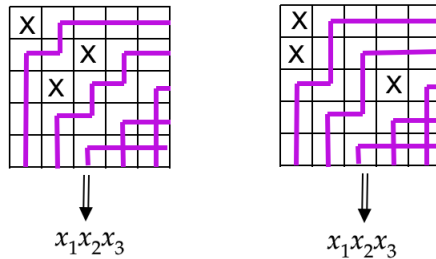


FIGURE 4. The permutation $w = 12543$ in S_5 has repeated monomial weights but no repeated diagram.

The integers in the tableau encode simple transpositions, which generate the symmetric group.

Example 3. We consider as an example the set of Edelman-Greene tableaux for $w = 214365$. We see that in cycle notation, $214365 = (12)(34)(56)$ and further that all of these transpositions commute with one another since they are disjoint. Figure 5 shows all possible increasing tableaux filled in with 1, 3, 5. Note that the repeated tableau shape is exactly the same shape as the repeated diagram that we see for $w = 214365$.

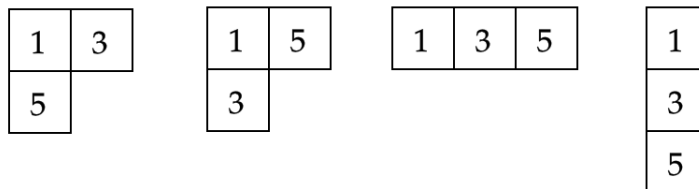


FIGURE 5. Pictured above are increasing tableaux for $w = 214365$.

Another related class of permutations are the *multiplicity free* permutations, defined by Billey and Pawlowski [BP14]. They are defined as follows. A permutation is in the set `MultFree` if it has no repeated Edelman-Greene tableaux shapes. The following results demonstrate that the set `MultFree` is also governed by pattern containment.

Theorem 3 ([BP14]). *If w contains v as a pattern and w is multiplicity free, then v is also multiplicity free.*

Conjecture 2 ([BP14]). *The set of multiplicity free permutations is closed under taking patterns, and the minimal patterns all occur in S_n for $n \leq 11$.*

It is clear that the set of non-multiplicity free permutations sits inside of the set of duplicitous permutations, i.e. $\overline{\text{MultFree}} \subseteq D$, since if a permutation has repeated Edelman-Greene tableaux shapes, then it has repeated Edelman Greene bumpless pipe dream diagrams with the same shape.

One of the original questions that we considered was the following: are there permutations with repeated diagrams where not all of the diagram boxes are in the upper left corner? We confirmed that there are indeed such permutations by generating a list of all duplicitous patterns in S_7 and comparing it with the conjectural list of patterns that cause permutations to be non-multiplicity free [BP14].

There is exactly one permutation in S_7 that is both duplicitous and multiplicity free, $w = 3216745$. Since this permutation avoids all of the patterns in Billey and Pawlowski's list, it must be multiplicity-free. However, it does have repeated diagrams in its bumpless pipe dreams, as shown in Figure 6. The purple diagram box is not in the upper left corner, meaning that these repeated diagrams are not from Edelman-Greene bumpless pipe dreams, as expected.

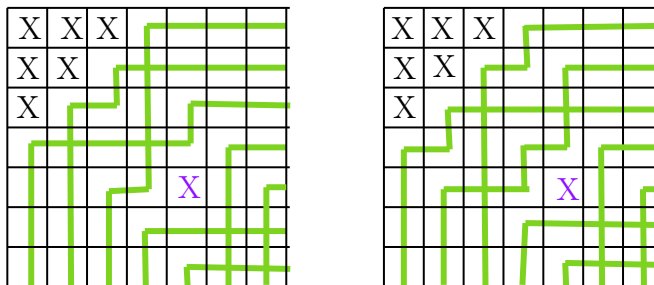


FIGURE 6. Repeated diagrams for $w = 3216745$.

Thus we arrive at the situation shown in Figure 7, where we know that the outer and inner circles are governed by pattern containment. The following theorem also tells us that all of the elements in $D \cap \overline{\text{MultFree}}$ are governed by pattern containment.

Definition 9. Given a permutation $v \in S_n$, the set $\mathcal{EG}(v)$ is the set of Edelman-Greene tableaux corresponding to v .

Theorem 4 ([BP14]). *Let v, w be permutations with w containing v as a pattern. There is an injection $i : \mathcal{EG}(v) \hookrightarrow \mathcal{EG}(w)$ such that if $P \in \mathcal{EG}(v)$, then $\text{shape}(P) \subseteq \text{shape}(i(P))$. Moreover, if P, P' have the same shape, so do $i(P), i(P')$.*

Corollary 1. *Given permutations v, w , if v is not a multiplicity-free permutation and w contains v as a pattern, then w is duplicitous.*

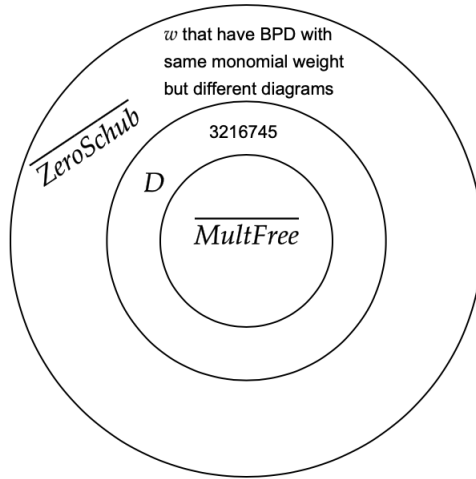


FIGURE 7. The outer and inner circles governed by pattern containment, and we hope that D is governed by pattern containment as well.

Proof. Since v is not multiplicity-free, it has multiple Edelman-Greene tableaux of the same shape. Since Edelman-Greene BPDs are in bijection with Edelman-Greene tableaux, then by Theorem 4, w must also have multiple Edelman-Greene tableaux of the same shape. Thus, w has repeated BPD diagrams. \square

Thus we can focus all of our attention on the set $D \setminus \overline{\text{MultFree}}$. To begin, we only have the one example in S_7 . As such, our first goal is to find other permutations with this property and determine whether or not these patterns are causing duplicitous behavior via containment.

5. GENERATING EXAMPLES OF PERMUTATIONS WHICH ARE DUPLICITOUS AND MULTIPLICITY FREE

A *pivot* of a permutation is a southeast-most cell in its Rothe diagram. We observe that the example 3216745 differs from one of the known duplicitous pattern in S_6 , 321654, simply by removing a south-east most box in its diagram (see Figure 8). Notice that there is an injection from $\text{BPD}(321654)$ to $\text{BPD}(3216745)$

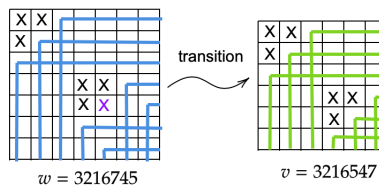


FIGURE 8. 3216745 and 321654 are related by deleting a pivot

which is almost diagram preserving. Under this map, the diagrams only differ by adding a cell in position $(5, 5)$.

From this follows a more general observation. Suppose v is obtained from w by deleting a pivot. Then, if v is duplicitous then so is w . Thus we can get many new examples of duplicitous permutations by taking known duplicitous permutations and adding new pivots. Of those new examples, we can then begin to generate a new list of permutations that are both duplicitous and multiplicity free. We state the following proposition without proof.

Proposition 1. *Fix w with a pivot in cell (a, b) . If $D(v)$ is obtained from the diagram of $D(w)$ by deleting cell (a, b) , then:*

(1) *There is an injection $\iota : \text{BPD}(v) \rightarrow \text{BPD}(w)$ such that*

$$D(\iota(P)) = D(P) \cup \{(a, b)\}.$$

(2) *If v is duplicitous, then so is w .*

Part (1) of this proposition is similar to a special case of the diagrammatic interpretation of transition found in [Las02] in terms of alternating sign matrices.

The hope is that we can generate many more such examples by adding new pivots to other known duplicitous permutations. We note in particular that 321654 is a minimal pattern that causes permutations to be non-multiplicity-free [BP14], and thus we use Billey and Pawlowski's lists of minimal patterns in S_9 , S_{10} , and S_{11} to generate more examples.

We wrote a code that takes in a list of known duplicitous permutations and deletes a pivot in all possible ways. We also ran an exhaustive check for all duplicitous permutations in S_8 and S_9 . Both the exhaustive list and the adding pivots method generated the same list of minimal duplicitous multiplicity-free patterns in S_8 . Thus, somewhat surprisingly, the method of adding pivots found all minimal duplicitous and multiplicity free patterns in S_8 . The two patterns that we found are

(9) 24137856 and 31427856.

Notice that the Rothe bumpless pipe dreams for these permutations are transposes of one another.

6. CURRENT RESULTS

Our first large-scale check with the code was feeding it the exhaustive list of duplicitous permutations in S_8 and generating all possible examples in S_9 via this method. After removing all permutations containing the known duplicitous patterns in S_6 , S_7 and S_8 and any non-multiplicity-free permutations, we obtained the following two permutations:

(10) 351289467 and 341728956.

Note again that the Rothe bumpless pipe dreams for these permutations are transposes of one another.

We can again ask the question: is this everything, or are there more examples that we are missing via this method?

If our pattern containment hypothesis is true, then it must be the case that all patterns in S_9 containing the patterns 3216745, 24137856, and 31427856 are duplicitous. To check this statement, we generated a list of all permutations in S_9 containing each of these patterns and compared it to the exhaustive list of duplicitous patterns in S_9 . We concluded that pattern containment does hold for S_n for $n \leq 9$ and that the method of adding pivots is missing many duplicitous permutations in S_9 .

To get a sense of exactly how many permutations we are missing by adding pivots: there were a total of 65 permutations in S_9 containing 24137856, 24 of which were missed by our methods. 10 of those permutations are multiplicity-free, which leaves 14 minimal patterns that this method does not find, just based on one pattern containment. This check tells us that this method is good for finding examples but certainly not for finding all examples.

It is also worth noting that when we generated a new list of duplicitous permutations from the non-multiplicity-free permutations in Billey and Pawlowski's list in S_9 , we obtained two new permutations that are in S_9 that we did not find from the exhaustive list in S_8 or from looking at permutations containing the permutations in S_7 and S_8 . The permutations are

$$(11) \quad 341279856 \text{ and } 341289576.$$

6.1. What are the largest minimal patterns? We are specifically interested in permutations that are both duplicitous and multiplicity free, since we know that non-multiplicity free patterns are governed by pattern containment by Theorem 3 from [BP14]. Based on our exhaustive checks through S_9 , the current confirmed list of minimal duplicitous and multiplicity patterns through S_9 is given by the permutations listed below.

3216745	341279856	341289657	341297856	341927856	351289647
24137856	341287956	341289675	341728956	351289467	351289674
31427856	341289576	341289756	341827956		

We would like to answer the question of whether or not the list of minimal patterns is finite, if the set is indeed governed by patterns. Thus it would be useful to know how far we can keep pushing this method into larger symmetric groups and obtaining new minimal patterns. From looking at the list of patterns that Billey and Pawlowski conjecture to cause permutations to be non-multiplicity-free, we used the code to generate classes of examples in S_n for $n = 9, 10, 11, 12$. We generated these lists sequentially and included all previous results in the list of patterns to be thrown out in the next check. However, as was described above,

we know that we do not have an exhaustive list of patterns in S_n for $n \geq 9$. Thus, just because a new permutation does not contain any of the patterns in our list does not mean that it is minimal.

Using the method of adding pivots applied to Billey and Pawlowski's list of patterns in S_{11} , we found 6 candidates for minimal permutations. We wrote a code that finds all patterns in S_{n-1} within a permutation in S_n . We can determine whether any of the patterns in S_{11} contained in the S_{12} permutations are duplicitous; if so, those are new candidates for minimal patterns; if not, we can repeat the process until we arrive at S_9 where we know all minimal patterns. The code to find the smaller patterns runs efficiently, but unfortunately we do not currently have an efficient code for determining whether large ($n \geq 10$) permutations are duplicitous, so we have not yet found conclusive results about the S_{12} permutations.

6.2. Work towards proving pattern containment. In Section 6, we discussed computational methods to verify that pattern containment holds for smaller examples and how we generated lists of minimal patterns that we conjecture cause duplicitous behavior.

We have some ideas towards proving a statement about pattern containment, though none are complete. We briefly summarize them here.

Definition 10. Given two permutations $w \in S_n$ and $v \in S_m$, we say that $w \times v$ is the permutation given by taking the permutation in S_{n+m} that has w in the first n entries and v in the last m entries, with n added to each of the last m entries.

Example 4. If $w = 2143$ and $v = 321$ then $w \times v = 2143765$.

There are some cases of pattern containment that are straightforward. For example, if we stabilize (meaning replace w with $w \times 1$ or back-stabilize (replace w with $1 \times w$) a permutation in D , we get another permutation in D .

Lemma 2. Fix $w \in S_n$ and $v \in S_m$. If w is duplicitous, then $w \times v$ and $v \times w$ are duplicitous as well.

Proof. Note that the Rothe bumpless pipe dream for w will show up in the upper left $n \times n$ subgrid of $w \times v$, and we can use the same repeated diagram as we would have used for w in the original case within this $n \times n$ grid to obtain a new repeated diagram. A similar argument works for $v \times w$. \square

The first question that we explored was the following: in general, when we have a permutation w containing a pattern v , can we find the pipes associated to v and droop them in the same ways as we would in the Rothe bumpless pipe dream for v to get a repeated diagram?

The answer is no; often you end up with a non-reduced bumpless pipe dream when you do this procedure.

As an example, consider the permutation 2135476 that contains 214365, a known duplicitous permutation whose duplicate diagrams are shown in Figure 3(a).

However, if we naively apply the same droops in the larger permutation, we end up with non-reduced bumpless pipe dreams, as is shown in Figure 9.

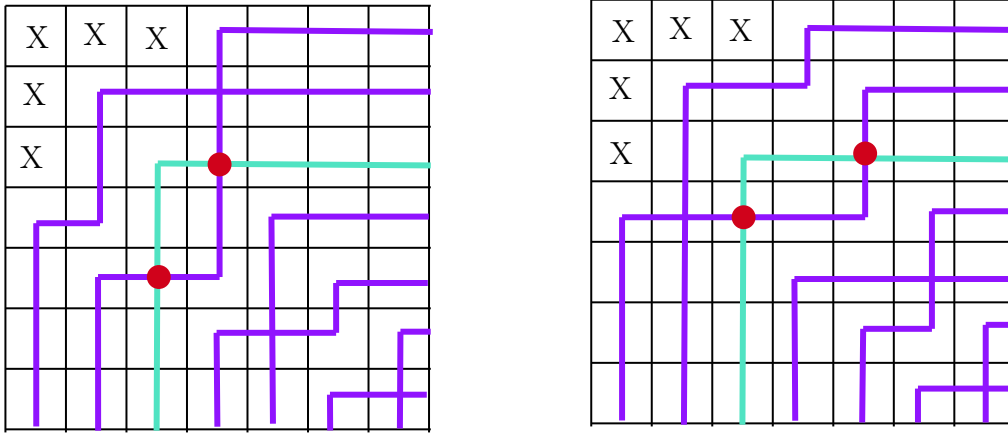


FIGURE 9. Applying droops to 2135476 the same way we would for 214365. The result is non-reduced.

Thus, this naive approach does not work as one might hope. We explored trying to avoid the double-crossings in systematic ways, but we did not find any consistent way to obtain a duplicate diagram given any such permutation, even though we were always able to find a repeated diagram by pushing the pipes upward and left in a variety of ways.

One future question to explore would be the following: If at each point where the bumpless pipe dream is non-reduced we instead add a “double elbow” tile as shown in Figure 10, then we could explore whether there is a way to systematically push the top pipe up and left to obtain reduced diagrams.

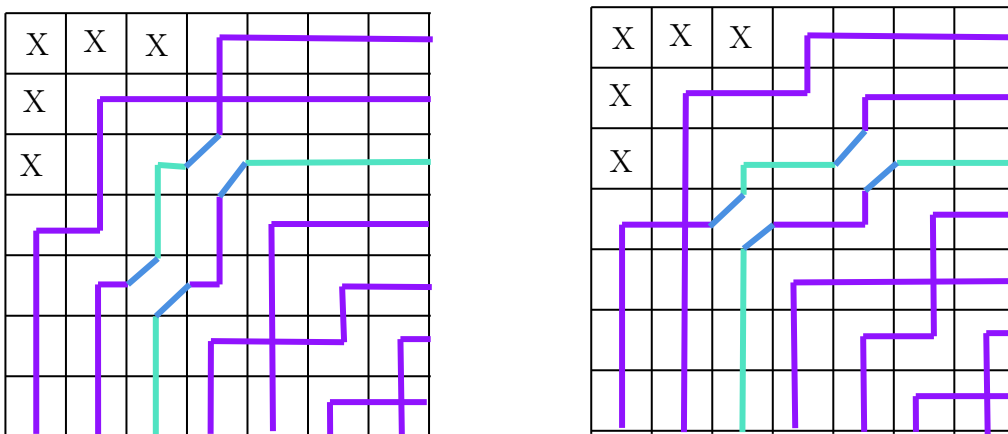


FIGURE 10. Replacing non-reduced diagrams with double-elbow tiles.

6.3. Exploring diagram containment. Another method that we explored was trying to determine how useful it is to look at the original diagram within the diagram of the larger permutation. The following proposition deals with a special case of permutations containing 321654. Although all permutations containing 321654 are known to be duplicitous since 321654 has repeated Edelman-Greene diagrams, there is no explicit map that explains why the repeated Edelman-Greene diagram shapes are preserved through pattern containment.

We look at a particular type of permutation in order to try to understand how pattern containment might relate to diagram containment. The techniques used here may be useful to prove more general results about why pattern containment governs repeated diagrams (if it indeed does).

Definition 11. The longest permutation S_n is given by

$$w_0^{(n)} := n \ n - 1 \ n - 2 \ \cdots \ 2 \ 1.$$

We consider permutations of the form $w_0^{(n)} \times w_0^{(m)}$, where $m, n \geq 3$. Note that we know that such permutations are duplicitous because all such permutations contain 321654 as a pattern. Here we give a construction for building the repeated diagrams that is based on the repeated diagram for 321654 = $w_0^{(3)} \times w_0^{(3)}$.

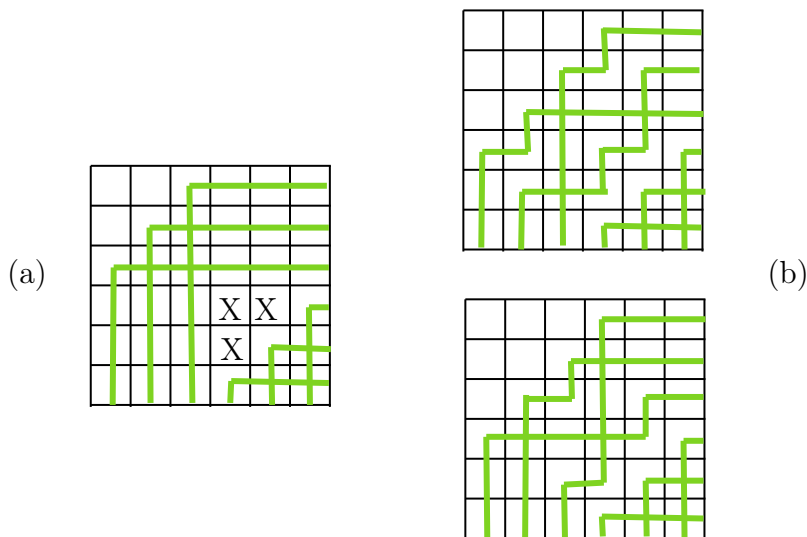


FIGURE 11. (a) Original BPD for 321654. (b) Repeated diagrams for 321654.

Take the Rothe diagram for $w = w_0^{(m)} \times w_0^{(n)}$. In the bottom $n \times n$ subgrid, there are $\frac{n(n-1)}{2}$ blank tiles stacked as a right triangle; pick the three boxes in the upper left corner as shown in Figure 12 to correspond to the three boxes marked in Figure 11(a). We see an exact copy of the diagram for 321654 if we take the subgrid in rows 1 to 3 and $m + 1$ to $m + 3$ and columns $m - 2$ to $m + 3$, as

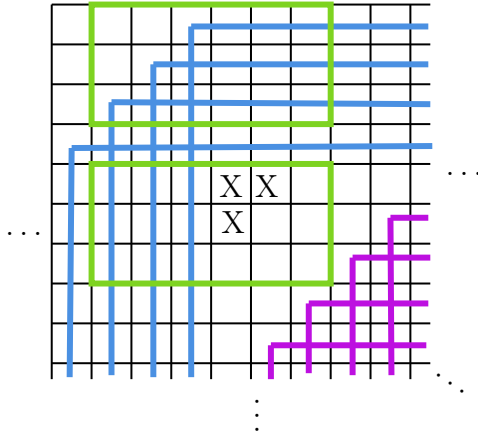


FIGURE 12. Above is a generalized diagram for the Rothe diagram of $w_o^{(m)} \times w_o^{(n)}$.

shown in 12. Then we can fill in the subgrid in exactly the same two ways as in 11(b). Within the subgrid, we will get a repeated diagram since we are copying from repeated diagrams. Outside of the subdiagram, the key is that all of the boxes between the two pieces of the subdiagram (specifically, rows 4 through m , columns $m - 2$ to $m + 3$) are covered by the other fixed pipes. Thus it does not matter that the pipes enter and exit the upper and lower halves of the subdiagram in different ways because none of those boxes will be in the diagram anyways. We conclude that $w_o^{(m)} \times w_o^{(n)}$ is a duplicitous permutation for any $m, n \geq 3$.

As mentioned above, any time we have pattern containment in a permutation, we can select the subgrid with columns given by the indices at which the pattern occurs and rows given by the outputs of those indices in the permutation. We can then perform the same sets of droops within the subdiagram, and get repeated diagrams; the question then is whether or not:

- (a) We can connect the parts of the subdiagram without introducing differences in the diagram. For example, we may get different diagram boxes if pipes enter and exit parts of the subdiagram in different ways.
- (b) The diagram remains reduced when we perform the same droops. In general this does not happen, as we saw above.

7. CONCLUSIONS

In order to determine what conditions cause permutations to be duplicitous, we used computational methods to find examples of duplicitous multiplicity-free permutations and to check whether pattern containment holds for relatively small permutations. In addition, we explored a variety of ideas relating to proving pattern containment results about D in general. We verified using computer checks that pattern containment holds through S_9 (i.e. all permutations in S_8

containing 3216745 are duplicitous, and all permutations in S_9 containing the three minimal patterns in S_7 and S_8 are duplicitous). Thus, though we do not yet have the full pattern containment result for the set D , we have ample evidence that such a result may hold and ideas towards proving a complete statement.

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