Computing examples of Hurwitz correspondences in cyclic and non-cyclic portraits.

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Abstract

A Hurwitz correspondence is a multi-valued self-map of the moduli space of genus zero curves with n marked points, $M_{0,n}$. Considering two n-marked curves $(C, a_1, ..., a_n)$ and $(D, b_1, ..., b_n)$, given a map ρ from $\{1, ..., n\}$ to itself and some ramification data, one obtains a Hurwitz correspondence on $M_{0,n}$ which sends a marked curve $(D, b_1, ..., b_n)$ to any of the marked curves $(C, a_1, ..., a_n)$ for which we have a rational map $\phi : C \to D$ with $a_i \to b_{\rho(i)}$ that satisfies the required ramification data. The dynamical degree of a Hurwitz correspondence h is the maximal eigenvalue of the induced map $(h)_*$ on homology. We study the case of the induced map on the divisor class group with a particular focus on ramification types, as studied by Koch and Roeder. This experimental study considers the eigenvalues for several Hurwitz correspondences for maps ϕ of low degree. Our main experimental findings are that $\forall \lambda_i$, where λ_i is not a dynamical degree but is an eigenvalue of the induced map on homology $(h)_*$, then $1 \leq |\lambda_i| \leq k$, where k is the global degree of the rational map ϕ. Our results also suggest that when $k < \lambda_{max}$, where λ_{max} is the dynamical degree for a given induced map $(h)_*$ and k is the global degree of the rational map ϕ, then the multiplicity of λ_{max} is 1.

1 Definitions and background

First, we will begin by defining n-pointed smooth rational curves and the moduli space of curves.

Definition 1. *An n-pointed rational curve, $(C, p_1, ..., p_n)$, is a projective rational curve C equipped with a choice of n distinct points, called the marked points.*

Definition 2. *For $n \geq 3$, there is a fine moduli space, $M_{0,n}$, for the problem of classifying n-pointed smooth rational curves up to isomorphism.*

Example 1. When $n = 3$, there is a unique isomorphism from any smooth rational curve with three marked points (C, p_1, p_2, p_3) to $(\mathbb{P}^1, 0, 1, \infty)$. In this case $M_{0,3}$ is a single point.

For $n = 4$, every rational curve with four distinct marked points $(C, p_1, p_2, p_3, p_4)$is isomorphic to $(\mathbb{P}^1, 0, 1, \infty, q)$ for some $q \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. This shows that $M_{0,4}$ is isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

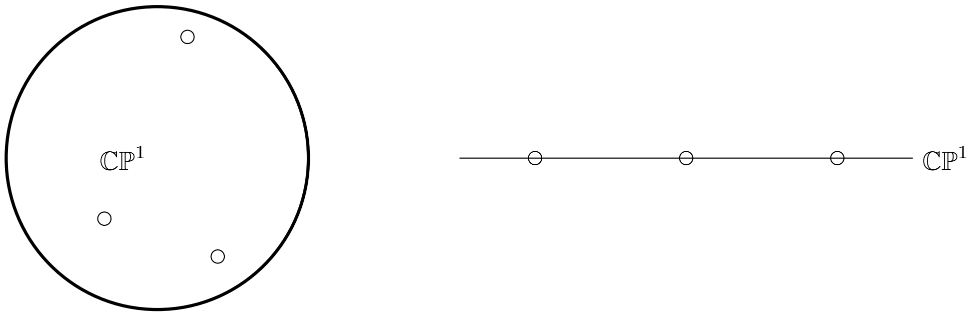


Figure 1: Two ways of representing $M_{0,4}$ where the sphere and the line are one copy of $\mathbb{CP}^1$ and the empty circles represent removed points $0, 1, \infty$.

In general, we obtain the space $M_{0,n}$ via an isomorphism to the cartesian product of $n-3$ copies of $M_{0,4}$ minus all the possible diagonals.

$$M_{0,n} = \underbrace{M_{0,4} \times ... \times M_{0,4}}_{n-3} \setminus \bigcup diagonals$$

For this reason, we define $M_{0,n}$ as a smooth moduli space of dimension $n-3$.

Example 2. We show the example of n=5 in an attempt to exemplify the generalization presented above. We know that $M_{0,5} = M_{0,4} \times M_{0,4} \setminus$ Diagonals. In this case, we have that every rational curve with five distinct marked points $(C, p_1, p_2, p_3, p_4, p_5)$ is isomorphic to $(C, 0, 1, \infty, q_1, q_2)$ for some $q_1, q_2 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, such that $q_1 \neq 0, 1, \infty$, $q_2 \neq 0, 1, \infty$ and $q_1 \neq q_2$, after applying a Möbius transformation. This can be represented in the following diagram.

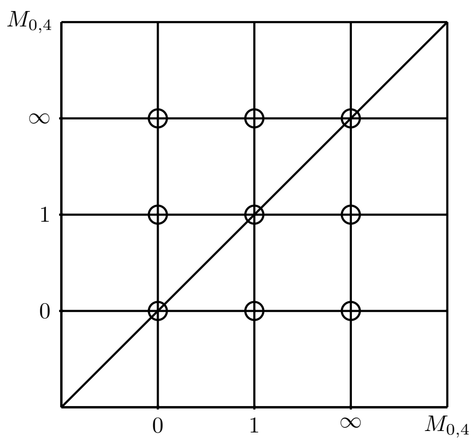


Figure 2: In this figure we show the grid formed by $M_{0,4} \times M_{0,4} \setminus$ Diagonals. The vertical, horizontal and diagonal lines correspond to taking away the points $0, 1, \infty$ from each copy of $M_{0,4}$. This corresponds to the uniqueness condition that $q_1 \neq 0, 1, \infty$, $q_2 \neq 0, 1, \infty$ and $q_1 \neq q_2$.

To do further study of the Hurwitz correspondences between moduli spaces of n-pointed rational curves we need to consider a suitable compactification of this space, namely we need to compactify $M_{0,n}$. To do this, we will consider the approach of using trees of projective lines, which was suggested by Knudsen and Mumford [5]. We will define a tree of projective lines as follows:

Definition 3. *A tree of projective lines is a connected curve with the properties:*

- *Each irreducible compononent is isomorphic to a projective line.*
- *The points of intersection of the components are ordinary double points.*
- *There are no closed circuits. That is, if any node is removed, the curve becomes disconnected.*

These three properties are equivalent to saying that the curve has arithmetic genus zero.

With this definition of a tree of projective lines, we define a stable curve as follows.

Definition 4. *A stable n-pointed rational curve is a tree C of projective lines, with n distinct marked points which are smooth points of C such that every irreducible component has at least 3 special points, where a special point is a marked point or a node.*

Example 3. In Figure 3 we show three examples of trees of projective lines with 5 marked points where 2 of the examples are stable and the other is not.

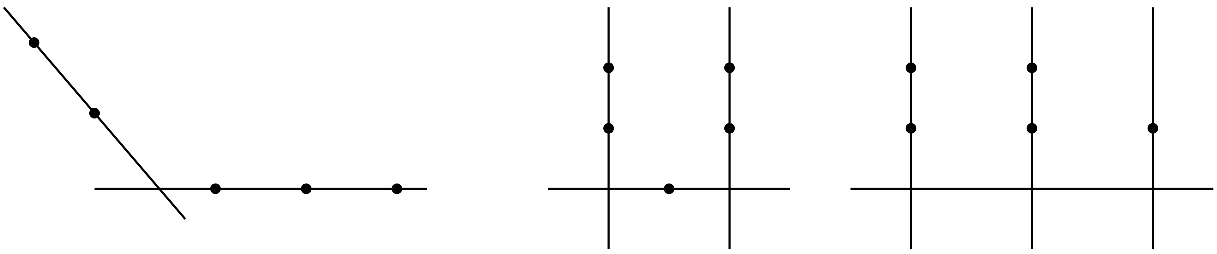


Figure 3: We see three examples of trees of projective lines with 5 marked points. The pictures in the left and the center are stable and the last one is not.

Given a stable $n+1$-pointed curve, we can associate this curve to an n-pointed curve by removing the p_{n+1} point. When we remove this point, we might get an unstable curve, and to get back a stable curve, there are two different procedures we can follow.

- Case 1: Suppose we have the point p_{n+1} in a component with just two other nodes, such that when we forget point p_{n+1}, we are left with a component joining two more components without a marked point on it, then we contract the nodes of that component into just one.
- Case 2: Suppose we have the point p_{n+1} in one component with just one node and another marked point p_k. Then, when we forget point p_{n+1}, the component where this point lied is contracted and the node of the intersection is now the marked point p_k.

Example 4.

With the construction of the trees of projective lines, we can construct $\overline{M_{0,n}}$. The elements of this space are classified by stratum. Each of this stratum corresponds to trees of projective lines that have δ nodes, which is the same as the codimension of the stratum. We can justify this claim by considering

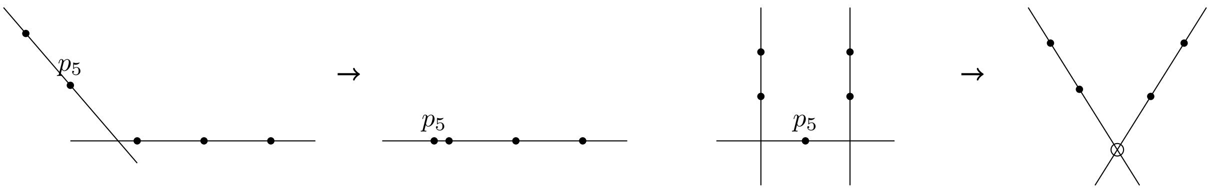


Figure 4: In the left picture, we forget point p_5 and stabilize following case 2. In the right picture, we forget point p_5 and stabilize following case 1.

the degrees of freedom for the components of the trees of projective lines. First, consider that for a tree with δ nodes, we have $n + 2\delta$ special points when counting all components. From this number of special points, we will fix 3 special points in each of the $\delta + 1$ components, and so we have that the degrees of freedom for any curve is $n + 2\delta - 3 * (\delta + 1) = n - 3 - \delta$. Now, since in total we have $n - 3$ degrees of freedom, then $n - 3 - (n - 3 - \delta) = \delta$ is the codimension of the given curve.

Example 5. We can see as an examples the stratification of $\overline{M_{0,6}}$.

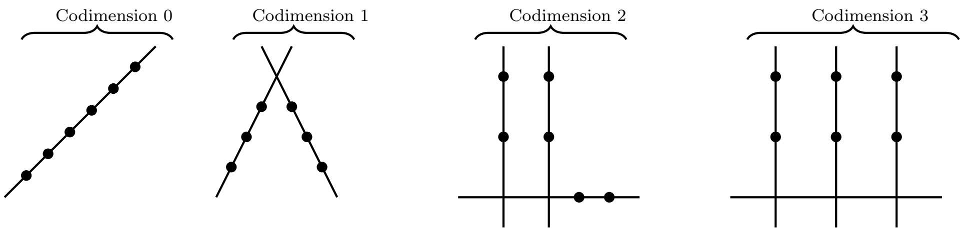


Figure 5: In Figure 5, we see the different stratum for $\overline{M_{0,n}}$, when $n = 6$. In this case, we only show one possible configuration for the marked points in each of the stratum but there are several ways to distribute the number of marked points along all the components of the curve.

We call the stratum of codimension 1 "boundary divisors". There will be an irreducible boundary divisor, denoted $D(A|B)$, for each partition $[n] = A \cup B$ where A and B are dosjoint and $|A| \geq 2$, $|B| \geq 2$, and in the stratum of codimension 1, one of the components of the curve has A marked points and the other component has B marked points.

Now, to study maps between $\overline{M_{0,n}}$ to $\overline{M_{0,n}}$, we need to understand the homology group $H_{2(n-4)}(x)$ of the divisors class group.The following description gives all relations between boundary divisors.

- $D(A|B) = D(A^c|B^c)$
- $D(A|B) = 0$ if $A = 0$ or $A = [n]$
- $\sum_{\substack{i,j \in A \\ k,l \in B}} D(A|B) = \sum_{\substack{i,k \in A \\ j,l \in B}} D(A|B) = \sum_{\substack{i,l \in A \\ j,k \in B}} D(A|B)$

With the background we have presented, we can define Hurwitz correspondences, which are multi-valued self-maps of the moduli space $M_{0,n}$.

First we will define what a Hurwitz space is and from there we will go on to define a Hurwitz correspondence.

Definition 5. *We follow the definition found in [3]. First fix the data:*

- *A and B are finite sets with at least 3 elements (representing the marked points on source and target curves)*
- *a degree d of the mapping, where d is a positive integer.*
- *F: $A \to B$ a map between the marked points on source and target curve.*
- *br: $B \to \{$partitions of d$\}$ (branching)*
- *rm: $A \to \mathbb{Z}^{>0}$ (ramification)*

where we satisfy the following two conditions:

- $\sum_{b\in B}(d -$ *length of br(b)* $= 2d - 2$
- *for all $b \in B$, the multiset $(rm(a))_{a\in F^{-1}(b)}$ is a submultiset of br(b)*

With this background we define a ***Hurwitz space*** *as a smooth quasiprojective variety $\mathcal{H} = \mathcal{H}(A, B, d, F, br, rm)$ which parametrizes isomorphisms $f : C \to D$ up to isomorphism, where*

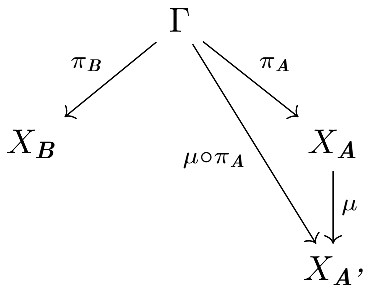
- *C and D are connected genus zero curves with A-marked and B-marked points respectively*
- *f has degree d*
- *we have that for all $a \in A$, $f(a) = F(A)$*
- *for all $b \in B$, the branching of f over b is given by the partition br(b)*
- *for all $a \in A$, the local degree of f at a is equal to $rm(a)$.*

With these concepts, we follow [3] and define what a Hurwitz correspondence is:

Definition 6. *Let* ***A'*** *be an arbitrary subset of* ***A*** *with at least 3 elements. We have a forgetful morphism $\mu : \mathcal{M}_{0,\boldsymbol{A}} \to \mathcal{M}_{0,\boldsymbol{A'}}$. Let Γ be a union of connected components of some Hurwitz Space $\mathcal{H}$. Suppose we have that $X_{\boldsymbol{A}}$,$X_{\boldsymbol{A'}}$ and $X_{\boldsymbol{B}}$ are smooth projective compactifications of $\mathcal{M}_{0,\boldsymbol{A}}$, $\mathcal{M}_{0,\boldsymbol{A'}}$ and $\mathcal{M}_{0,\boldsymbol{B}}$ respectively. Then, we have that $\mathcal{H}$ admits two maps, the first one is π_B to $\mathcal{X}_{\boldsymbol{B}}$ which sends elements in the Hurwitz Space to a source curve and the second one is π_A to $\mathcal{X}_{\boldsymbol{A}}$ which sends elements in the Hurwitz Space to a target curve.*

This implies we have the following rational correspondence:

$$(\Gamma, \pi_{\boldsymbol{B}}, \mu \circ \pi_A) = X_{\boldsymbol{B}} \rightrightarrows X_{\boldsymbol{A'}}$$



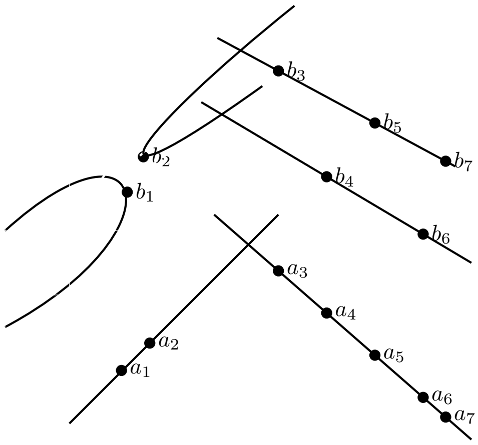
As noted in [3], if the Hurwitz Space $\mathcal{H}$ is not empty, then the map $\pi_{\mathbf{B}}$ is a finite covering map and $\pi_{\mathbf{A}} \circ \pi_{\mathbf{B}}^{-1}$ would define a multivalued map from $\mathcal{M}_{0,\mathbf{A}}$ to $\mathcal{M}_{0,\mathbf{B}}$.

Finally, we present the concept of an admissible cover, following the definition presented in [3]. Moduli spaces of admissible covers will be used to parametrize ramified maps between nodal curves.

Definition 7. *Fix a Hurwitz space* $(\boldsymbol{A}, \boldsymbol{B}, \mathrm{d}, \mathrm{F}, br, rm)$. *An* $(\boldsymbol{A}, \boldsymbol{B}, \mathrm{d}, \mathrm{F}, br, rm)$*-admissible cover is a map of curves* $f : C \to D$*, where*

1. *D is a $\boldsymbol{B}$-marked genus zero curve*
2. *C is a connected nodal genus zero curve, with an injection from $\boldsymbol{A}$ into C.*
3. *$f : C \to D$ is a map of degree d, such that:*
   - *for all $a \in \boldsymbol{A}$, $f(a) = F(a)$*
   - *for all $b \in \boldsymbol{B}$, the branching f over b is equal to $br(b)$*
   - *for all $a \in \boldsymbol{A}$, the local degree of f at a is equal to $rm(a)$*
   - *if η is a node on C, then $f(\eta)$ is a node on D*
   - *(Balancing condition) if η is a node between C_1 and C_2, where C_1 and C_2 are irreducible components of C, then $f|_{C_1}$ and $f|_{C_2}$ are distinct components of D and the local degree of $f|_{C_1}$ on η is equal to the local degree of $f|_{C_2}$ on η.*

Example 6. We show one example of an admissible cover sending a curve in $\mathcal{M}_{0,7}$ to a curve in $\mathcal{M}_{0,7}$.



We have the following admissible cover of the Hurwitz correspondence, where the map has degree 2 and where the points are sent as follows:

$$a_1 \xrightarrow{2} b_1$$
$$a_2 \xrightarrow{2} b_2$$
$$a_3 \rightarrow b_3$$
$$a_4 \rightarrow b_4$$
$$a_5 \rightarrow b_5$$
$$a_6 \rightarrow b_6$$
$$a_7 \rightarrow b_7$$

Figure 6: In this picture we see an admissible cover for the Hurwitz correspondence parametrizing the squaring map. In the picture, we see the ramification (rm) is the following $rm(a_1) = rm(a_2) = 2$ and $rm(a_i) = 1$ for $i = 3, 4, 5, 6, 7$. We say that this map has 2 special points, a_1 and a_2.

In this work, we will denote a Hurwitz correspondence between moduli space of genus zero curves $\overline{M_{0,A}}$ and $\overline{M_{0,B}}$ by (H_0, π_A, π_B) and define it by

$$\begin{array}{ccc} & H_0 & \\ \pi_{\mathbf{B}} \swarrow & & \searrow \pi_{\mathbf{A}} \\ \overline{M_{0,B}} & & \overline{M_{0,A}} \end{array}$$

where $M_{0,B}$ is a moduli space of genus zero curves with marked points $b_1, ..., b_n$; $M_{0,A}$ is a moduli space of genus zero curves with marked points $a_1, ..., a_n$; in our study we focus on a Hurwitz space H_0 parametrizing maps of degree 2, where the map from H_0 to $\overline{M_{0,B}}$ is given by a 'target curve' map π_B and the map from H_0 to $\overline{M_{0,A}}$ is given by a 'source curve' map π_A. H_0 parametrizes maps with global degree 2 and has branching $(2, 2, 1)$ as shown by following mapping of points:

$$a_1 \xrightarrow{2} b_1$$
$$a_2 \xrightarrow{2} b_2$$
$$a_i \rightarrow b_i \text{ for } 3 \leq i \leq n$$

We also denote another Hurwitz correspodence by (H_ρ, π'_A, π'_B)

$$\begin{array}{ccc} & H_\rho & \\ & \swarrow^{\pi'_{\mathbf{A}}} \quad \searrow^{\pi'_{\mathbf{B}}} & \\ \overline{M_{0,A}} & & \overline{M_{0,B}} \end{array}$$

where $M_{0,B}$ and $M_{0,A}$ are as defined above, the 'target curve' map is π'_A, the 'source curve' map is π'_B and H_ρ is a Hurwitz space parametrizing degree 1 maps. This correspondence is described by the following mapping of points: $b_{\rho(i)} \to a_i$ for $1 \leq i \leq n$ and for some $\rho \in S_n$.

We studied rational maps $f : X^{n-3} \to X^{n-3}$ and the dynamics of the induced map on homology $f_* : H_{2(n-4)}(X) \to H_{2(n-4)}(X)$ where we are following the notation and terminology found in [3]. Consider X^{n-3} to be isomorphic to the compactification $\overline{M_{0,n}}$. Specifically, we study the push-forward (the induced) map on homology of the composition of these correspondences $(H_\rho \circ H_0)_* : H_{2(n-4)}(\overline{M_{0,B}}) \to H_{2(n-4)}(\overline{M_{0,B}})$ on the divisor class group, where H_ρ and H_0 are defined as above. Particularly, we study the dynamical degree of this induced map. In [3], we see the definition of the dynamical degrees as follows:

$$\lambda_k(f) := \lim_{n\to\infty} ||(f^n)^* : H^{k,k}(X;\mathbb{C}) \to H^{k,k}(X;\mathbb{C})||^{1/n}$$

where $0 \leq k \leq N$, f is a rational map $f : X \to X$ and it induces a pullback action $f^* : H^{k,k}(X,\mathbb{C}) \to H^{k,k}(X;\mathbb{C})$.

The Hurwitz correspondence $(H_\rho, \pi'_A, \pi'_B) : \overline{M_{0,B}} \rightrightarrows \overline{M_{0,A}}$ is an automorphism of $M_{0,A}$, induced by a permutation $\rho \in S_n$, where n is the number of marked points in curves in $M_{0,B}$ and $M_{0,A}$. When computing the dynamical degree for moduli space of rational curves with n marked points and genus zero for large n, to prevent the number of permutations ρ to grow exponentially, we classify them according to whether the points with ramifications 2 are either in the same or in different cycles of the permutation.

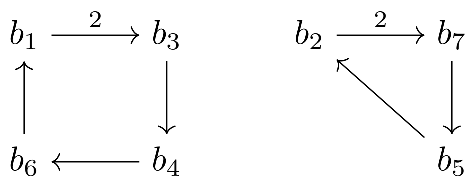
In [1], the authors show that the dynamical degree of a Hurwitz correspondence is equal to the largest eigenvalue of the matrix representing the induced map on homology $(H_\rho \circ H_0)_*$. Since the main focus of this work is to look at the dynamical degree, we focus on a quotient Ω_B^{n-4} of the homology group $H_{2(n-4)}(\overline{M_{0,B}})$, where the quotient is described and fully defined in [3]. In the rest of the text when we refer to the quotient of an induced map, we refer to Ω_B^{n-4}. The action of this quotient can be represented by an $n \times n$ matrix whose largest eigenvalue is the dynamical degree. For this reason, we compute this $n \times n$ quotient matrix for different Hurwitz correspondences and study all the eigenvalues obtained from this matrix, paying particular attention to the behavior of the dynamical degree.

Example 7. Let $n = 7$, consider the moduli space $\mathcal{M}_{0,7}$ and the permutation ρ that sends $1 \to 3 \to 4 \to 6 \to 1$ and $2 \to 7 \to 5 \to 2$. Let H_0 and H_ρ be Hurwitz correspondences defined as above and consider an arbitrary curve of genus 0 with 7 marked points $(C, a_1, ..., a_7) \in M_{0,7}$ and an arbitrary curve of genus 0 with 7 marked points $(D, b_1, ..., b_7) \in M_{0,7}$. Then suppose that under the induced map on homology

$(H_\rho \circ H_0)_*$, the points on a marked curve in $M_{0,7}$ are transformed as follows:

$$a_1 \to b_3$$
$$a_3 \to b_4$$
$$a_4 \to b_6$$
$$a_6 \to b_1$$
$$a_2 \to b_7$$
$$a_7 \to b_5$$
$$a_5 \to b_2$$

The induced map $(H_\rho \circ H_0)_*$ generates the following cyclic portraits.



Remark 1. *As seen in [1],the map $H_0^{-1} : \mathcal{M}_{0,A} \to \mathcal{M}_{0,B}$ can be seen as a squaring map on $M_{0,N}$ when thought of as an open subset of $\mathbb{P}^{N-3}$ where the coordinates are mapped as follows $(p_1, ..., p_n) \to (p_1^2, ..., p_n^2)$*

We have mainly studied Hurwitz correspondences that are constructed as the composition of the Hurwitz correspondences H_0 and H_ρ for some $\rho \in S_n$, where H_0 and H_ρ are defined as above. We studied the induced map on homology of these correspondences and how the composition of these induced maps acted on the basis elements of the homology group. We obtained the following proposition and present a the sketch of a proof for it. For a complete proof, please refer to [4].

Proposition 1. *Considering the Hurwitz correspondence between a source curve with n marked points and a target curve with n marked points parametrizing maps of degree 2 with the following ramification*

$$a_1 \overset{2}{\mapsto} b_1$$
$$a_2 \overset{2}{\mapsto} b_2$$
$$a_3 \mapsto b_3$$
$$\vdots$$
$$a_n \mapsto b_n$$

We claim the action of such a Hurwitz correspondence on the basis of the homology group behaves as follows:

$$\Delta_{1,2,k} \rightarrow \Delta_{1,2,k} \text{ for } k \subset 3,..,n \text{ and } |k| < n-4$$
$$\Delta_{1,j} \rightarrow 2\Delta_{1,j} \text{ for } j \subset 3,...,n \text{ and } |j| \leq n-2$$

Proof. We will consider the two cases shown above :

Case $\Delta_{1,2,k}$:

For this part of the proof we consider $k \subset \{3,..,n\}$ such that $|k| < n-4$. In this case, the target curve, D_1 has the marked points b_1, b_2 and some other set of k marked points, where as mentioned before $k \subset \{3,..,n\}$ such that $|k| < n-4$ and in the other component D_2 we have the other $n - |k| - 2$ marked points. In the case where the Hurwitz correspondence parametrizes maps of degree 2, the cover of D_1, name it C_1 in the source curve, has points a_1 and a_2 covering the points b_1 and b_2 with ramification 2, as shown in Figure 7. For the proof, we will consider the following cases:

Case 1: The cover of component D_2 in the target curve, call it C_2 in the source curve, has one component with a fixed marked point, d_3, with the other $n - |k| - 3$ marked points on that same component. This is shown in Case 1 in Figure 7. In this case, one component of C_2 is not stable, and so when stabilizing we lose the unstable component. Therefore, we have that the degrees of freedom of the space of possibilites for component C_1 is $|k| + 3 - 3 = |k|$ and the degrees of freedom of the space of possibilities for component C_2 is $n - |k| - 4$, which implies the space of source curves, (space of possibilities for C_1) $\times$ (space of possibilities for C_2), has $n-4$ degrees of freedom. In the same fashion, the space of target curves, (space of possibilities for D_1) $\times$ (space of possibilities for D_2), has $n-4$ degrees of freedom. Since there is no reduction in the degrees of freedom, the curve $\Delta_{1,2,k}$ maps to $\Delta_{1,2,k}$.

Case 2: In this case, the cover of component D_2, namely C_2 in the source curve, has one component with exactly one marked point a_4, and other $n - |k| - 4$ marked points on it, and the other component has only the marked point a_3. The second component of C_2 is not stable and when we stabilize it, the marked point contracts to a marked point below a stack structure, exactly below the node between C_1 and C_2. In this case, the degrees of freedom of the space of possibilities for component C_1 is $|k| + 4 - 3 - 1 = |k|$, since even though we have $|k| + 4$ special/marked points to consider, when we fix either the node or the marked point below it, the other point will be automatically fixed by the stack construction and so we have one free marked point less. Also, considering that the space of possibilities for component C_2 has $n - |k| - 4 + 2 - 3 = n - |k| - 5$ degrees of freedom, we can conclude that the space of source curves (space of possibilities for C_1) $\times$ (space of possibilities for C_2) has $n-5$ degrees of freedom. However, the space of target curves (space of possibilities for D_1) $\times$ (space of possibilities for D_2) still has $n-4$ degrees of freedom and so, since the mapping reduces the degrees of freedom of the space of curves in this case, we conclude the map sends $\Delta_{1,2,k}$ to 0.

Case 3: In this case, in the cover of component D_2 in the target curve, namely C_2, each component of C_2 has at least 2 marked points. This means that both components are stable and so we have a stable curve with n marked points and 2 nodes. This implies that the space of source curves, (space of possibilities for C_1) $\times$ (space of possibilities for C_2) has $n-3-2 = n-5$ degrees of freedom, but the space of target curves (space of possibilities for D_1) $\times$ (space of possibilities for D_2) still has $n-4$ degrees of freedom. Since the mapping from the space of source curves to the space of target curves reduces the degrees of freedom of these spaces we conclude that the map sends $\Delta_{1,2,k}$ to 0 in this case.

After studying these cases, we conclude that the squaring map for the rational curves applied to some basis elements behaves as $\Delta_{1,2,k} \mapsto \Delta_{1,2,k}$.

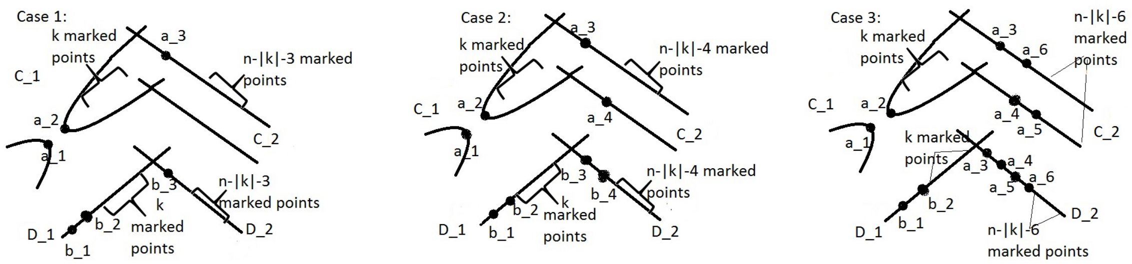


Figure 7: The three possible cases for $\Delta_{1,2,k}$.

Case $\Delta_{1,p}$:

For this part of the proof we consider $p \subset \{3, ..., n\}$ such that $1 \leq |k| \leq n-2$ and we will refer to Figure 8. In this case, in the target curve, in one of the components, D_1, there are the marked point b_1 and some subset of k marked points, where as mentioned before $k \subset \{3, ..., n\}$, and in the other component, D_2, we have the other $n-|k|-2$ marked points plus the marked point b_2. When the Hurwitz correspondence parametrizes maps of degree 2, the cover of the component D_1 in the target curve, name it C_1, has a point a_1 covering the point b_1 with ramification 2 and the cover of the component D_2 in the target curve, name it C_2, has a point a_2 covering the point b_2 with ramification 2. Because we need to have a cover with points a_1 and a_2 with ramification of 2, we have a source curve of the form shown in Figure 2, where the node intersecting the components C_1 and C_2 of the source curve is of degree 2. Notice also that for any allowed cardinality of the subset k, the components C_1 and C_2 of the source curve are always stable. Thus, when we calculate the degrees of freedom of space of possible curves for C_1, we get $|k|+2-3 = |k|-1$ and the degrees of freedom for the space of curves for C_2 is $n-|k|-2+2-3 = n-|k|-3$, which allows us to conclude that the space of source curves, (space of possibilities for C_1) $\times$ (space of possibilities for C_2), has $n-4$ degrees of freedom, which is the same as the degrees of freedom of the space of target curves $D_1 \times D_2$. Thus, since there is no change in the degrees of freedom when considering this Hurwitz correspondence, we conclude that the map sends $\Delta_{1,2,p} \mapsto 2 * \Delta_{1,2,p}$. In this case, the factor 2 comes from the fact that the degree of the node in the target curve is 2.

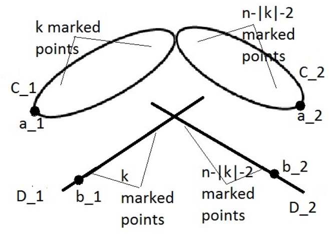


Figure 8: Image of the curves for $\Delta_{1,p}$.

□

2 Computation results and conjectures

2.1 All permutations

The first results obtained are the eigenvalues where we considered the quotient of the induced map $(H_\rho \circ H_0)_*$ and look at the behavior of all the eigenvalues of the quotient given in [3] for every possible Hurwitz correspondence for a fixed n. In the main text we present only three images.

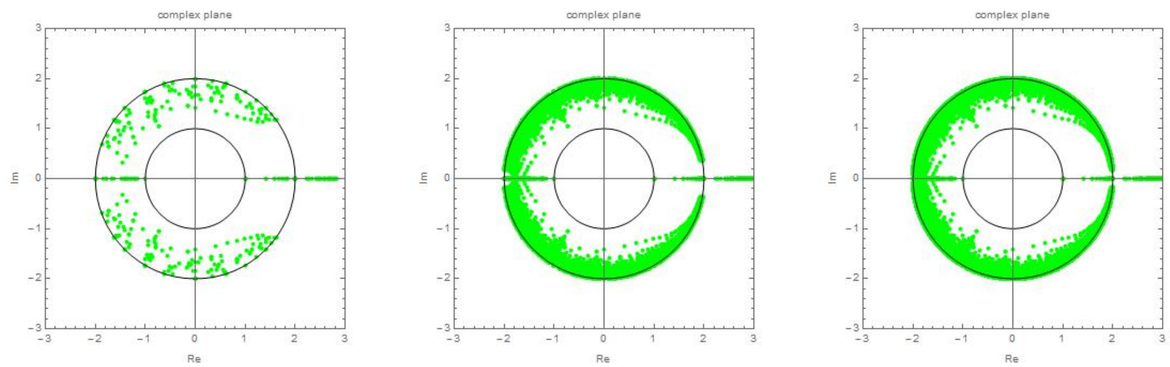


Figure 9: The left picture shows all the eigenvalues, including the dynamical degrees and its Galois conjugates, when n = 10, the middle when n= 35 and the right when n = 70.

As can be seen in Figure 9, most of the eigenvalues are inside the annulus $1 \leq |z| \leq k$, where $k = 2$ is the degree of the maps parametrized by H_0 and the only eigenvalues that are not inside this annulus are real and positive.

Given these observations, we made a similar computation for Hurwitz correspondences that parametrize maps of degree k. We show results of this computation in Figure 10.

As can be seen from these examples, most of the eigenvalues lie in the annulus $1 \leq |z| \leq k$, with the only difference, compared to the case when $k = 2$, being that the annulus has a bigger outer radius.

Additionally, we tested whether the points that lie outside the annulus shown in Figure 9 and Figure 10 correspond to the dynamical degree of the permutations. To test this idea, we plotted all the eigenvalues except the maximum, the dynamical degree, for each permutation and show some basic results in Figure

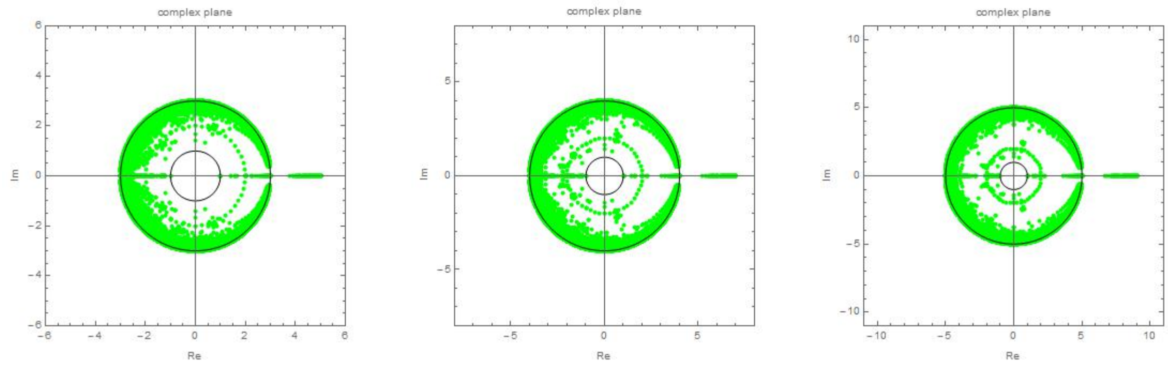


Figure 10: We fix n = 50. The left picture shows the eigenvalues for Hurwitz correspondences parametrizing maps of degree 3, the middle for Hurwitz correspondences parametrizing maps of degree 4 and the right for Hurwitz correspondences parametrizing maps of degree 5.

11.

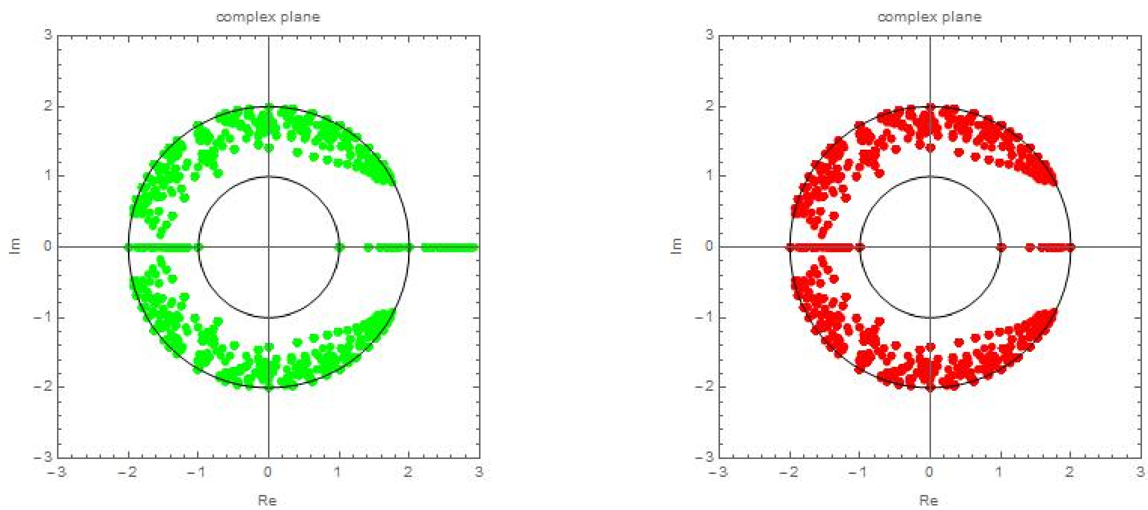


Figure 11: The picture to the left is the plot of all the eigenvalues of the quotient matrix for $n = 13$. The picture to the right is the plot of all the eigenvalues for the quotient matrix, except for the largest one, for $n = 13$.

From Figures 11 and 12, we see that when we do not plot the dynamical degree for each permutation, then every single eigenvalue is inside the annulus $1 \leq |z| \leq k$, where k is the degree of the map. After doing computations of multiplicities, we also notice that for these examples the multiplicities of the dynamical degree is always equal to 1. With these observations in mind, we propose the following conjectures.

Conjecture 1. *Let $n \geq 5$, where n is the number of marked points in a smooth rational curve. Consider the induced map $(f_{\rho_i})_* := (H_{\rho_i} \circ H_0)_* : H_{2(n-4)}(X^{n-3}) \to H_{2(n-4)}(X^{n-3})$, where $\rho_i \in S_n$ and H_0 parametrizes maps of degree k. Let Λ be the set of eigenvalues of $(f_{\rho_i})_*$. If $\lambda_s \in \Lambda$ for $1 \leq s \leq |S_n|$ such that $\lambda_s \neq \max \Lambda$, then $1 \leq |\lambda_s| \leq k$.*

Conjecture 2. *For $n \geq 5$, where n is the number of marked points in a smooth rational curve. Consider*

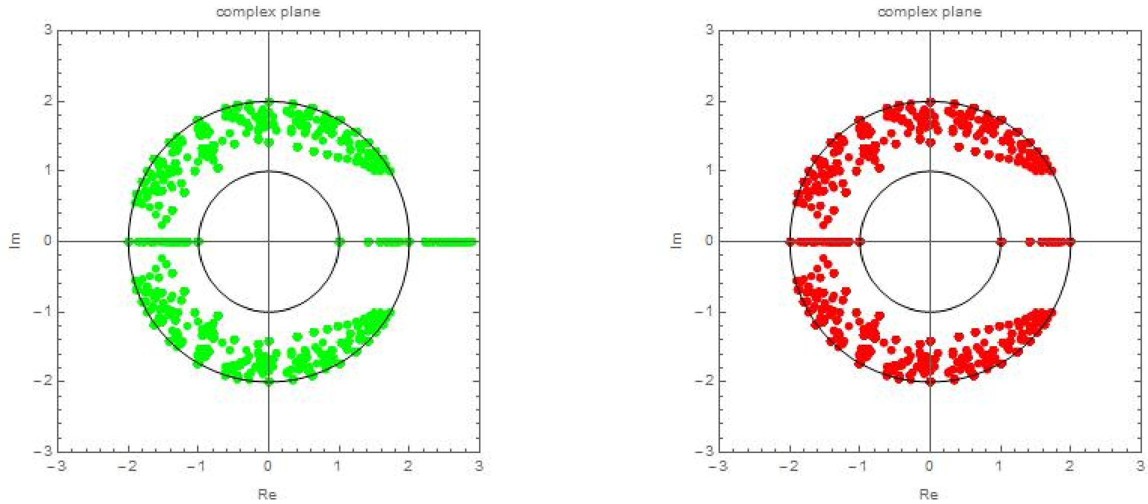


Figure 12: The picture to the left is the plot of all the eigenvalues of the quotient matrix for $n = 12$. The picture to the right is the plot of all the eigenvalues for the quotient matrix, except for the largest one, for $n = 12$.

the induced map $(f_{\rho_i})_* := (H_{\rho_i} \circ H_0)_* : H_{2(n-4)}(X^{n-3}) \to H_{2(n-4)}(X^{n-3})$*, where* $\rho_i \in S_n$*,* $1 \le i \le |S_n|$ *and* H_0 *parametrizes maps of degree* k*. Let* Λ *be the set of eigenvalues of the map* $(f_{\rho_i})_*$*. If* $\lambda_s \in \Lambda$ *for* $1 \le s \le |S_n|$ *such that* $\lambda_s = \max \Lambda$*,* $\lambda_s \in \mathbb{R}$ *and* $k < |\lambda_s|$*, then multiplicity of* λ_s *is 1.*

Remark 2. *Let* Λ *be the set of eigenvalues of the matrix representing the action of the induced map* $(f_{\rho_i})_* := (H_{\rho_i} \circ H_0)_* : H_{2(n-4)}(X^{n-3}) \to H_{2(n-4)}(X^{n-3})$ *and* $\Lambda_{ND} \subset \Lambda$ *be the set of eigenvalues of the matrix representing the action of the induced map* $(f_{\rho_i})_*$ *without considering the largest eigenvalue of such map (without the dynamical degree). We notice that for every Hurwitz correspondence, for each* $\rho_i \in S_n$*, for every* $\lambda_s \in \Lambda_{ND}$ *such that* $Arg(\lambda_s) = \min(Arg(\Lambda_{ND}))$*, then* $\lim_{n\to\infty} Arg(\lambda_s) = 0$*.*

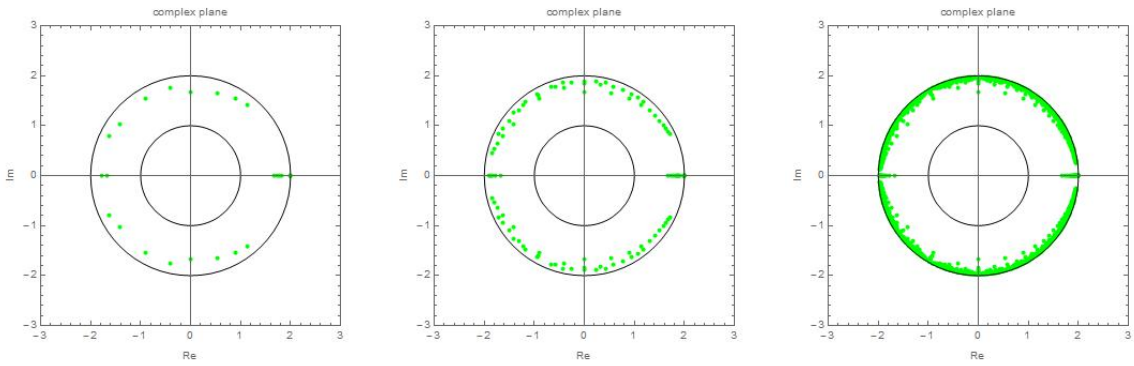
2.2 Special permutations

Finally, we study the behavior of specific cycles. In this part of the text, we study three cases:

1. For $n > 5$, the cycles where the special point 1 is in a cycle by its own and the remaining $n - 1$ points are in one cycle.
2. For $n > 5$, the cycles where all the n points are in one cycle and the special points a_1 and a_2 have no points between them.
3. For $n > 5$, the cycles where all n points are in one cycle and there is a number d of points between a_1 and a_2 such that $0 < d < n - 2$.

Case 1:

In this case, the eigenvalues show that as we increase the number n of points, the eigenvalues tend to go to the circle of radius $|z|= k$



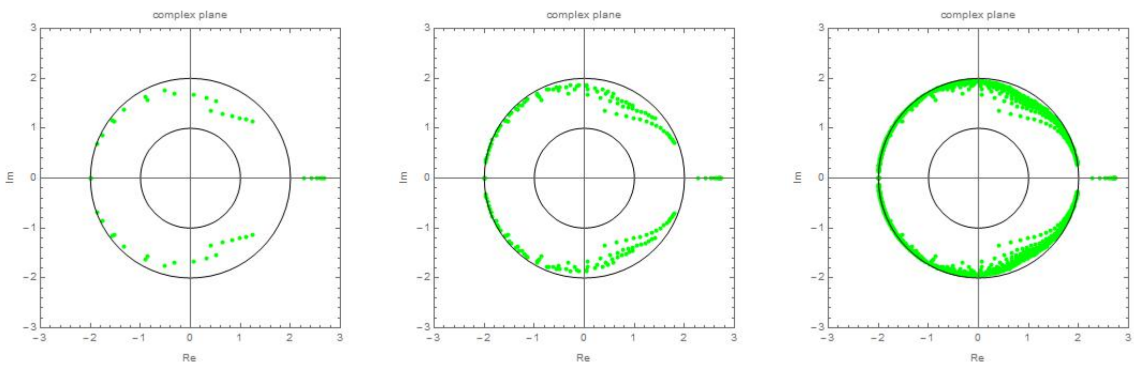
(a) n=10 (b) n=20 (c) n=50

Figure 13: We see three figures corresponding to the first case of permutations described in the list above.

In Figure 13, we see that none of the eigenvalues are outside the annulus, suggesting that the dynamical degree for these permutations is exactly 2.

Case 2:

In this case, as we increase n, the pictures in Figure 14 show that the eigenvalues seem to have a limiting behavior towards the circle of radius k.



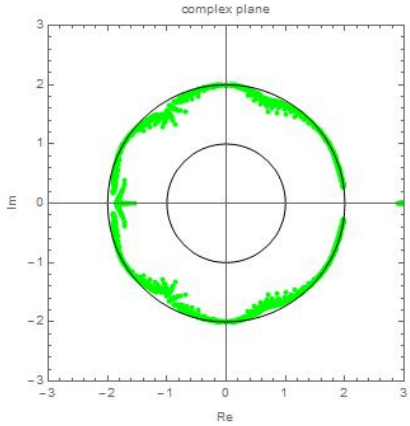
(a) n=10 (b) n=20 (c) n=50

Figure 14: We see three figures corresponding to the second case of permutations described in the list above.

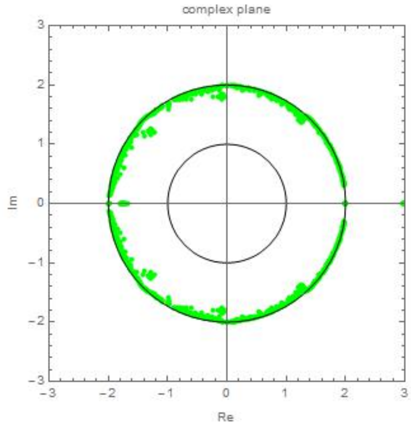
Case 3:

In this last case, we have an interesting behavior of the eigenvalues, depending on whether the number of marked points n is even or odd.

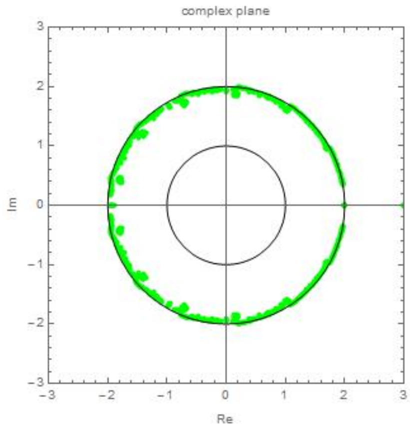
Remark 3. *For every n, we see in Figure 15 that the points seem to accumulate in lumps at certain angles around the circle. In the pictures above we tend to see a pattern. For n even, we see that there are $\frac{n}{2}$ lumps of points along the circle of radius $|z| = k$ and for n odd, it seems that there are $\lfloor \frac{n}{2} \rfloor$ lumps of points along the circles of radius $|z| = k$.*



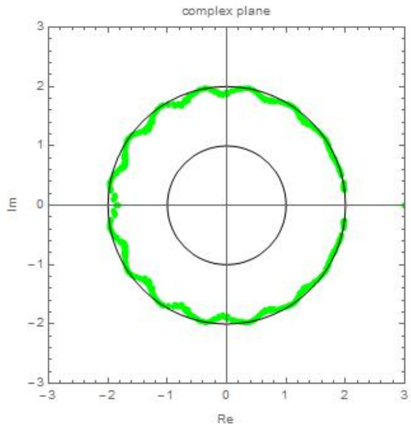
(a) n=10



(b) n=15



(c) n=25



(d) N=30

Figure 15: We see three figures corresponding to the third case of permutations described in the list above.

2.3 Non-cyclic portraits

We made a small study of Hurwitz correspondences with non-cyclic portraits. To represent the non-cyclic portraits, we will use the notation $i \to j$, which means that under the correspondence, special point i is sent to j. Also, if we have the expression $* \to i$, it means that some arbitrary non-special point is sent to special point i under this correspondence. Specifically, we study the cases where our portraits have to distinct cycles of the following form:

1. $1 \to 1$
   $* \to 2 \to 3 \to 4 \to ... \to n \to 3$
2. $1 \to 1$
   $* \to 2 \to 3 \to 4 \to ... \to n \to 4$
3. $1 \to 1$
   $* \to 2 \to 3 \to 4 \to ... \to n \to n$

In the few cases we study, we fixed n and got the eigenvalues of the quotient matrix defined in [3], obtaining the following results:

1. For $n = 20$, we get the following eigenvalues $(2, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

2. For $n = 20$, we get the following eigenvalues $(2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

3. For $n = 20$, we get the following eigenvalues $(2, (-1)^{7/9}2^{17/18}, -(-1)^{7/9}2^{17/18},$ $-(-1)^{1/9}2^{17/18}, (-1)^{1/9}2^{17/18}, -2^{17/18}, 2^{17/18}, (-1)^{8/9}2^{17/18}, -(-1)^{8/9}2^{17/18}, -(-1)^{2/9}2^{17/18},$ $(-1)^{2/9}2^{17/18}, -(-1)^{4/9}2^{17/18}, (-1)^{4/9}2^{17/18}, (-1)^{5/9}2^{17/18}, -(-1)^{5/9}2^{17/18}, -(-1)^{1/3}2^{17/18},$ $(-1)^{1/3}2^{17/18}, (-1)^{2/3}2^{17/18}, -(-1)^{2/3}2^{17/18}, 0)$

2.4 Future directions

The following steps in this project will be to prove the conjectures presented in section 2.1 and try to formalize the observations seen in section 2.2. Also, we will try to do similar analysis as in sections 2.1 and 2.2 in the case when the Hurwitz correspondence H_0 parametrizes map of degree 3 with different specified branching, like $(2, 2, 2, 2)$ or $(3, 2, 2, 1)$. Finally, we will try to expando and find patterns in the case of the non-cyclic portraits.

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