Properties of a Chain-Like Linkage

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Abstract

Networks of rigid bonds connected by rotors have various applications such as material science and structural engineering. In this article, we explore the properties of a chain-like linkage, which behaves like a topological mechanical insulator whose zero-energy modes are localized at the edge.

1 Introduction

The chain-like linkage that we are interested in is a long chain with N rotors and two kinds of bonds. The distance between adjacent rotors are fixed, which is d. The radius of the rotors are r. The absolute value of the initial angle of the red bonds and the chain is θ $(0 < \theta < \frac{\pi}{2})$. The blue bonds connect all the adjacent red rotors, each has length $l = \sqrt{d^2 + 4\sin^2\theta}$.

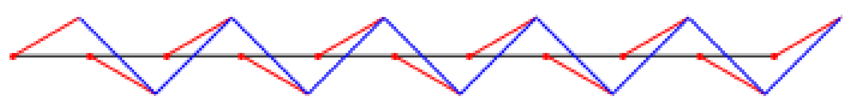


Figure 1: The Chain-Like Linkage

The chain-like linkage is not really one dimensional, as it can move in a two-dimensional plane. Since every free side of the red bonds has one degree of freedom, and the N sites are connected by $N-1$ blue bonds, so the system has one zero mode. Figure 1 shows that the zero mode is at the right side of the system. We call the side with the zero mode the free side, and the other side the rigid side.

An interesting question would be: If we ignore the frictions of the system, is the rigid side that "rigid"? What will happen if we wiggle the rigid side?

A natural way to tackle the question is to change the angle of the first rotor a little bit, and since the length of the blue bonds are fixed, it is expected that we can mathematically know the angle of the second rotor. We repeat the process over and over again, then we know the information of all the following up rotors.

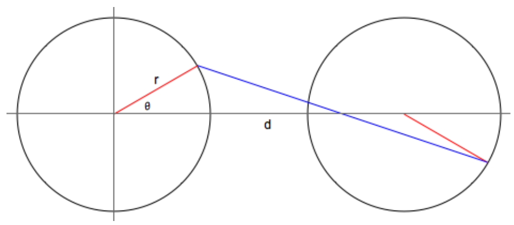


Figure 2: Two-Rotor System

2 Two-Rotor System

First of all, we should deal with the Two-Rotor Situation, which means to know the angle of the $(n+1)th$ rotor mathematically provided the angle of the nth rotor.

Suppose the long chain is the x-axis, and the angle of the nth rotor is x, while the angle of the $(n+1)th$ rotor is y. To make the system easier to calculate, we make r the unit of length, so that $r = 1$, and we let $t = \sin\theta$.

Since l is fixed, we get the math constraint:

$$(\cos y - \cos x + d)^2 + (\sin y - \sin x)^2 = l^2 = 4t^2 + d^2$$

$$-\pi < x \leq \pi, \quad -\pi < y \leq \pi$$

The equation could be solved by Mathematica 10.0, and there are two branches of solution, which are:

$$y_1 = f_1(x) = \tan^{-1}\left(-\frac{d\left(-4t^2+\cos(2x)+3\right)-2\left(d^2-2t^2+1\right)\cos(x)}{2\left(d^2-2d\cos(x)+1\right)} - \frac{\sqrt{\sin^2(x)\left(d^2-d\left(d\cos(2x)+8t^2\cos(x)\right)-8t^4+8t^2\right)}}{\sqrt{2}\left(d^2-2d\cos(x)+1\right)}, \frac{\sin(x)\left(-2d\cos(x)-4t^2+2\right)}{2\left(d^2-2d\cos(x)+1\right)} + \frac{(\cot(x)-d\csc(x))\sqrt{\sin^2(x)\left(d^2-d\left(d\cos(2x)+8t^2\cos(x)\right)-8t^4+8t^2\right)}}{\sqrt{2}\left(d^2-2d\cos(x)+1\right)}\right)$$

and

$$y_2 = f_2(x) = \tan^{-1}\left(\frac{\left(d^2 - 2t^2 + 1\right)\cos(x)}{d^2 - 2d\cos(x) + 1}\right.$$

$$-\frac{\sqrt{2}\sqrt{\sin^2(x)\left(d^2 - d\left(d\cos(2x) + 8t^2\cos(x)\right) - 8t^4 + 8t^2\right) + d\left(-4t^2 + \cos(2x) + 3\right)}}{2\left(d^2 - 2d\cos(x) + 1\right)},$$

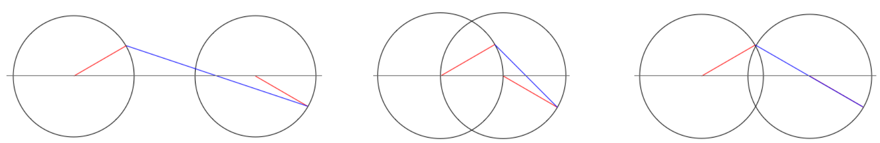
$$\frac{\sin(x)\left(-d\cos(x) - 2t^2 + 1\right)}{d^2 - 2d\cos(x) + 1}$$

$$\left. + \frac{\csc(x)(d - \cos(x))\sqrt{\sin^2(x)\left(d^2 - d\left(d\cos(2x) + 8t^2\cos(x)\right) - 8t^4 + 8t^2\right)}}{\sqrt{2}\left(d^2 - 2d\cos(x) + 1\right)}\right)$$

After getting the solution analytically, the first question in our concern is, which branch to choose. A simple way to test is to let $x = \theta$, and choose the branch that makes the corresponding $y = -\theta$.

Figure 3(c) shows the condition when $y1 = y2$, and the two branches are the same. Since the blue bond is the diameter of the right circle, $d = 2\cos\theta$.

When $d > 2\cos\theta$, which corresponds to Figure 3(a), y_1 is the right branch; when $d < 2\cos\theta$, which corresponds to Figure 3(b), y_2 is the right branch. Since the difference is critical, we call $C_1 = 2\cos\theta$ the first critical length.



(a) $d > 2\cos\theta$ (b) $d < 2\cos\theta$ (c) $d = 2\cos\theta$

Figure 3: Different Bond-Relations

When the red bond of the first circle gets wiggled, it is crucial to know how much the red bond of the second circle moves. This means, starting from $x = \theta$, we change the value of x a little bit, and we want to know the new y.

In Figure 3(a), if x has a tiny positive increment, then y will have a tiny negative increment; if x has a tiny negative increment, then y will have a tiny positive increment; the absolute angle increases or decreases synchronously. In Figure 3(b), it is a different story. If x has a tiny positive increment, then y will have a tiny positive increment; if x has a negative increment, then y will have a tiny negative increment; the red bonds move up or down synchronously. The dynamics are different, and can be proved geometrically.

In Figure 3(c), if x is greater than θ, then the dynamics are similar to those in Figure 3(a). However, it is obvious that if x has a tiny negative increment, the red bond of the first circle will enter the second circle, and since the diameter is the longest segment in a circle, it is impossible to put the blue bond inside the circle, which means the solutions above are imaginary. And the physical interpretation of this condition is that we could not move the red bond down at all, it just halts there. This suggests the existence of the Forbidden Region which the red bonds could not enter.

By symmetry, we know the Forbidden Region would be a sector symmetric about the x-axis, so that it could denoted as $(-\theta_c, \theta_c)$, where θ_c is the critical angle. In Figure 3(c), $\theta_c = \theta$.

The reason that the Forbidden Region occurs is that when the absolute value of x is small, the blue bond tends to be too long to be on the second circle.

An interesting question would be, does the Forbidden Region always exist? The answer is no, since we can easily figure out an extreme condition: $d = 0$. The Forbidden Region disappears when $x = 0$ is allowed.

When d is small enough, there exists at least one point (p, q) on the second circle such that the distance between $(1, 0)$ and (p, q) is l. The largest such d will be obtained when $(p, q) = (d - 1, 0)$, since $(d - 1, 0)$ is the farthest from $(1, 0)$. This can be illustrated by Figure 4(a).

When $(p, q) = (d - 1, 0)$,

$$|2 - d| = l = \sqrt{d^2 + 4\sin^2\theta}$$

.

$$d = \cos^2\theta$$

.

We call $D_2 = \cos^2\theta$ the second critical length.

(a) $d = \cos^2\theta, x = 0$

(b) $d > \cos^2\theta, x = \theta_c$

Figure 4: Bond-Relations at Critical Points

When $d \leq D_2$, the Forbidden Region does not exist; when $d > D_2$, the Forbidden Region exists, and it is interesting to know the critical angle θ_c.

When $x = \theta_c$, $y_1 = y_2$, so the circle centered at $(\cos\theta_c, \sin\theta_c)$ with radius l should be internal tangent to the second circle.

$$\sqrt{(\cos\theta_c - d)^2 + \sin^2\theta_c} + 1 = l = \sqrt{d^2 + 4\sin^2\theta}$$

The equation could be solved by Mathematica 10.0, and the real positive solution is :

$$\theta_c = \cos^{-1}\left(\frac{\sqrt{d^4 - 2d^2\cos 2\theta + 2d^2} - 2d\sin^2\theta}{d^2}\right)$$

3 Many-Rotor System

Now we have known that the Two-Rotor System behaves differently with different value of d. It is expected that the Many-Rotor System (the 1D-Chain) would behave differently with different d. The main strategy of extracting the information of the whole chain is to do the function iteration: Applying function f to θ_1 (the angle of the first circle) $(n-1)$ times, we know θ_n. There are mainly three different behaviors.

We suppose $\delta\theta > 0$.

(1): $d \leq D_2$.

If $\theta_1 = \theta_0 + \delta\theta$, the corresponding odd-numbered and even-numbered angles are plotted as Figure 5.

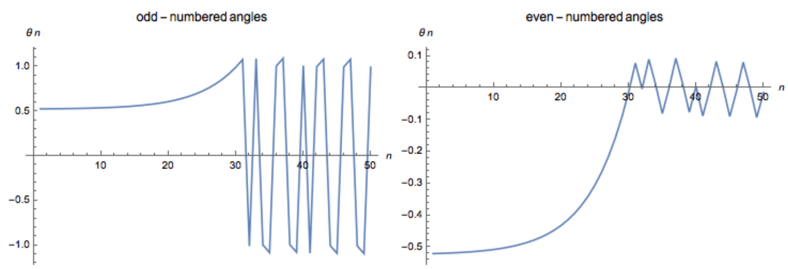


Figure 5: $d \leq D_2, \theta_1 > \theta_0$

If $\theta_1 = \theta_0 - \delta\theta$, the corresponding odd-numbered and even-numbered angles are plotted as Figure 6.

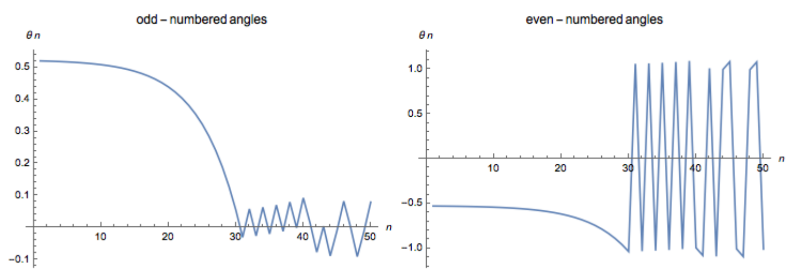


Figure 6: $d \leq D_2, \theta_1 < \theta_0$

When d becomes smaller, the oscillation pattern appears later.

(2): $D_2 < d \leq D_1$.

The system will halt after several rotors, because it reaches the critical angle θ_c very fast. The angles changed from the equilibrium places exponentially grow.

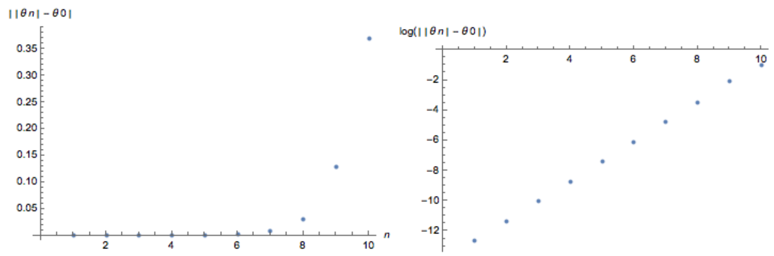


Figure 7: $D_2 < d \leq D_1$

Changing the direction of wiggling will not change the main picture. The only difference is that when $\delta\theta > 0$, with other parameters fixed, one more rotor will change the angle.

(3): $d > D_1$.

If $\theta_1 = \theta_0 - \delta\theta$, the dynamics are the same as (2).

If $\theta_1 = \theta_0 + \delta\theta$, a soliton will appear.

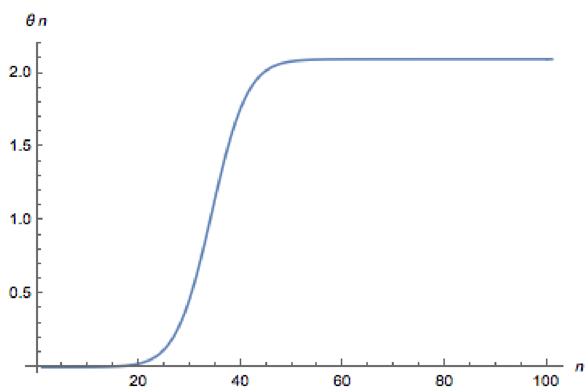


Figure 8: $d > D_1$

4 References

Bryan Gin-ge Chen, Nitin Upadhyaya, and Vincenzo Vitelli. "Nonlinear conduction via solitons in a topological mechanical insulator."Proceedings of the National Academy of Sciences111.36 (2014): 13004-13009.

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