Local cohomology of $L_{(1,1,...,1)}$ module

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Abstract

L_λ is one kind of important modules in representation stability. In the article [5], all local cohomology of L_λ module over $k[x_0, ..., x_n]$ are studied if k is of characteristic 0. In this paper, local cohomology of special L_λ will be computed in an elementary way regardless of characteristic of k.

1 Background

The goal of this section is introducing the way of computing local cohomology of graded modules over polynomial rings.

Theorem 1. *[4, 17.3] Let M be a finitely generated Z-graded module over the polynomial ring of finite variables and let $\mathcal{F} = \widetilde{M}$ be the corresponding coherent sheaf on $X = P_k^n$. Let $m = (x_0, ...x_n)R$, then:*
(a): for $i \geq 1$, $H^i(X, \mathcal{F}(t) \cong [H_m^{i+1}(M)]_t$, so that $H_m^{i+1}(M) \cong \bigoplus_t H^i(X, \mathcal{F}(t))$
(b): there is a short exact sequence as follows

$$0 \to H_m^0(M) \to M \to \bigoplus_t H^0(X, \mathcal{F}(t)) \to H_m^1(M) \to 0$$

Proof. Let $\mathcal{U} = \{U_i = D_+(x_i)\}$ be an open affine covering of P_k^n. By general facts of Cech cohomology, we have exact sequence as follows,

$$0 \to \bigoplus_i \mathcal{F}(U_i) \to \bigoplus_{i<j} \mathcal{F}(U_i \cap U_j) \to \cdots \to \mathcal{F}(U_0 \cap U_1 \cap \cdots \cap U_n) \to 0$$

It is not hard to find the following two complexes isomorphic

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_i \mathcal{F}(U_i) & \longrightarrow & \bigoplus_{i<j} \mathcal{F}(U_i \cap U_j) & \longrightarrow \dots \longrightarrow & \mathcal{F}(U_0 \cap \cdots \cap U_n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_i M_{x_i} & \longrightarrow & \bigoplus_{i<j} M_{x_i x_j} & \longrightarrow \cdots \longrightarrow & M_{x_0 \dots x_n} & \longrightarrow & 0 \end{array}$$

the bottom complex is almost the same as the Koszul Complex, $K^{\bullet}(x_0, ..., x_n; M)$: one only has to drop the first term of the koszul complex, and shift the number by one.
Moreover, if one takes the t-th graded piece of this complex one obtains the Cech complex for $\mathcal{F}(t)$ with respect to the same open cover. Now there is an obvious map of $[M_0]$ into the global sections of $\widetilde{M}$, and $[M]_t$ into the global sections of $\widetilde{M} \otimes O_X(t)$ for every t. We have already seen that $\oplus_t H^i(X, \mathcal{F}(t))$ may be identified with the cohomology of the complex $K^{\bullet}(x_0, ..., x_n; M)$ truncated at the beginning. This implies the isomorphism in (a) at once and also yields an exact sequence:

$$0 \to H^0_m(M) \to M \to \bigoplus_t H^0(X, \mathcal{F}(t)) \to H^1_m(M) \to 0$$

□

From the above theorem, one way to compute local cohomology of L_λ is considering its sheafified version, $\widetilde{L_\lambda}$ on projective space P^n_k. Then the only job is computing the sheaf cohomology of $\widetilde{L_\lambda} \otimes O_X(m)$ for any integer m.
By general facts in representation theory, we have

$$\widetilde{L_\lambda} = S_\lambda(\Omega_{X/k} \otimes O_X(1))$$

where S_λ is Schur Functor. We also define $S_\lambda(\mathcal{F})$ for any sheaf O_X-module as sheaf associate to the presheaf

$$U \longrightarrow S_\lambda(\mathcal{F}(U))$$

On P^1_k, all computations are easy. David Mumford solved this problem on P^2_k. People still don't know much for higher dimensional projective space when characteristic of k is not 0. So, In this paper, I assume $n \geq 3$. I try to compute local cohomology of $L_{(1,1,...,1)}$. In order to do this, by Theorem 1, we simply need to understand

$$H^i(X, \bigwedge^q(\Omega_{X/k}(1)) \otimes O_X(m)) = H^i(X, \bigwedge^q(\Omega_{X/k}) \otimes O_X(m+q))$$

.

2 sheaf cohomology of $O_X(m)$

Lemma 1 (1, III.5.1). *Let A be a Noetherian ring, and let $X = P^n_A$, with $n \geq 1$, then:*
(a): $H^i(X, O_X(m)) = 0$ for $0 < i < n$
(b): there is a perfect pairing of finitely generated free A module

$$H^0(X, O_X(m) \times H^n(X, O_X(-m-n-1) \to H^n(X, O_X(-n-1)) \cong A$$

Proof. It is a well-known fact. ☐

Theorem 2. *Take $P_k^n = Proj(k[x_0, ..., x_n])$.*
If $m \geq 0$,

$$dim_k(H^0(X, O_X(m))) = \binom{n+m}{n}$$

other sheaf cohomology group vanish
If $-n \leq m \leq -1$, all sheaf cohomology group vanish
If $m \leq -n-1$,

$$dim_k(H^n(X, O_X(m)) = \binom{-m-1}{n}$$

other sheaf cohomology group vanish

Proof. By Grothendieck Vanishing Theorem, $H^i(X, O_X(m)) = 0$ for $i > n$. By Part (a) of the above lemma, only $H^0(X, O_X(m))$ and $H^n(X, O_X(m))$ might not be zero. Since on projective space P_k^n, it is obvious that the

$$H^0(X, O_X(m)) \cong [k[x_0, ..., x_n]]_m$$

as vector space over k, then by the perfect pairing above, it is easy to get

$$dim_k(H^0(X, O_X(m))) = dim_k(H^n(X, O_X(-m-n-1)))$$

By discussion, it is easy to get result above. ☐

3 sheaf cohomology of $\Omega_{X/k} \otimes O_X(m)$

Lemma 2 (1, II.8.13)**.** *Let A be a ring, let $Y = SpecA$, and let $X = P_A^n$. Then there is an exact sequence of sheaves on X,*

$$0 \to \Omega_{X/k} \to O_X(-1)^{\oplus n+1} \to O_X \to 0$$

Proof. This short exact sequence is well known; it is called the *Euler Sequence*. ☐

Theorem 3. *Let $\Omega_{X/k} \otimes O_X(m)$ be a sheaf on $P_k^n = proj(k[x_0, ..., x_n])$, then:*
(a): If $m \geq 1$,

$$dim_k(H^0(X, \Omega_{X/k} \otimes O_X(m))) = (m-1)\binom{m+n-1}{n-1}$$

other sheaf cohomology group vanish
(b): If $m = 0$,

$$dim_k(H^1(X, \Omega_{X/k} \otimes O_X(m))) = 1$$

other sheaf cohomology group vanish
(c): If $-n+1 \leq m \leq -1$, all sheaf cohomology group vanish
(d): If $m \leq -n$,

$$dim_k(H^n(X, \Omega_{X/k} \otimes O_X(m))) = \frac{(m-1)n}{m} \binom{-m}{n}$$

other sheaf cohomology group vanish

Proof. By the above lemma, there is a short exact sequence,

$$0 \to \Omega_{X/k} \to O_X(-1)^{\oplus n+1} \to O_X \to 0$$

Since $O_X(m)$ is a line bundle, then $- \otimes O_X(m)$ is an exact functor. Thus, we get an exact sequence as follows,

$$0 \to \Omega_{X/k} \otimes O_X(m) \to O_X(m-1)^{\oplus n+1} \to O_X(m) \to 0$$

This short exact sequence induces a long exact sequence, and by Grothendieck Vanishing Theorem, the long exact sequence has two parts

$$0 \to H^0(X, \Omega_{X/k}(m)) \to \bigoplus_{n+1} H^0(X, O_X(m-1)) \to H^0(X, O_X(m)) \to H^1(X, \Omega_{X/k}(m)) \to 0$$

$$0 \to H^n(X, \Omega_{X/k} \otimes O_X(m)) \to H^n(X, O_X(m-1)^{\oplus n+1}) \to H^n(X, O_X(m)) \to 0$$

Since we have the following diagram commutes,

$$\begin{array}{ccc} \bigoplus_{n+1} H^0(X, O_X(m-1)) & \longrightarrow & H^0(X, O_X(m)) \\ \downarrow & & \downarrow \\ \bigoplus_{n+1} [k[X_0, ..., X_n]]_{m-1} & \longrightarrow & [k[x_0, ..., x_n]]_m \end{array}$$

where the map $\theta : \bigoplus_{n+1} [k[x_0, ..., x_n]]_{m-1} \to [k[x_0, ..., x_n]]_m$ sends $(f_0, ..., f_n) \to x_0 f_0 + \cdots x_n f_n$ where f_i is a homogeneous polynomial of degree $m-1$.
when $m \geq 1$, the map θ is surjective, then the first sheaf cohomology group vanish. In addition, it is easy to get the n-th sheaf cohomology vanish. Also,

$$dim_k(H^0(X, \Omega_{X/k}(m))) = (n+1) dim_k(H^0(X, O_X(m-1))) - dim_k(H^0(X, O_X(m)))$$

When $m = 0$, by Theorem 2, we know that $H^0(X, O_X(-1) = 0$, then all sheaf cohomology of $\Omega_{X/k}$ vanishes except the first. We have

$$dim_k(H^1(X, \Omega_{X/k})) = dim_k(H^0(X, O_X)) = 1$$

when $-n+1 \leq m \leq -1$, by Theorem 2, we have all sheaf cohomology of $O_X(m-1)$ and $O_X(m)$ vanish. Thus, all sheaf cohomology of $\Omega_{X/k} \otimes O_X(m)$ vanish.

When $m \leq -n$, By theorem 2, we know all sheaf cohomology of $O_X(m-1)$ and $O_X(m)$ vanish except the n-th sheaf cohomology group. We have

$$dim_k(H^n(X, \Omega_{X/k}(m) = (n+1)dim_k(H^n(X, O_X(m-1)) - dim_k(H^n(X, O_X(m)))$$

Therefore, we have the result above □

4 sheaf cohomology of $\bigwedge^q(\Omega_{X/k}) \otimes O_X(m+q)$

Before getting into the main theorem, I am going to exhibit how people use Borel-Weil-Bott Theorem to compute sheaf cohomolgy of $S_\lambda(\Omega_{X/k}(1)) \otimes O_X(m)$, when characteristic of k is 0. This method can be found in [3].
***Bott's Algorithm**. Let $\lambda = (\lambda_1, ..., \lambda_n)$ be a partition of q where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and $\lambda_i \geq 0$ for $1 \leq i \leq n$. let $\alpha = (\alpha_1, ..., \alpha_{n+1}) = (m, \lambda_1, ..., \lambda_n)$, we try to make α weakly decreasing by the following moves: for $1 \leq i \leq n$ with $\alpha_i < \alpha_{i+1}$, we can replace (α_i, α_{i+1}) by $(\alpha_{i+1} - 1, \alpha_i + 1)$. If this move doesn't do anything, then we say we are "stuck".*

Theorem 4 (Borel-Weil-Bott's Theorem)**.** *Let $P_k^n = proj(k[x_0, ..., x_n])$ where $char(k) = 0$. Run Bott's Algorithm for $\alpha = (m, \lambda_1, ..., \lambda_n)$*
(a) During the process, if it ever get stuck, then for all i,

$$H^i(S_\lambda(\Omega_{X/k}(1)) \otimes O_X(m)) = 0$$

(b) If get a weakly decreasing sequence β after i moves, then,

$$H^i(X, S_\lambda(\Omega_{X/k}(1)) \otimes O_X(m)) = S_\beta(k^n)$$

with other sheaf cohomology group vanish.

Proof. It is a special case of Borel-Weil-Bott's theorem in chapter 5 of [3]. □

Remark 1. *You might notice that if $q > n$, and the partition of q can not be put into n-tuples (for example $(1, 1, ..., 1)$)as what Bott's Algorithm requires. However, if this case happens, $S_\lambda(\Omega_{X/k}(1))$ is zero sheaf. There is nothing to worry.*

Definition 1. *Let $T_\lambda^{i,m} = dim_k(H^i(X, S_\lambda(\Omega_{X/k}(1)) \otimes O_X(m)))$. Let $B_\lambda^{i,m}$ denote the dimension of vector space gotten by running Bott's algorithm on $\alpha = (m, \lambda_1, ..., \lambda_n)$ regardless of characteristic of k. Thus, when $char(k) = 0$, $T_\lambda^{i,m} = B_\lambda^{i,m}$.*

Theorem 5. *Let $\lambda = (1, 1, ..., 1)$ be a partition of q where $q \leq n$,*

$$T_{(1,1,...,1)}^{i,m} = B_{(1,1,...,1)}^{i,m}$$

for arbitrary field k.

Proof. Consider the Euler Sequence,

$$0 \to \Omega_{X/k} \to O_X(-1)^{\oplus n+1} \to O_X \to 0$$

Since $\Omega_{X/k}$ and $O_X(-1)$ are vector bundles, and $O_X(1)$ is line bundle, then we get an induced short exact sequence,

$$0 \to \bigwedge^q \Omega_{X/k} \to \bigwedge^q ((O_X(-1)^{\oplus n+1}) \to \bigwedge^{q-1} (\Omega_{X/k}) \to 0$$

By tensoring with $O_X(m+q)$, we get a short exact sequence, as follows,

$$0 \to \bigwedge^q (\Omega_{X/k}(1)) \otimes O_X(m) \to \bigoplus_{\binom{n+1}{q}} O_X(m) \to \bigwedge^{q-1} (\Omega_{X/K}(1)) \otimes O_X(m+1) \to 0$$

In article [2], "A smooth projective variety X over a field is said to satisfy Bott vanishing if

$$H^j(X, (\bigwedge^i \Omega_{X/k}) \otimes \mathcal{L}) = 0$$

for all ample line bundles $\mathcal{L}$, all $i \geq 0$, and all $j > 0$. Bott proved this when X is projective space."

When $m+q \geq 1$. By the result from article [2], we know all except the 0-th sheaf cohomology of $\bigwedge^q (\Omega_{X/k}(1)) \otimes O_X(m)$ vanish. But from the induced long exact sequence, we have,

$$T^{0,m}_{1^q=(1,1,\ldots,1)} + T^{0,m+1}_{1^{q-1}=(1,1,\ldots,1)} = \binom{n+1}{q} dim_k(H^0(X, O_X(m))$$

When $m+q \leq -1$. We have the following exact sequences,

$$0 \to H^0(X, \bigwedge^{q-1} \Omega_{X/k}(1) \otimes O_X(m+1)) \to H^1(X, \bigwedge^q \Omega_{X/k}(1) \otimes O_X(m)) \to 0$$

$$\ldots$$

$$0 \to H^{n-2}(X, \bigwedge^{q-1} \Omega_{X/k}(1) \otimes O_X(m+1)) \to H^{n-1}(X, \bigwedge^q \Omega_{X/k}(1) \otimes O_X(m)) \to 0$$

$$0 \to H^{n-1}(X, \Omega^{q-1}_{X/k}(m+q)) \to H^n(X, \Omega^q_{X/k}(m+q)) \to \bigoplus_{\binom{n+1}{q}} H^n(X, O_X(m)) \to H^n(X, \Omega^{q-1}_{X/k}(m+q)) \to 0$$

Notice that $H^i(X, \Omega_{X/k} \otimes O_X(m+q)) = 0$ for all $0 \leq i \leq n-1$. To simplify, denote $m+q = s$, where $s \leq -1$. Assume that for all $1 \leq b \leq q-1$,

$H^i(X, \bigwedge^b \Omega_{X/k} \otimes O_X(s)) = 0$ for all $0 \leq i \leq n-1$. By exact sequences above, we have $T^{n-1,m}_{1^q} = T^{n-2,m+1}_{1^{q-1}}, \dots, T^{1,m}_{1^q} = T^{0,m+1}_{1^{q-1}}$. According to the induction Hypothesis, it is true that when $s \leq -1$ $H^i(X, \bigwedge^b \Omega_{X/k} \otimes O_X(s)) = 0$ for all $b \in Z$, and all $0 \leq i \leq n-1$.

Therefore, the last exact sequence becomes a short exact sequence, and there is a recurrence $T^{n,m}_{1^q} + T^{n,m+1}_{1^{q-1}} = \binom{n+1}{q} dim_k(H^n((X, O_X(m)))$.

when $m+q=0$, what we should know is sheaf cohomology of $\bigwedge^q \Omega_{X/k}$. However, it is a well-known fact that $H^i(X, \bigwedge^q \Omega_{X/k}) = k$ if $i = q$, and zero otherwise.

□

From the above computations, we have found recurrence formulas for all $T^{i,m}_{1^q=(1,1,\dots,1)}$, *which is independent of the characteristic of* k. *Thus, we have proved that Borel-Weil-Bott's theorem works for exterior power case. Next, let's compute the exact dimension of* $H^i(X, \bigwedge^q(\Omega_{X/k}) \otimes O_X(m+q))$.

Theorem 6. *Let* $X = P^n_k = proj(k[x_0, ..., x_n])$
If $m \geq 1$, *then* $dim_k(H^0(X, (\bigwedge^q \Omega_{X/k}) \otimes O_X(m+q))) = dim_k(S_{(m,1,1,\dots,1)}(k^n))$ *,and other sheaf cohomology groups vanish.*
If $-n \leq m \leq 0$, *then all sheaf cohomology groups vanish.*
If $m \leq -n-1$, *then* $dim_k(H^n(X, (\bigwedge^q \Omega_{X/k}) \otimes O_X(m+q))) = dim_k(S_{(0,\dots,0,1,..,1,m+n)})(k^n))$, *and other sheaf cohomology groups vanish.*

Proof. Running Bott's algorithm on $(m, 1, .., 1, 0, ..., 0)$, which is very easy. □

5 Conclusion

In this paper, I computed local cohomology of a special L_λ*-module over* $k[x_0, ..., x_n]$ *in a surprisingly elementary way. I don't know if it is in literature. However, since all techniques used here is trivial, I think everyone wants to compute can do the same thing. So, I will just think this paper as an expository article. Hopefully, it is useful for someone.*

Finally, I want to express my appreciation to professor Andrew Snowden for his generous help. Without his help, this paper will never be done.

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