ON THE PSEUDOSPECTRUM OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH COMPLEX POTENTIAL

ZHAI XIYU, USTC, MENTOR: PROFESSOR ALEJANDRO URIBE

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1. Introduction

The class of Schödinger operators we're concerned with in this paper is

$$L_h = -h^2\frac{d^2}{dx^2} + V(x), 0 < h < 1 \tag{1}$$

acting on $L^2(X)$, where X is a circle S^1, an interval $[a, b]$ or $\mathbb{R}$.

Convention: we shall denote the function space as H. And we shall mean L^2-norm by $\|\cdot\|$ and L^2-inner product by $\langle\cdot,\cdot\rangle$.

We're going to study the pseudospectrum of these operators. Given a positive increasing function $f(h)$ with $\lim\limits_{h\to 0+} f(h) = 0$, the $f(h)$-pseudospectrum of L_h is defined to be

$$\{\lambda \in \mathbb{C} : \exists C > 0, \forall h_0 > 0, \exists 0 < h < h_0, \psi_h \in H, \|\psi_h\| = 1, \|(L_h - \lambda)\psi_h\| \leq Cf(h)\} \tag{2}$$

The h^∞-pseudospectrum is the intersection of all h^N-pseudospectrum for $N > 0$.

In 1999, E. Brian Davies [cite] proved the following result:

Theorem (E.B. Davies, 1999)**.** For $V \in C^\infty(\mathbb{R}), \Im V'(x_0) \neq 0$, then for $\lambda \in \mathbb{C}, \Re\lambda > \Re V(x_0), \Im\lambda = \Im V(x_0)$, λ belongs to the h^∞-pseudospectrum of L_h.

In 2001, Zworski [cite] showed the result in fact follows immediately from the now standard results in microlocal analysis described in Chapter 26 of [cite].

2003, K.P. Starov [cite] generalized this result of Davies to the following:

Theorem (Karel Pravda-Starov,2003)**.** For $V \in C^\infty(\mathbb{R}), x_0 \in \mathbb{R}, \exists p \in \mathbb{N}, \Im V^{(1)}(x_0) = \cdots = \Im V^{(2p)}(x_0) = 0, \Im V^{(2p+1)}(x_0) \neq 0$, then for $\lambda \in \mathbb{C}, \Re\lambda > \Re V(x_0), \Im\lambda = \Im V(x_0)$, λ belongs to the h^∞-pseudospectrum of L_h.

In other words, the result of Davies (1999) deals with a simple zero and that of Karel Pravda-Starov deals with a zero of odd order. This extension points out the close link between the pseudospectrum and sign changes of $\Im V - \Im\lambda$. One can easily deduce from this result that "when $\Im V$ is **real analytic**, a sufficient condition to conclude that $\lambda \in \mathbb{C}, \Re\lambda > \Re V(x_0)$ belongs to the pseudospectrum is to require a sign change of $\Im V - \Im\lambda$" at x_0, as observed by Karel Pravda-Starov in [cite].

Therefore, a natural question is that when $\Im V$ is only C^∞, can a sign change of $\Im V - \Im\lambda$ guarantee that λ belongs to the pseudospectrum?

For example,consider the following potential

$$V = \begin{cases} ie^{-\frac{1}{x-1}}, & x > 1 \\ 0, & 0 \leq x \leq 1 \\ -ie^{\frac{1}{1+x}}, & x < 0 \end{cases} \tag{3}$$

.

TBA: There should be a figure depicting V, haha.

V belongs to $C^\infty(\mathbb{R})$ and vanishes over $[0,1]$, positive over $(1,+\infty)$, and negative over $(-\infty,0)$. There is a change of sign of $\Im V$ from $-\infty$ to $+\infty$, but the derivatives of all order vanishes over $[0,1]$. Then the previous results of E.B. Davies and K.P. Starov say nothing about whether $\lambda > 0$ belongs to the pseudospectrum.

Here we prove a more general statement showing that the change of sign principle holds true for C^∞-potentials in dimension one:

Theorem. (Main result) Let $k \in \mathbb{Z}_+, V \in C^{k+2}([a,b])$, with $\Im V(a) \neq \Im V(b)$, then for any $\lambda \in \mathbb{C}$ with $\Im\lambda$ lies between $\Im V(a)$ and $\Im V(b)$ and $\Re\lambda > \max_{\substack{x\in(a,b) \\ \Im V(x)=\Im\lambda}} \Re V(x)$, there exists a family of quasimodes $\{\psi_h\}_{0<h<1} \subset C_c^2((a,b))$ and a constant $C > 0$ such that

$$\|\psi_h\|_{L^2(a,b)} = 1, \|(L_h - \lambda)\psi_h\|_{L^2(a,b)} \leq Ch^{k+1}, \forall 0 < h < 1 \tag{4}$$

It follows that λ belongs to the h^k-pseudospectrum of L^h.

If $V \in C^\infty([a,b])$, then λ belongs to the h^∞-pseudospectrum of L^h.

In particular for the above case equation(3), our results show that for $\lambda > 0$, λ belongs to the pseudospectrum.

Our proof of this result mainly uses the WKB construction of quasimodes, as in those of E.B. Davies and K.P. Starov. However, we utilize certain techniques in Mathematical Analysis to adapt the WKB construction to general potentials.

The new feature of our WKB method is that the quasimode ψ_h we constructed might not concentrate on one single point x_0 but might concentrated over a whole interval as $h \to 0^+$.

Here is a comparison between the previous WKB methods and ours:

- In previous WKB constructions, given $\lambda \in \mathbb{C}$, one first find a single point x_0 satisfying $\Im V(x_0) = \Im\lambda, \Re V(x_0) < \Re\lambda$ and a nondegenerate condition on derivatives of $\Im V$ at this point x_0. Then one construct an appropriate phase $S(x) = \pm\int_{x_0}^{x} \sqrt{\lambda - V(x)}$ which can localize the quasimode to the single point x_0.
- In our WKB construction, given $\lambda \in \mathbb{C}$, we don't fix a single point x_0 in the first place. Instead we first fix an interval I over which $\Im V - \Im\lambda$ experiences a sign change and $\Re V(x) < \Re\lambda$ for all $x \in I$. We then integrate $\pm\sqrt{\lambda - V(x)}$ over the interval I to get a function $\tilde{S}$ over I, serving as the phase function plus a constant. Then we show $\tilde{S}$ can only attain minimum in the interior of I and choose the phase function $S(\cdot)$ to be $\tilde{S}(\cdot) - \min\limits_{x\in I} S(x)$, and can show the quasimode constructed has the required property. In short, **we utilize the whole interval instead of a single point to construct a quasimode**, which enables us to deal with more general potentials.

The proof itself is elementary, but can hope to be generalized to other situations, maybe operators in higher dimensions. In higher dimensions, we might need to construct quasimode whose wavefront set is more complicated then one dimensional case.

2. Proof of the main theorem

Assume the condition of the main theorem. Is this OK?

2.1. Restriction to smaller interval.

In our main theorem, we only assume that $\Re\lambda > \Re V(x)$ if $x \in (a, b), \Im V(x) = \Im\lambda$. We want to strengthen this condition by limiting to a subinterval $[a', b'] \subset [a, b]$. The strengthened condition can guarantee the well-definedness of $\sqrt{\lambda - V}$. To do that, we prove the following lemma:

Lemma 2.1. There exists $[\alpha, \beta] \subset [a, b]$, such that $(\Im V(\alpha) - \Im\lambda)(\Im V(\beta) - \Im\lambda) < 0$ and for any $x \in [\alpha, \beta], \Re V(x) < \Re\lambda$.

Proof. The set $\{x \in [a, b] : \Im V(x) = \Im\lambda\}$ is a compact set. Since $\Re V$ is continous, $\sup\limits_{\substack{x\in[a,b]\\ \Im V=\Im\lambda}} \Re V(x) = \Re V(x_0)$ for some $x_0 \in [a, b], \Im V(x_0) = \Im\lambda$. Now denote $\epsilon = \Re\lambda - \sup\limits_{\substack{x\in[a,b]\\ \Im V=\Im\lambda}} \Re V(x) = \Re\lambda - \Re V(x_0) > 0$ by the given condition.

$\Im V$ is continous over the interval $[\alpha, \beta]$, then it must be uniformly continuous. So we can find $n \in \mathbb{Z}_+$, such that for any $x, y \in [a, b], |x - y| < \frac{b-a}{n}, |\Re V(x) - \Re V(y)| < \frac{\epsilon}{2}$.

Now make a partition of $[a, b], a = x_0 < x_1 < \cdots < x_{n+1} = b$, with $x_i = a + \frac{i(b-a)}{n+1}$.

Consider the sequence $\{\Im V(x_i) - \Im\lambda\}_{i=0}^{n+1}$. The beginning term and the ending term are nonzero and have different signs. □

Proof. Let $\epsilon = \Re\lambda - \sup\limits_{\substack{a\le x\le b\\ \Im V(x)=\Im\lambda}} \Re V(x)$. We have $\epsilon > 0$. Otherwise there is a sequence $\{x_n\}_{n=1}^{\infty} \subset [a, b], \Im V(x_n) = \Im\lambda$ such that $\Re\lambda - \Re V(x_n) \to 0$ as $n \to \infty$. Choose a convergent subsequence $\{x_{n_k}\}$ and let the limit be x, then $x \in [a, b], \Im V(x) = \Im\lambda$ and $\Re\lambda - \Re V(x) = 0$, contradicting our assumption that "$\Re\lambda > \Re V(x)$ if $x \in (a, b), \Im V(x) = \Im\lambda$".

Since $V(\cdot)$ is continuous over $[a, b]$, then it is equicontinuous and there is $\delta > 0$ such that for any $x, y \in [a, b], |x - y| < \delta, |V(x) - V(y)| < \frac{\epsilon}{2}$.

Here we use binary search to find the desired interval.

Start from $a_0 = a, b_0 = b$. We proceed as follows:

1. Whether $b_k - a_k \geq \delta$? If so, turn to step 2; if not, choose $x \in (a_k, b_k)$ with $\Im V(x) = \Im\lambda$ and end the loop.
2. Whether there exists $a_k + \frac{\delta}{2} \leq x \leq b_k - \frac{\delta}{2}$ such that $\Im V(x) \neq \Im\lambda$? If the answer is yes, choose such an x and turn to step 3; if no, stop the loop.
3. Whether $(\Im V(a_k) - \Im\lambda)(\Im V(x) - \Im\lambda) < 0$? If so, let $a_{k+1} = a_k, b_{k+1} = x$; if not, let $a_{k+1} = x, b_{k+1} = b_k$. Turn to step 1.

One sees immediately that $(\Im V(a_k) - \Im\lambda)(\Im V(b_k) - \Im\lambda) < 0$ always holds and $b_k - a_k \leq b_{k-1} - a_{k-1} - \frac{\delta}{2}$. As a result, the loop must end after finitely many steps.

- If the loop ends in step 1, we get $b_k - a_k < \delta$, together with the point $x, a_k < x < b_k$ chosen satisfying $\Im V(x) = \Im\lambda$.

Then $\Re\lambda = \epsilon + \max_{\substack{y\in(a,b)\\ \Im V(y)=\Im\lambda}} \Re V(y) \geq \epsilon + \Re V(x)$.

Since $b_k - a_k < \delta$, for any $y \in [a_k, b_k]$, $|\Re V(y) - \Re V(x)| < \frac{\epsilon}{2}$, then $\forall y \in [a_k, b_k], \Re V(y) < \Re V(x) + \frac{\epsilon}{2} < \Re\lambda$.

- If the loop ends in step 2, we get $b_k - a_k \geq \delta, \forall a_k + \frac{\delta}{2} \leq y \leq b_k - \frac{\epsilon}{2}, \Im V(y) = \Im\lambda$.

  Then $\forall a_k + \frac{\delta}{2} \leq y \leq b_k - \frac{\epsilon}{2}, \Re\lambda = \epsilon + \max_{\substack{y'\in(a,b)\\ \Im V(y')=\Im\lambda}} \Re V(y') \geq \epsilon + \Re V(y) > \Re V(y)$.

  Now for $b_k - \frac{\delta}{2} < y \leq b_k$, $|y - (b_k - \frac{\delta}{2})| < \delta$, then $|V(y) - V(b_k - \frac{\epsilon}{2})| < \frac{\epsilon}{2}$. Then $\Re V(y) < \Re V(b_k - \frac{\epsilon}{2}) + \frac{\epsilon}{2} < \Re\lambda$.

  Similarly for $a_k \leq y < a_k + \frac{\delta}{2}$, we have $\Re V(y) < \Re\lambda$.

  In summary, $\forall y \in [a_k, b_k]$, $\Re V(y) < \Re\lambda$.

Therefore, (a_k, b_k) obtained in the end of loop is the interval we want to find. □

For convenience, we shall assume $a' = 0, b' = 1$ in the following.

2.2. Existence of appropriate phase function.

Lemma 2.2. There exists a complex-valued function $S \in C^{k+1}([0,1])$ such that

- $S'^2 + V = \lambda$
- $\min_{0\leq x\leq 1} \Im S(x) = 0$,
- $\Im S(0), \Im S(1) > 0$

Proof. Use the holomorphic branch of the function $z \mapsto \sqrt{z}$ over $\mathbb{C} \setminus (-\infty, 0]$. In other words, for z not lying on the negative axis, $\sqrt{z} = \sqrt{|z|}e^{\frac{1}{2}\arg z}, |\arg z| < \pi$.

Since $\Re(\lambda - V) > 0$, we have a well-defined function $\sqrt{\lambda - V}$ over $[0,1]$.

Let $\tilde{S}(x) = \operatorname{sgn}(\Im V(0) - \Im V(1)) \int_0^x \sqrt{\lambda - V(t)}dt$QUESTION: Integral for $0 \leq x \leq 1$.

Here $\operatorname{sgn} x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$. Then $\tilde{S} \in C^{k+1}([0,1])$.

Claim: $\Im\tilde{S}$ doesn't achieve minimal value on end points.

Note that $\Im\tilde{S}'(0) = \operatorname{sgn}(\Im V(0) - \Im V(1))\Im\sqrt{\lambda - V(0)}$.

Since $\Re(\lambda - V(0)) > 0, \operatorname{sgn}(\Im(\lambda - V(0))) = \operatorname{sgn}(\Im V(1) - \Im V(0))$, we have

$$\operatorname{sgn}\Im\sqrt{\lambda - V(0)} = \operatorname{sgn}(\Im V(1) - \Im V(0)) \tag{5}$$

Then $\Im\tilde{S}'(0) = \operatorname{sgn}(\Im V(0) - \Im V(1))\Im\sqrt{\lambda - V(0)} < 0$, implying $\Im\tilde{S}$ can't achieve minimal value at 0.

Similarly, $\Im\tilde{S}'(1) > 0$, implying $\Im\tilde{S}$ can't achieve minimal value at 1.

From these, we know $\Im\tilde{S}(0), \Im\tilde{S}(1)$ is strictly larger than the minimum of $\Im\tilde{S}$ over $[0,1]$.

Then take $S = \tilde{S} - \min_{x\in[0,1]} \Im\tilde{S}(x)$. Then S satisfies the required properties. □

2.3. **Succeeding terms in WKB approximation.** Now fix a phase function S constructed in the previous subsection, we want to construct functions $\psi_0, \psi_1, \cdots, \psi_k$ such that $\psi := e^{iS/h}(\psi_0 + h\psi_1 + \cdots + h^k\psi_k)$ solves $(L_h - \lambda)u = 0$ up to h^{k+1} order, i.e.

$$(L_h - \lambda)\psi = O(h^{k+1}\|\psi\|) \tag{6}$$

Direct computation gives us

$$\begin{aligned}
(L_h - \lambda)\psi &= -h^2\frac{d^2\psi}{dx^2} + (V-\lambda)\psi \\
&= -h^2\frac{d^2}{dx^2}\left(e^{iS/h}(\psi_0 + h\psi_1 + \cdots + h^k\psi_k)\right) + (V-\lambda)\psi \\
&= -h^2\left(-\frac{S'^2}{h^2}\psi + \frac{iS''}{h}\psi + 2i\frac{S'}{h}e^{iS/h}(\psi_0' + \cdots + h^k\psi_k') + e^{iS/h}(\psi_0'' + \cdots + h^k\psi_k'')\right) + (V-\lambda)\psi \\
&= (S'^2 + V - \lambda - ihS'')\psi - 2ihS'e^{iS/h}(\psi_0' + \cdots + h^k\psi_k') - h^2e^{iS/h}(\psi_0'' + \cdots + h^k\psi_k'') \\
&= -ihS''\psi - 2ihS'e^{iS/h}(\psi_0' + \cdots + h^k\psi_k') - h^2e^{iS/h}(\psi_0'' + \cdots + h^k\psi_k'') \\
&= -2ihe^{iS/h}\sqrt{S'}\frac{d}{dx}(\sqrt{S'}(\psi_0 + \cdots + h^k\psi_k)) - h^2e^{iS/h}(\psi_0'' + \cdots + h^k\psi_k'') \\
&= -he^{iS/h}\left(2i\sqrt{S'}\frac{d}{dx}\left(\sqrt{S'}\psi_0\right) + \sum_{j=1}^{k}h^j\left(2i\sqrt{S'}\frac{d}{dx}\left(\sqrt{S'}\psi_j\right) + \psi_{j-1}''\right) + h^{k+1}\psi_k''\right)
\end{aligned}$$

We obtain the following set of equations

$$\begin{cases} 2i\sqrt{S'}\frac{d}{dx}\left(\sqrt{S'}\psi_0\right) = 0 \\ 2i\sqrt{S'}\frac{d}{dx}\left(\sqrt{S'}\psi_j\right) + \psi_{j-1}'' = 0, j = 1, \cdots, k \end{cases} \tag{7}$$

Then we define $\psi_0 = \frac{1}{\sqrt{S'}}, \psi_j = \frac{i}{2\sqrt{S'}}\int_0^x \frac{\psi_{j-1}''}{\sqrt{S'}}dx$.

Note that $S' = \pm\sqrt{\lambda - V} \neq 0$ over $[0,1]$ since $\Re\lambda > \Re V$ over $[0,1]$. Then ψ_0 is well-defined, and obviously a C^{k+2} function. Then $\psi_1 = \frac{i}{2\sqrt{S'}}\int_0^x \frac{\psi_0''}{\sqrt{S'}}$ is a C^{k+1} function. Inductively, ψ_j is C^{k+2-j} function.

Then for $\psi = e^{iS/h}(\psi_0 + \cdots + h^k\psi_k)$, $(L_h - \lambda)\psi = -h^{k+2}e^{iS/h}\psi_k''$ and $\|(L_h - \lambda)\psi\| \leq h^{k+2}\|\psi_k''\|$.

2.4. **Cutoff function.** Since $\Im S(0), \Im S(1) > 0$, by continuity there exists $0 < \delta < 0.1$ and a constant $\epsilon > 0$, such that $\Im S(x) > \epsilon, \forall x \in [0, 2\delta]\bigcup[1-2\delta, 1]$. Then we take a function $\eta \in C_c^\infty(0,1)$, with $\eta|_{[\delta,1-\delta]} \equiv 1$ and take the quasimode $\tilde{\psi} = \psi\eta$. We shall prove the quasimode $\tilde{\psi}$ satisfies $\|(L_h - \lambda)\tilde{\psi}\| = O(h^{k+1}\|\tilde{\psi}\|)$.

2.5. **Estimates.** To prove

$$\|(L_h - \lambda)\tilde{\psi}\| = O(h^{k+1}\|\tilde{\psi}\|) \tag{8}$$

we shall show that $\|\tilde{\psi}\| \gtrsim h$ and $\|(L_h - \lambda)\tilde{\psi}\| = O(h^{k+2})$.

- We first give a lower bound to $\|\tilde{\psi}\|$ (i.e. an upper bound to the normalization constant).

  To begin with,

$$\|\tilde{\psi}\| \geq \|\eta e^{-\Im S/h} \max\{0, |\psi_0| - (h|\psi_1| + \cdots + h^k|\psi_k|)\}\|$$

  We have $\psi_0 = \frac{1}{\sqrt{S'}}$ continuous and nonzero over $[0,1]$, and $\psi_i, i = 1, \cdots, k$ bounded, then there exists $\epsilon > 0, 0 < h_1 < 1$, such that for $0 < h < h_1$, the following holds over the whole interval

$$|\psi_0| - (h|\psi_1| + \cdots + h^k|\psi_k|) > \epsilon \tag{9}$$

  .

  Then for $0 < h < h_1$, $\|\tilde{\psi}\| \geq \epsilon\|\eta e^{-\Im S/h}\|$.

  Take $x_0 \in (0,1)$ such that $S(x_0) = 0$, let $M = \sup_{x\in[0,1]} |S'(x)| < +\infty$, then $\Im S(x) \leq M|x - x_0|$. Then $\|\tilde{\psi}\| \geq \|\eta e^{-M|x-x_0|/h}\| \geq Ch$ for some $C > 0$.
- Then we estimate $\|(L_h - \lambda)\tilde{\psi}\|$.

$$\begin{aligned}\|(L_h - \lambda)\tilde{\psi}\| &= \|(L_h - \lambda)(\eta\psi)\| \\ &= \|\eta(L_h - \lambda)\psi - h^2\psi\frac{d^2\eta}{dx^2} - 2h^2\frac{d\eta}{dx}\frac{d\psi}{dx}\| \\ &\leq \|\eta(L_h - \lambda)\psi\| + \|h^2\psi\frac{d^2\eta}{dx^2}\| + \|2h^2\frac{d\eta}{dx}\frac{d\psi}{dx}\|\end{aligned}$$

  We estimate the three terms one by one:

  - $\|\eta(L_h-\lambda)\psi\| = \|\eta h^{k+2}\psi_k''\| \leq h^{k+2}\|\eta\|_{L^\infty}\|\psi_k''\| = O(h^{k+2})$ since $\psi_k \in C^2([0,1])$.
  - Note that for $x \in \mathrm{supp}\frac{d^2\eta}{dx^2}$, we have $\Im V(x) > \epsilon$.

    Then

$$\begin{aligned}\|h^2\psi\frac{d^2\eta}{dx^2}\| &= \|h^2\frac{d^2\eta}{dx^2}e^{iS/h}(\psi_0 + \cdots + h^k\psi_k)\| \\ &\leq \|h^2 e^{-\epsilon/h}(|\psi_0| + \cdots + |\psi_k|)\frac{d^2\eta}{dx^2}\| \\ &\leq h^2 e^{-\epsilon/h}\left\|\frac{d^2\eta}{dx^2}\right\|_{L^\infty}\sum_{i=0}^{k}\|\psi_i\| \\ &= O(h^{k+2})\end{aligned}$$

  - Note again that for $x \in \mathrm{supp}\frac{d\eta}{dx}$, we have $\Im V(x) > \epsilon$, then

$$\begin{aligned}\left|2h^2\frac{d\eta}{dx}\frac{d\psi}{dx}\right| &= \left|2h^2\frac{d\eta}{dx}\left(\frac{iS'}{h}\psi(x) + e^{iS/h}(\psi_0' + \cdots + h^k\psi_k')\right)\right| \\ &\leq \left|2h^2\frac{d\eta}{dx}\right|\left(\frac{|S'|}{h}e^{-\epsilon/h}\sum_i|\psi_i(x)| + e^{-\epsilon/h}\sum_i|\psi_i'(x)|\right)\end{aligned}$$

Then $\|2h^2\frac{d\eta}{dx}\frac{d\psi}{dx}\| \leq 2he^{-\epsilon/h}\left\|\frac{d\eta}{dx}\right\|_{L^\infty}\left(\|S'\|_{L^\infty}\sum_{i=1}^{k}\|\psi_i(x)\| + \sum_{i=1}^{k}\|\psi_i'(x)\|\right) = O(h^{k+2})$.

In summary, $\|(L_h - \lambda)\tilde{\psi}\| = O(h^{k+2})$.

Now we've completed the proof of our main theorem.

3. Application: h^N-pseudospectrum for purely imaginary smooth potential over a circle

Theorem. Let $L_h = -h^2\frac{d^2}{dx^2} + V(x)$ be a differential operator over a circle S^1, with $V(x) \in C^\infty(S^1)$, **purely imaginary** and **nonconstant**. Then the h^N-pseudospectrum for $N > 2$ is an open strip:

$$\{\lambda \in \mathbb{C} : \Re\lambda > 0, \min \Im V < \Im\lambda < \max \Im V\} \tag{10}$$

In what follows, we shall prove this in three steps:

1. Use our main theorem to show that h^N-pseudospectrum contains $\{\lambda \in \mathbb{C} : \Re\lambda > 0, \min \Im V < \Im\lambda < \max \Im V\}$
2. Show that the pseudospectrum lies in the closure of numerical image, $\{\lambda \in \mathbb{C} : \Re\lambda \geq 0, \min \Im V \leq \Im\lambda \leq \max \Im V\}$.
3. Show that the boundary doesn't belong to the pseudospectrum by giving a lower bound.

3.1. Application of main theorem.

Given $\lambda \in \mathbb{C}, \Re\lambda > 0, \Im\lambda \in (\min \Im V, \max \Im V)$, we want to show that λ lies in the h^N-pseudospectrum.

Now take an interval $[a, b]$ out of S^1, with $\Im V(a) = \min \Im V, \Im V(b) = \max \Im V$. Since $V \in C^\infty([a, b]), \Im V(a) < \Im\lambda < \Im V(b), \Re\lambda > 0 = \Re V(x), \forall x \in [a, b]$, the condition of our main theorem is satisfied, then it follows that λ belongs to the h^N-pseudospectrum of L_h for all $N > 0$ and therefore for $N = \infty$.

3.2. Numerical image.

Definition 1 (Numerical image)**.** The Numerical image of the operator family $L_h(0 < h < 1)$ is defined to be the set

$$\{\langle\psi, L_h\psi\rangle_{L^2(S^1)} : 0 < h < 1, \psi \in C^\infty(S^1), \|\psi\| = 1\} \tag{11}$$

Proposition 1. The h^N-pseudospectrum of L_h is contained in the closure of numerical image of L_h.

Proof. Assume λ lies in the h^N-pseudospectrum of L_h, then there is a constant $C > 0$ so that for any $h_0 > 0$, there exists $0 < h < h_0$ and $\|\psi_h\| = 1$ such that $\|(L_h - \lambda)\psi_h\| \leq Ch^N$.

Then $|\langle\psi_h, (L_h - \lambda)\psi_h\rangle| \leq \|\psi_h\|\|(L_h - \lambda)\psi_h\| \leq Ch^N$.

Then $|\langle\psi_h, L_h\psi_h\rangle - \lambda| \leq \|\psi_h\|\|(L_h - \lambda)\psi_h\| \leq Ch^N \leq Ch_0^N$.

Taking $h_0 \to 0^+$, we get that λ lies in the closure of the numerical image of L_h. □

Proposition 2. The numerical image of L_h is contained in the set

$$\{\omega \in \mathbb{C} : \Re\omega \geq 0, \min \Im V \leq \Im\omega \leq \max \Im V\} \tag{12}$$

Proof. Let ω lie in the numerical image, then for some $0 < h < 1$, $\psi_h \in C^\infty(S^1)$, $\|\psi_h\| = 1$, $\omega = \langle\psi_h, L_h\psi_h\rangle$.

Then $\omega = \int_{S^1} \overline{\psi}(-h^2\frac{d^2\psi}{dx^2} + V\psi) = \int_{S^1} h^2|\frac{d\psi}{dx}|^2 + V|\psi|^2$.

Since V is purely imaginary, $\Re\omega = \int_{S^1} h^2|\frac{d\psi}{dx}|^2 \geq 0$, and $\Im\omega = \int_{S^1} V|\psi_h|^2 \in [\min V, \max]$. □

We then easily get the following result:

Corollary. The h^N-pseudospectrum of L_h lies in the set

$$\{\omega \in \mathbb{C} : \Re\omega \geq 0, \min \Im V \leq \Im\omega \leq \max \Im V\} \tag{13}$$

3.3. **Boundary behavior: obstruction to the existence of quasimode.** What is left now is the boundary behavior, i.e. to show the followings don't belong to the pseudospectrum

(1) $\lambda = it, \min \Im V \leq t \leq \max \Im V$
(2) $\lambda = a + i \max \Im V, a > 0$
(3) $\lambda = a + i \min \Im V, a > 0$

The strategy is to prove first an inequality inspired by uncertainty principle, and then derive an estimate for $\|(L_h - \lambda)\psi\|$ for any $\psi \in H, \|\psi\| = 1$, which prohibits the possible of a quasimode.

1. $\lambda = it, \min \Im V \leq t \leq \max \Im V$.
First, let's prove an inequality inspired by uncertainty principle:

Lemma 3.1. Let $C = \frac{1}{2\pi}(\|(V - it)\| + \|(V - it)\|_{L^\infty})$. Then $\forall \psi \in C^\infty(S^1)$, we have the following inequality:

$$\|(V - it)\psi\| + C \left\| \frac{d\psi}{dx} \right\| \geq \|V - it\| \|\psi\| \tag{14}$$

Remark. *Taking* $\|\psi\| = 1$, *we can see that either* $\|(V - it)\psi\| \geq \frac{1}{2}\|V - it\| > 0$ *or* $\left\|\frac{d\psi}{dx}\right\| \geq \frac{1}{2C}\|V - it\| > 0$. *This means that either the position of a particle doesn't concentrate near the set* $V^{-1}(it)$ *or the momentum of a particle deviates from zero. So this inequality can be interpreted as a type of uncertainty principle.*

Proof. Let $\psi = \psi_0 + \psi_1$ be an orthogonal decomposition with $\psi_0 \equiv \int_{S^1} \psi$ (here S^1 is of perimeter 1). Via Fourier transformation, we can easily deduce that $\|\frac{d^k}{dx^k}\psi_1\| \geq (2\pi)^k \|\psi_1\|$ for $k \geq 1$.
We have

$$\begin{aligned}\|(V - it)\psi\| &= \|(V - it)(\psi_0 + \psi_1)\| \\ &\geq \|(V - it)\psi_0\| - \|(V - it)\psi_1\| \\ &\geq \|(V - it)\| |\psi_0| - \|(V - it)\|_{L^\infty} \|\psi_1\| \\ &= \|(V - it)\| \|\psi_0\| - \|(V - it)\|_{L^\infty} \|\psi_1\|\end{aligned}$$

Note that $\left\|\frac{d\psi}{dx}\right\| = \|\frac{d\psi_0}{dx} + \frac{d\psi_1}{dx}\| = \|\frac{d\psi_1}{dx}\| \geq 2\pi\|\psi_1\|$.

$$\begin{aligned}\|(V-it)\psi\| + C\left\|\frac{d\psi}{dx}\right\| &\geq (\|(V-it)\|\|\psi_0\| - \|(V-it)\|_{L^\infty}\|\psi_1\|) \\ &+ \frac{1}{2\pi}(\|(V-it)\| + \|(V-it)\|_{L^\infty})2\pi\|\psi_1\| \\ &\geq \|(V-it)\|(\|\psi_0\| + \|\psi_1\|) \\ &\geq \|(V-it)\|(\|\psi_0 + \psi_1\|) \\ &= \|(V-it)\|\|\psi\|\end{aligned}$$

□

Now we prove the following proposition which excludes $\lambda = it$ from pseudospectrum.

Proposition 3. There exists a constant $C > 0$, such that for any $\psi \in C^\infty(S^1), \|\psi\| = 1$, we have

$$\|(L_h - it)\psi\| \geq Ch^2 \tag{15}$$

Proof. Fix an arbitrary $\psi \in C^\infty(S^1), \|\psi\| = 1$. Denote $\|(L_h - it)\psi\|$ by τ. We want to give a lower bound to τ via the above inequality. We're going to use τ to give an upper bound of $\|(V-it)\psi\|$ and $\left\|\frac{d\psi}{dx}\right\|$ in order to give a lower bound of τ.

(i) Estimate of $\|\frac{d\psi}{dx}\|$ with τ.Note that

$$\begin{aligned}\|h\frac{d}{dx}\psi\|^2 &= -\int h^2\overline{\psi}\frac{d^2\psi}{dx^2} \\ &= \Re(-\int h^2\overline{\psi}\frac{d^2\psi}{dx^2} + (V-it)|\psi|^2) \\ &= \Re\langle\psi, (L_h - it)\psi\rangle \\ &\leq \|\psi\|\|(L_h - it)\psi\| \\ &= \tau\end{aligned}$$

Then $\|\frac{d\psi}{dx}\| \leq \frac{\sqrt{\tau}}{h}$.

(ii) Before estimation of $\|(V-it)\psi\|$, we shall estimate $\|\frac{d^2\psi}{dx^2}\|$ using a method similar to a priori estimates in elliptic pde theory.
On the one hand, by Cauchy inequality

$$\langle -\frac{d^2\psi}{dx^2}, (L_h - it)\psi\rangle \leq \left\|\frac{d^2\psi}{dx^2}\right\| \cdot \|(L_h - it)\psi\| \leq \tau\left\|\frac{d^2\psi}{dx^2}\right\|$$

On the other hand, by partial integration and Cauchy inequality

$$\begin{aligned}
\langle -\frac{d^2\psi}{dx^2}, (L_h - it)\psi\rangle &= h^2 \left\| \frac{d^2\psi}{dx^2} \right\|^2 - \int_{S^1} (\frac{d^2}{dx^2}\overline{\psi})(V - it)\psi \\
&= h^2 \left\| \frac{d^2\psi}{dx^2} \right\|^2 + \int_{S^1} (V - it) \left| \frac{d\psi}{dx} \right|^2 + \int_{S^1} V'\psi \frac{d\overline{\psi}}{dx} \\
&\geq h^2 \left\| \frac{d^2\psi}{dx^2} \right\|^2 - \|V - it\|_{L^\infty} \left\| \frac{d\psi}{dx} \right\|^2 - \|V'\|_{L^\infty} \|\psi\| \left\| \frac{d\psi}{dx} \right\| \\
&\geq h^2 \left\| \frac{d^2\psi}{dx^2} \right\|^2 - \|V - it\|_{L^\infty} \frac{\tau}{h^2} - \|V'\|_{L^\infty} \frac{\sqrt{\tau}}{h}
\end{aligned}$$

Combining the two above, we have

$$h^2 \left\| \frac{d^2\psi}{dx^2} \right\|^2 - \|\frac{d^2\psi}{dx^2}\|\tau \leq \|V - it\|_{L^\infty} \frac{\tau}{h^2} + \|V'\|_{L^\infty} \frac{\sqrt{\tau}}{h} \tag{16}$$

Then

$$h^2(\left\| \frac{d^2\psi}{dx^2} \right\| - \frac{\tau}{2h^2})^2 \leq \|V - it\|_{L^\infty} \frac{\tau}{h^2} + \|V'\|_{L^\infty} \frac{\sqrt{\tau}}{h} + \frac{\tau^2}{4h^2} \tag{17}$$

Now we get the estimate for second derivative as:

$$\left\| \frac{d^2\psi}{dx^2} \right\| \leq \frac{\tau}{2h^2} + \frac{1}{h}\sqrt{\|V - it\|_{L^\infty} \frac{\tau}{h^2} + \|V'\|_{L^\infty} \frac{\sqrt{\tau}}{h} + \frac{\tau^2}{4h^2}} \tag{18}$$

Assume without loss of generality that $\tau \leq h^2 < 1$.
Let $C_1 = \sqrt{1 + \|V - it\|_{L^\infty} + \|V'\|_{L^\infty}}$.
Then (18) can be simplified in such a way:

$$\begin{aligned}
\left\| \frac{d^2\psi}{dx^2} \right\| &\leq \frac{\tau}{2h^2} + \frac{1}{h}\sqrt{\|V - it\|_{L^\infty} \frac{\tau}{h^2} + \|V'\|_{L^\infty} \frac{\sqrt{\tau}}{h} + \frac{\tau}{4h^2}} \\
&= \frac{\tau}{2h^2} + \frac{1}{h}\sqrt{(\frac{1}{4} + \|V - it\|_{L^\infty})\frac{\tau}{h^2} + \|V'\|_{L^\infty} \frac{\sqrt{\tau}}{h}} \\
&\leq \frac{\tau}{2h^2} + \frac{1}{h}\sqrt{(1 + \|V - it\|_{L^\infty})\frac{\sqrt{\tau}}{h} + \|V'\|_{L^\infty} \frac{\sqrt{\tau}}{h}} \\
&\leq \frac{\tau}{2h^2} + C_1 \frac{\sqrt[4]{\tau}}{h^{3/2}} \\
&\leq \frac{\tau^{3/4}}{2\sqrt{h}} \frac{\sqrt[4]{\tau}}{h^{3/2}} + C_1 \frac{\sqrt[4]{\tau}}{h^{3/2}} \\
&\leq (C_1 + 1) \frac{\sqrt[4]{\tau}}{h^{3/2}}
\end{aligned}$$

(iii) Now we can estimate $\|(V-it)\psi\|$ via triangular inequality:

$$\begin{aligned}\|(V-it)\psi\| &= \|(L_h - it + h^2\frac{d^2}{dx^2})\psi\| \\ &\leq \|(L_h-it)\psi\| + h^2\left\|\frac{d^2\psi}{dx^2}\right\| \\ &\leq \tau + h^2(C_1+1)\frac{\sqrt[4]{\tau}}{h^{3/2}} \\ &\leq \sqrt[4]{\tau}h^{3/2} + (C_1+1)\sqrt[4]{\tau}\sqrt{h} \\ &\leq (C_1+2)\sqrt[4]{\tau}\sqrt{h}\end{aligned}$$

(iv) Finally, we can estimate τ.
Let $C_2 = \frac{1}{2\pi}(\|(V-it)\| + \|(V-it)\|_{L^\infty})$, then by the above lemma:

$$\|(V-it)\psi\| + C_2\left\|\frac{d\psi}{dx}\right\| \geq \|(V-it)\|$$

Using the above results, we have

$$\|(V-it)\psi\| + C_2\left\|\frac{d\psi}{dx}\right\| \leq (C_1+2)\sqrt[4]{\tau}\sqrt{h} + C_2\frac{\sqrt{\tau}}{h} \leq (C_1+C_2+2)\sqrt[4]{\tau} \tag{19}$$

Then $\tau \geq \left(\frac{\|(V-it)\|}{C_2+C+2}\right)^4$.

Therefore, we get $\tau \geq \left(\frac{\|(V-it)\|}{C_2+C+2}\right)^4$ assuming $\tau \leq h^2$.

In conclusion,$\tau \geq C_3 h^2$ for $0<h<1$, where $C_3 = \max(1, \left(\frac{\|(V-it)\|}{C_2+C+2}\right)^4)$. Our proof is complete.

□

Remark. *This result can't be improved in certain sense. We show for a specific potential V the existence of a family $\psi_h \in C^\infty(S^1), \|\psi_h\| = 1, 0<h<1$ such that $\|(L_h-\lambda)\psi_h\| = O(h^2)$.*

Just take V such that supp $V \subset [0,\frac{1}{2}]$ and $\lambda = 0$. Take $\psi_h = \eta$, where $\eta \in C^\infty(S^1), \|\eta\| = 1$ and supported in $[\frac{1}{2},1]$. Then $\|(L_h-\lambda)\psi_h\| = h^2\|\frac{d^2}{dx^2}\eta\| = O(h^2)$.

Note: The following hasn't been completed yet.

2. $\lambda = a + i\max\Im V, a>0$ doesn't belong to the pseudospectrum for $N>1$.
To show this, we first prove an "uncertainty principle" type inequality.

Lemma 3.2. For a constant $a>0$, and a fixed continuous function $K \not\equiv 0$ over S^1, there exists $A,B,D>0$ such that the following inequality holds for any $0<h<0.1\sqrt{a}, \psi \in C^\infty(S^1), \|\psi\| = 1$:

$$\frac{A}{\sqrt{h}}\|(-h^2\frac{d^2}{dx^2} - a)\psi\| + B\|K\psi\| \geq D \tag{20}$$

Proof. First choose $n \in \mathbb{N}$ such that $|n^2h^2 - a|$ is as small as possible. Then $|n - \frac{\sqrt{a}}{h}| < 1, n - \frac{1}{2} > 0.1\sqrt{a}/h$.

For any $m \in \mathbb{Z}, m \neq \pm n$, then $|m^2 - n^2| \geq 2n - 1$. Then we have

$$|m^2h^2 - a| \geq \frac{1}{2}(|m^2h^2 - a| + |n^2h^2 - a|) \tag{21}$$

$$\geq \frac{1}{2}|m^2 - n^2|h^2 \tag{22}$$

$$\geq (n - \frac{1}{2})h^2 \tag{23}$$

$$\geq 0.1\sqrt{a}h \tag{24}$$

For an aribitrary $\psi \in C^\infty(S^1)$, make the orthogonal decomposition $\psi = \psi_0 + \psi_1$ where ψ_0 is the projection of ψ into the space spanned by the two functions $e^{i2n\pi x}, e^{-i2n\pi x}$.

Note that

- ψ_0 is periodic with period $\frac{1}{n}$.
- $\|(-h^2\frac{d^2}{dx^2} - a)\psi_1\| \geq 0.1\sqrt{a}h\|\psi_1\|$ via Fourier decomposition.

Then we have

$$\begin{aligned}\|(-h^2\frac{d^2}{dx^2} - a)\psi\| &= \sqrt{\|(-h^2\frac{d^2}{dx^2} - a)\psi_0\|^2 + \|(-h^2\frac{d^2}{dx^2} - a)\psi_1\|^2} \\ &\geq \|(-h^2\frac{d^2}{dx^2} - a)\psi_1\| \\ &\geq 0.1\sqrt{a}h\|\psi_1\|\end{aligned}$$

For the second term, we have $\|K\psi\| \geq \|K\psi_0\| - \|K\psi_1\|$.

Now we need some aid from elementary mathematical analysis:

Claim: For a fixed continuous function $K \not\equiv 0$ over the one-dimensional torus S^1, there exists a constant $C > 0$, such that for any $m \in \mathbb{N}, \psi_m \in \mathbb{C}e^{imx} + \mathbb{C}e^{-imx}, \|\psi_m\| = 1, \|K\psi_m\| > C$.

(The claim is very easy to prove, since $\lim_{m\to\infty} \|K\psi_m\| = \|K\|$ uniformly for any sequence $\{\psi_m\}, \psi_m \in \mathbb{C}e^{imx} + \mathbb{C}e^{-imx}, \|\psi_m\| = 1$. QUESTION: Do I need to explain this in detail?)

As a result of this claim, there is a constant $C > 0$ depending only on K such that $\|K\psi_0\| \geq C_K\|\psi_0\|$.

Then we get

$$\|K\psi\| \geq C_K\|\psi_0\| - \|K\|_{L^\infty}\|\psi_1\| \tag{25}$$

Now for $A, B > 0$, we get

$$\frac{A}{\sqrt{h}}\|(-h^2\frac{d^2}{dx^2} - \Re\lambda)\psi\| + B\|(V - i\Im\lambda)\psi\| \geq AC_{V,\lambda}\|\psi_0\| + (A\sqrt{0.1\sqrt{\Re\lambda}} - B(\max\Im V - \min\Im V))\|\psi_1\| \tag{26}$$

For A large enough and using $\|\psi_0\|^2 + \|\psi_1\|^2 = 1$, we can find $D > 0$, such that

$$\frac{A}{h}\|(-h^2\frac{d^2}{dx^2} - \Re\lambda)\psi\| + B\|(V - i\Im\lambda)\psi\| \geq D \tag{27}$$

Here A, B, D depends only on λ, V not on h, ψ. □

Proposition 4. There exists a constant $C > 0$, such that $\|(L_h - \lambda)\psi\| \geq Ch\|\psi\|$ for any $\psi \in C^\infty(S^1)$.

Proof. Now fix $0 < h < 0.1\sqrt{\Re\lambda}$.

First choose $n \in \mathbb{N}$ such that $|n^2h^2 - \Re\lambda|$ is as small as possible. Then $|n - \frac{\sqrt{\Re\lambda}}{h}| < 1$.

For any $m \in \mathbb{Z}, m \neq \pm n$, $|m^2h^2 - \Re\lambda| \geq \frac{1}{2}(|m^2h^2 - \Re E| + |n^2h^2 - \Re\lambda|) \geq \frac{1}{2}|m^2 - n^2|h^2 \geq (n - \frac{1}{2})h^2 \geq 0.1\sqrt{Re\lambda}h$.

For an aribitrary $\psi \in C^\infty(S^1)$, make the orthogonal decomposition $\psi = \psi_0 + \psi_1$ where ψ_0 is the projection of ψ into the space spanned $e^{i2n\pi x}, e^{-i2n\pi x}$. Note that ψ_0 is periodic with period $\frac{1}{n}$.

Firstly, we have

$$\begin{aligned}\|(-h^2\frac{d^2}{dx^2} - \Re\lambda)\psi\|^2 &= \|(-h^2\frac{d^2}{dx^2} - \Re\lambda)\psi_0\|^2 + \|(-h^2\frac{d^2}{dx^2} - \Re\lambda)\psi_1\|^2 \\ &\geq \|(-h^2\frac{d^2}{dx^2} - \Re\lambda)\psi_1\|^2 \\ &\geq 0.1\sqrt{\Re\lambda}h\|\psi_1\|^2\end{aligned}$$

Secondly, we have $\|(V - i\Im\lambda)\psi\| \geq \|(V - i\Im\lambda)\psi_0\| - \|(V - i\Im\lambda)\psi_1\|$.

Use the following lemma,

Lemma 3.3. For a fixed nonnegative, nonconstant continuous function K over the one-dimensional torus S^1, there exists a constant $C > 0$, such that for any $m \in \mathbb{N}, \psi \in \mathbb{C}e^{imx} + \mathbb{C}e^{-imx}$, $\|\psi\| = 1$, $\|K\psi\| > C$.

there is a constant $C_{V,\lambda}$ depending only on $(\max_{x\in S^1} \Im V(x)) - \Im V$, such that $\|(V - i\Im\lambda)\psi_0\| \geq C_{V,\lambda}\|\psi_0\|$.

Then we get

$$\|(V - i\Im\lambda)\psi\| \geq C_{V,\lambda}\|\psi_0\| - (\max \Im V - \min \Im V)\|\psi_1\| \tag{28}$$

Now for $A, B > 0$, we get

$$\frac{A}{\sqrt{h}}\|(-h^2\frac{d^2}{dx^2} - \Re\lambda)\psi\| + B\|(V - i\Im\lambda)\psi\| \geq AC_{V,\lambda}\|\psi_0\| + (A\sqrt{0.1\sqrt{\Re\lambda}} - B(\max \Im V - \min \Im V))\|\psi_1\| \tag{29}$$

For A large enough and using $\|\psi_0\|^2 + \|\psi_1\|^2 = 1$, we can find $D > 0$, such that

$$\frac{A}{\sqrt{h}}\|(-h^2\frac{d^2}{dx^2} - \Re\lambda)\psi\| + B\|(V - i\Im\lambda)\psi\| \geq D \tag{30}$$

Here A, B, D depends only on λ, V not on h, ψ.

Now denote $\tau = \|(L_h - \lambda)\psi\|$.
We have $\int_{S^1}(\Im\lambda - \Im V)|\psi|^2 = -\Im\langle\psi, (L_h - \lambda)\psi\rangle \leq \|\psi\|\tau = \tau$.
Since $\Im\lambda - \Im V \geq 0$, we have

$$\begin{aligned}\|(i\Im\lambda - V)\psi\|^2 &= \int_{S^1}(\Im\lambda - \Im V)^2|\psi|^2 \\ &\leq (\max\Im V - \min\Im V)\int_{S^1}(\Im\lambda - \Im V)|\psi|^2 \\ &\leq (\max\Im V - \min\Im V)\tau\end{aligned}$$

Then $\|(V - i\Im\lambda)\psi\| \leq \sqrt{(\max\Im V - \min\Im V)\tau}$.
Since $(L_h - \lambda)\psi = (-h^2\frac{d^2}{dx^2} - \Re\lambda)\psi + (V - i\Im\lambda)\psi$,

$$\|(-h^2\frac{d^2}{dx^2} - \Re\lambda)\psi\| \leq \sqrt{(\max\Im V - \min\Im V)\tau} + \tau \tag{31}$$

Assume $\tau \leq \max\Im V - \min\Im V$, then

$$\|(-h^2\frac{d^2}{dx^2} - \Re\lambda)\psi\| \leq 2\sqrt{(\max\Im V - \min\Im V)\tau} \tag{32}$$

Use the "uncertainty relation" (26), we get

$$(\frac{2A}{\sqrt{h}} + B)\sqrt{(\max\Im V - \min\Im V)\tau} \geq D \tag{33}$$

Then

$$(2A + B)\sqrt{(\max\Im V - \min\Im V)\tau} \geq D\sqrt{h} \tag{34}$$

Then we get $\tau \gtrsim h$. □

Remark. *This can't be improved. We now show for a specific potential* V *the existence of a family* $\psi_h \in C^\infty(S^1), \|\psi_h\| = 1, 0 < h < 1$ *such that* $\|(L_h - \lambda)\psi_h\| = O(h)$.

Just take $V \leq 0$ *such that supp* $V \subset [0, \frac{1}{2}]$ *and* $\lambda = a^2 > 0$. *Take* $\psi_h = e^{iax/h}\eta$, *where the fixed function* $\eta \in C^\infty(S^1), \|\eta\| = 1$ *and supported in* $[\frac{1}{2}, 1]$. *Then* $\|(L_h - \lambda)\psi_h\| = \|(h^2\frac{d^2}{dx^2} - a^2)(e^{iax/h}\eta)\| = \|h^2e^{iax/h}\frac{d^2\eta}{dx^2} + 2haie^{iax/h}\eta'\| \leq h^2\|\frac{d^2\eta}{dx^2}\| + 2ha\|\eta'\| = O(h)$.