Cyclic Sieving of Matchings

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Abstract

Let P_n denote the poset of matchings on $2n$ points on a circle, labeled $1, 2, \ldots, 2n$ in cyclic order, in which we define an element to be less than another element if it can be obtained by an uncrossing. Let $P_{n,k}$ denote the level of the poset with k crossings. Let $P_{n,k}$ denote the level of the poset with k crossings. We study the *cyclic sieving phenomenon* of $P_{n,k}$.

It is known that the triple $(P_{n,0}, g(q), C_{2n})$ exhibits cyclic sieving with $g(q) = \frac{1}{[n+1]_q}\binom{2n}{n}_q$. Motivated by this result, our project focuses on finding similar results for greater number of crossing. By studying to structure of $P_{n,1}$, we find that there exists a q-analog polynomial $f_n(q)$ such that $(P_{n,1}, f_n(q), C_{2n})$ exhibits cyclic sieving phenomenon. We have also attained combinatorial interpretations of the polynomial $f_n(q)$.

1 Main Result

We use the notation in [1], and we let $P_{n,k} = \{\tau \in P_n | c(\tau) = k\}$ be the subset of P_n with k crossings $(0 \leqslant k \leqslant \frac{n(n+1)}{2})$. Consider the operation σ_{2n} acting on the set P_n by $\tau \mapsto \sigma_{2n}(\tau)$, where

$$\sigma_{2n}(\tau)(i) = \begin{cases} \tau(i+1) & \text{if } i \neq 2n \\ \tau(1) & \text{if } i = 2n \end{cases} . \tag{1}$$

The rotation σ_{2n} generates the cyclic group C_{2n} with order $2n$. We let C_{2n} act on the poset P_n, and we study the one subset of P_n with one crossing $P_{n,1}$. We have proved via a counting argument that $|P_{n,1}| = \binom{2n}{n-2}$. Let $f_{n,1}(q)$ be the q-analog of $|P_{n,1}|$:

$$f_{n,1}(q) = \binom{2n}{n-2}_q = \frac{[2n]_q[2n-1]_q \cdots [n+3]_q}{[n-2]_q[n-3]_q \cdots [1]_q}, \tag{2}$$

The function $f_{n,1}(q)$ exhibits the cyclic sieving phenomenon for the operation of σ_{2n} on the set $P_{n,1}$.

Theorem 1. *Let $\xi_{2n} = e^{\frac{\pi i}{n}}$, which is a primitive root of the equation $x^{2n} - 1 = 0$, then $f_{n,1}(\xi_{2n}^j) = \#\{\tau \in P_{n,1} | \sigma_{2n}^j(\tau) = \tau\}$. In other words, $(P_{n,1}, f_{n,1}(q), C_{2n})$ exhibits the cyclic sieving phenomenon.*

2 Proof of the Main Result

Lemma 2.1. $|P_{n,1}| = \binom{2n}{n-2}$.

Proof. We first show that $|P_{n,1}| = \binom{2n}{n-2}$ via a bijection between choosing $n-2$ element subsets of $2n$ elements and the matching of $2n$ points on a circle. We have the labeling of the points as $\{1, \ldots, 2n\}$, and we choose a subset S of those points of size $n-2$. We then must get a matching with one crossing from this subset. Look at each point i in the subset. First, if the point $i+1 (\mod n)$ is not in S, then we connect i and $i+1$. Then for those points in S which are still not connected, we connect i and $i+3$ if we are able. We continue this process, next trying to connect i and $i+5$, increasing the target point each time by 2. The end result in this process will be that $2(n-2) = 2n-4$ of the points will be connected to some other point. Note that by construction there will be no crosses in this set. Then we have exactly one way to connect the remaining four vertices such that there is one cross. We give an example with $n = 6$ and $\leftrightarrow S = \{1, 2, 7, 10\}$.

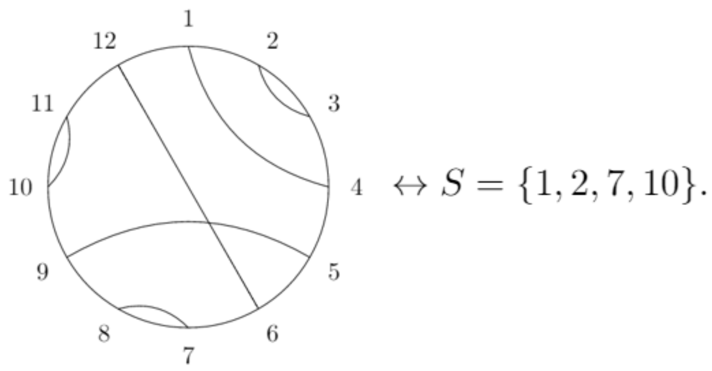

Figure 1: The bijective relation.

Now, if we are given a one-crossing matching $\tau \in P_{n,1}$. Assume the crossing is the intersection of the curves connecting (b_1, b_3) and (b_2, b_4), then τ can be written in the form of

$$\tau = (a_1, a_{n+1})(a_2, a_{n+2}) \dots (a_{n-2}, a_{2n-2})(b_1, b_3)(b_2, b_4), \tag{3}$$

where each pair (a_i, a_{i+n}) represents a non-crossing curve, and the length of the clockwise arc from a_i to a_{n+i} is not greater than the clockwise arc from a_{i+1} to a_i. This one-crossing matching corresponds to the $(n-2)$ point subset

$$S_\tau = \{a_1, a_2, \dots, a_{n-2}\}. \tag{4}$$

Thus, there is a bijective relation between the $n-2$ points subsets $\{a_1, a_2, \dots, a_{2n}\}$ and the one-crossing matchings in $P_{n,1}$. □

For any $\tau \in P_n$, let $d(\tau)$ be the smallest integer j such that $\sigma_{2n}^j(\tau) = \tau$, then $d(\tau)$ must be a divisor of $2n$. The following lemma asserts that there are only three possible values of $d(\tau)$ for $\tau \in P_{n,1}$. The figure below is illustrative of the lemma when $n = 6$.

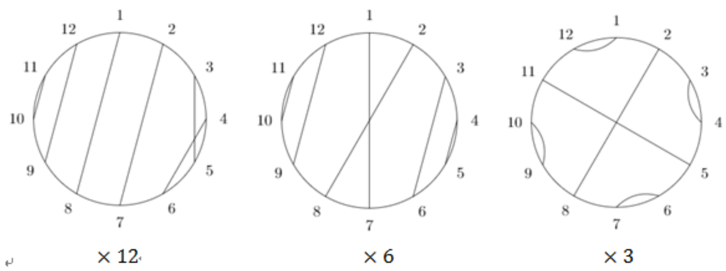

Figure 2: Some one-crossing matchings in P_6.

Lemma 2.2. *For any $\tau \in P_{n,1}$, $d(\tau) \in \{2n, n, \frac{n}{2}\}$. More precisely,*

$$\begin{cases} d(\tau) = 2n \text{ for all } \tau \in P_{n,1} & \text{if } n \text{ is odd} \\ d(\tau) \in \{2n, n\} \text{ for all } \tau \in P_{n,1} & \text{if } n \equiv 0 (\text{mod } 4) \\ d(\tau) \in \{2n, n, \frac{n}{2}\} \text{ for all } \tau \in P_{n,1} & \text{if } n \equiv 2 (\text{mod } 4) \end{cases}. \tag{5}$$

Proof. First note that $d(\tau)$ must be a divisor of $2n$. Now, assume $\tau \in P_{n,1}$, we use the bijection described earlier. Let $S_\tau = \{a_1, a_2, \dots, a_{n-2}\}$. Note that

$$S_{\sigma_{2n}(\tau)} = \{a_1 + 1, a_2 + 1, \dots, a_{n-2} + 1\} (\text{mod } 2n). \tag{6}$$

Since $\sigma_{2n}^{d(\tau)}(\tau) = \tau$. Hence, S_τ and $S_{\sigma_{2n}(\tau)}$ should represent the same residue class modulo $2n$. Thus $\sum_{i=1}^{n-2} a_i \equiv \sum_{i=1}^{n-2}(a_i + d(\tau))(\text{mod } 2n)$. Cancel the term $\sum_{i=1}^{n-2} a_i$ on both sides, and we get $2n|(n-2)d(\tau)$.

- If n is odd, then $n-2$ is also odd. Thus, $\gcd(2n, n-2) = 1 \Rightarrow 2n|d(\tau) \Rightarrow d(\tau) = 2n$.
- If n is even, then $n|\frac{n-2}{2} \cdot d(\tau)$.
  - If $4|n$, then $\frac{n-2}{2}$ is odd, and $\gcd(n, \frac{n-2}{2}) = 1$. Thus, $n|d(\tau) \Rightarrow d(\tau) \in \{2n, n\}$.
  - If $n \equiv 2(\text{mod } 4)$, then $\frac{n}{2}|\frac{n-2}{4} \cdot d(\tau)$ and $\gcd(\frac{n}{2}, \frac{n-2}{4}) = 1$. Thus, $\frac{n}{2}|d(\tau) \Rightarrow d(\tau) \in \{2n, n, \frac{n}{2}\}$.

□

Now, let

$$A_j = \{\tau \in P_{n,1} | \sigma_{2n}^j(\tau) = \tau\} \tag{7}$$

be the subset of one-crossing matchings fixed by σ_{2n}^j. Since $\sigma_{2n}^{2n} = 1$, we have $|A_{2n}| = |P_{n,1}| = \binom{2n}{n-2}$. Note that for $\tau \in P_{n,1}$ and each fixed $1 \leqslant j \leqslant 2n$, then $\sigma_{2n}^j(\tau) = \tau$ if and only if $d(\tau)|j$. So, A_j can also be written as

$$A_j = \{\tau \in P_{n,1} | \sigma_{2n}^j(\tau) = \tau\} = \{\tau \in P_{n,1} | d(\tau) \text{ is a divisor of } j\}. \tag{8}$$

To check whether $f_{n,1}(p)$ satisfies the conditions of cyclic sieving, it is sufficient to check $f_{n,1}(\xi_{2n}^j)$ agrees with $|A_j|$ at each j. The values of $|A_j|$ are evaluated in the following lemma.

Lemma 2.3. *$|A_{2n}| = \binom{2n}{n-2}$, and $|A_j| = 0$ for $j \notin \{\frac{n}{2}, n, 2n\}$. The values of A_n and $A_{\frac{n}{2}}$ depend on the value of n(mod 4):*

$$|A_n| = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{n}{\frac{n-2}{2}} & \text{if } n \text{ is even} \end{cases}, \tag{9}$$

$$|A_{\frac{n}{2}}| = \begin{cases} 0 & \text{if } n \text{ is odd or } 4|n \\ \binom{\frac{n}{2}}{\frac{n-2}{4}} & \text{if } n \equiv 2 \pmod 4 \end{cases}. \tag{10}$$

Proof. From the definition, we have $|A_{2n}| = |P_{n,1}| = \binom{2n}{n-2}$. Also, $|A_j| = 0$ for $j \notin \{\frac{n}{2}, n, 2n\}$ follows immediately from Lemma 2.2. Now, we calculate $|A_n|$ and $|A_{\frac{n}{2}}|$. Again, according to Lemma 2.2, we have $|A_n| = 0$ when n is odd and $|A_{\frac{n}{2}}| = 0$ when n is odd or $4|n$.
Now, when n is even, we divide the numbers $\{1, 2, \dots, 2n\}$ into n pairs as follow.

$$\begin{aligned} \beta_1 &= \{1, 1+n\}, \\ \beta_2 &= \{2, 2+n\}, \\ &\dots \\ \beta_n &= \{n, 2n\}. \end{aligned}$$

For $\tau \in P_{n,1}$, we use the bijective relation again. Let $S_\tau = \{a_1, a_2, \dots, a_{n-2}\}$, then $S_{\sigma_{2n}^n(\tau)} = \{a_1 + n, a_2 + n, \dots, a_{n-2} + n\}$. From the bijection, we know $\tau \in A_n$ if and only if $S_\tau = S_{\sigma_{2n}^n(\tau)}$ (mod $2n$), which means for each $1 \leqslant i \leqslant n$ we have $i \in S_\tau \Leftrightarrow i + n \in S_\tau$. Two examples are shown below. Hence,

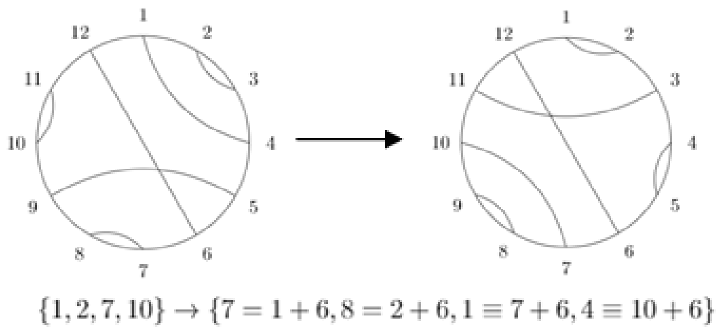

Figure 3: An element not in A_n $(n = 6)$.

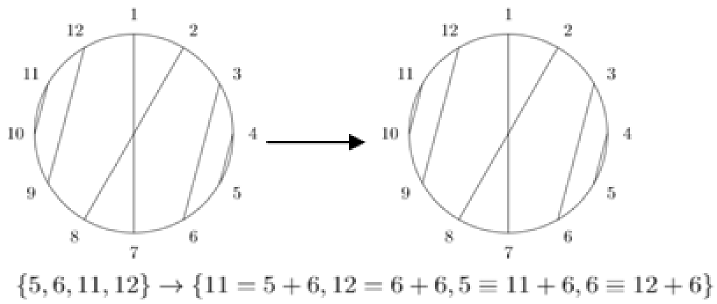

Figure 4: An element in A_n $(n = 6)$.

$$|A_n| = \#\text{ ways to pick } \frac{n-2}{2} \text{ pairs among } \beta_j\text{'s} = \binom{n}{\frac{n-2}{2}}. \tag{11}$$

Now, when $n \equiv 2 \pmod 4$, we divide the numbers $\{1, 2, \dots, 2n\}$ into $\frac{n}{2}$ parts as follow.

$$\begin{aligned}
\gamma_1 &= \{1, 1 + \frac{n}{2}, 1 + n, 1 + \frac{3n}{2}\}, \\
\gamma_2 &= \{2, 2 + \frac{n}{2}, 2 + n, 2 + \frac{3n}{2}\}, \\
&\dots \\
\gamma_{\frac{n}{2}} &= \{\frac{n}{2}, n, \frac{3n}{2}, 2n\}.
\end{aligned}$$

We now have $S_{\sigma_{2n}^{\frac{n}{2}}(\tau)} = \{a_1 + \frac{n}{2}, a_2 + \frac{n}{2}, \dots, a_{n-2} + \frac{n}{2}\}$. From the bijective relation, we know $\tau \in A_{\frac{n}{2}}$ if and only if $S_\tau = S_{\sigma_{2n}^{\frac{n}{2}}(\tau)} \pmod{2n}$, which means the following four statements are equivalent for each $1 \leqslant i \leqslant n$: 1) $i \in S_\tau$, 2) $i + \frac{n}{2} \in S_\tau$, 3)$i + n \in S_\tau$, and 4) $i + \frac{3n}{2} \in S_\tau$. Hence,

$$|A_{\frac{n}{2}}| = \#\text{ ways to pick } \frac{n-2}{4} \text{ parts among } \gamma_j\text{'s} = \binom{\frac{n}{2}}{\frac{n-2}{4}}. \tag{12}$$

□

Proof of Main Result. Now, we can use the results from previous lemmas to prove $f_{n,1}(\zeta_{2n}^j) = |A_j|$. First, note that $f_{n,1}(1) = \binom{2n}{n-2} = |A_{2n}|$ is clear. Then, we prove

$$f_n(-1) = \begin{cases} 0 = |A_n| & \text{if } n \text{ is odd} \\ \binom{n}{\frac{n-2}{2}} = |A_n| & \text{if } n \text{ is even} \end{cases}, \tag{13}$$

using the following facts.

Fact 1. *When m is odd,*

$$[m]_q(-1) = 1 + (-1)^1 + (-1)^2 + \cdots + (-1)^{m-1} = 1. \tag{14}$$

Fact 2. *When m is even,*

$$[m]_q = (1+q)(1+q^2+q^4+\cdots+q^{m-2}). \tag{15}$$

Consequently, for any two positive even integers m_1 and m_2, we have

$$\frac{[m_1]_q}{[m_2]_q}(-1) = \frac{1+(-1)^2+(-1)^4+\cdots+(-1)^{m_1-2}}{1+(-1)^2+(-1)^4+\cdots+(-1)^{m_2-2}} = \frac{m_1/2}{m_2/2}. \tag{16}$$

Now, when n is even, we have

$$\begin{aligned}
f_n(-1) = \binom{2n}{n-2}_{q=-1} &= \frac{[2n]_{q=-1}[2n-1]_{q=-1}\cdots[n+3]_{q=-1}}{[n-2]_{q=-1}[n-3]_{q=-1}\cdots[1]_{q=-1}} \\
&= \frac{[2n]_{q=-1}[2n-2]_{q=-1}\cdots[n+4]_{q=-1}}{[n-2]_{q=-1}[n-4]_{q=-1}\cdots[2]_{q=-1}}, \text{ (We have used Fact 1 here.)} \\
&= \frac{[2n]_{q=-1}}{[n-2]_{q=-1}}\frac{[2n-2]_{q=-1}}{[n-4]_{q=-1}}\cdots\frac{[n+4]_{q=-1}}{[2]_{q=-1}} \\
&= \frac{n(n-1)\cdots\frac{n+4}{2}}{\frac{n-2}{2}\frac{n-4}{2}\cdots 1}, \text{ (We have used Fact 2 here.)} \\
&= \frac{n}{\frac{n-2}{2}} = |A_n|.
\end{aligned}$$

But when n is odd, we have

$$\begin{aligned}
f_n(-1) = \binom{2n}{n-2}_{q=-1} &= \frac{[2n]_{q=-1}[2n-1]_{q=-1}\cdots[n+3]_{q=-1}}{[n-2]_{q=-1}[n-3]_{q=-1}\cdots[1]_{q=-1}} \\
&= \frac{[2n]_{q=-1}[2n-2]_{q=-1}\cdots[n+3]_{q=-1}}{[n-3]_{q=-1}\cdots[2]_{q=-1}}, \text{ (We have used Fact 1 here.)} \\
&= [2n]_{q=-1}\frac{[2n-2]_{q=-1}}{[n-3]_{q=-1}}\frac{[2n-4]_{q=-1}}{[n-5]_{q=-1}}\cdots\frac{[n+3]_{q=-1}}{[2]_{q=-1}} \\
&= 0\cdot\frac{(n-1)(n-2)\cdots\frac{n+3}{2}}{\frac{n-3}{2}\frac{n-5}{2}\cdots 1}, \text{ (We have used Fact 2 here.)} \\
&= 0 = |A_n|.
\end{aligned}$$

Using the same idea, we can prove $f_{n,1}(\zeta_{2n}^j) = |A_j|$. Thus, the proof is complete and $(P_{n,1}, f_{n,1}(q), C_{2n})$ exhibits the cyclic sieving phenomenon. □

References

[1] Thomas Lam. *The uncrossing partial order on matching is Eulerian.* Journal of Combinatorial Theory, Series A, 135(2015) 105-111