Understanding ε-Distorted Diffeomorphisms

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Abstract

This paper is devoted to studying and expanding the work of Steven Damelin and Charles Fefferman done in *Extentsions, Interpolations, and Matching in* $\mathbb{R}^D$ [1]. The work done is concerned with the matching of point sets. This includes the study of "ε-distorted diffeomorphisms" and the use of the Euclidean measure between distance sets to better understand these mappings.

1 Introduction

Matching sets of points in D-dimensional space $\mathbb{R}^D$ and aligning images are common problems to computer vision, Geographical Information Systems, and medical imaging. Difficulties that arise when there are conditions put on these point sets. Conditions discussed throughout this paper are labels matching points between the sets, equal pairwise distances between points in each set, equal distribution of distances.

The process of finding the transformations that correspond to these alignments and matchings has been studied in depth. Some works that delve into this topic are our main paper of discussion [1] and work of Werman and Weinshall [4].

2 Framework

We discuss finding motions to match sets throughout this research. The goal is to take point sets $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ and find a map Φ such that

$$\Phi(p_i) = q_i, \forall p_i \in P, q_i \in Q$$

In our investigation, we take Φ to be an orthogonal map, unless otherwise stated. An orthogonal transformation is a bijective map consisting of rotations, reflections, and compositions of these (Translations are also discussed, but can be viewed as trivial, since a translation of each point sets' centroid to the origin before any analysis takes place eliminates the need for translations in the transformation). The group of orthogonal transformations in $\mathbb{R}^D$ is called $O(D)$. These transformations are distance preserving and the determinant of an orthogonal transformation is ± 1.

The following is an example of an orthogonal transformation in two dimensions, a rotation through $\pi/6$ radians.

Example 1. Take the vector $\vec{x} \in \mathbb{R}^2$ and the transformation $\Phi \in O(2)$ of an arbitrary vector $\vec{y}$.

$$\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \Phi(\vec{y}) = \begin{bmatrix} cos(\pi/6) & -sin(\pi/6) \\ sin(\pi/6) & cos(\pi/6) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

To begin, look at the original length of the vector $\vec{x}$

$$|x| = \sqrt{(x_1^2 + x_2^2)} = \sqrt{(4^2 + 1^2)} = \sqrt{(16 + 1)} = \sqrt{17}$$

and the determinant of Φ

$$det(\begin{bmatrix} cos(30) & -sin(30) \\ sin(30) & cos(30) \end{bmatrix}) = (cos(\pi/6))(cos(\pi/6)) - (sin(\pi/6))(-sin(\pi/6))$$

$$= cos^2(\pi/6) + sin^2(\pi/6) = 1$$

Now apply Φ to $\vec{x}$

$$\Phi(\vec{x}) = \begin{bmatrix} cos(\pi/6) & -sin(\pi/6) \\ sin(\pi/6) & cos(\pi/6) \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(cos(\pi/6)) - 1(sin(\pi/6)) \\ 4(sin(\pi/6)) + 1(cos(\pi/6)) \end{bmatrix}$$

$$= \begin{bmatrix} 2\sqrt{3} - 1/2 \\ 2 + \sqrt{3}/2 \end{bmatrix}$$

To show that Φ is distance preserving, look at $|\Phi(\vec{x})|$

$$|\Phi(\vec{x})| = |x| = \sqrt{(2\sqrt{3} - 1/2)^2 + (2 + \sqrt{3}/2)^2)}$$

$$= \sqrt{(12 - 2\sqrt{3} + 1/4 + 4 + 2\sqrt{3} + 3/4)} = \sqrt{17}$$

As you can see, Φ is a distance preserving map with determinant 1.

3 Motivation

The problem we are working with is that of matching data sets. When given two sets of distinct data points in $\mathbb{R}^D$, how does one match these points?

We began looking at a question known as the "Procrustes Problem", which has been studied thoroughly and is discussed in [1].

3.1 Procrustes Analysis

Problem 1 (Procrustes Problem)**.** For a fixed $D \geq 1$, and given two sets of distinct points $P = \{p_1, \ldots, p_n\}, Q = \{q_1, \ldots, q_n\} \subset \mathbb{R}^D$, find a transformation $T \in O(D)$ such that $T(P) = Q$.

In this case, the Euclidean distance between points x and y $|x, y|$ are known for all points and $|p_i, p_j| = |q_i, q_j|$ for $1 \leq i, j \leq n$.
Theorem 1. Let $y_1, ..., y_n$ and $z_1, ..., z_n$ be two n point configurations in $\mathbb{R}^D$ with y_i, z_i distinct. Suppose that

$$|z_i - z_j| = |y_i - y_j|,\ 1 \leq i, j \leq n,\ i \neq j$$

Then there exists a Euclidean motion $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ such that $\Phi(y_i) = z_i$, for $i = 1, ..., n$.

This is the main result of the Procrustes Problem.

3.2 Damelin and Fefferman's work

In order to expand the scope of this analysis, we introduce the concept of an ε-distorted diffeomorphism[1].

[1] a diffeomorphism is an isomorphism of smooth manifolds

Definition 1. Let $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ be a diffeomorphism. We say that Φ is "ε-distorted" if for all $x \in \mathbb{R}^D$

$$(1+\varepsilon)^{-1} I \leq [\nabla\Phi(x)]^[T][\nabla\Phi(x)] \leq (1+\varepsilon)$$

Here, I denotes the identity matrix in $\mathbb{R}^D$

The following equivalent result to that of the Procrustes Problem is found in [1].

Theorem 2. Let $0 < \varepsilon < C$ and $1 \leq n \leq D$. Then there exists $\delta > 0$ small enough depending on ε such that the following holds: Let $y_1, ...y_n$ and $z_1, ...z_n$ be distinct n point configurations in $\mathbb{R}^D$. Suppose that

$$(1+\delta)^{-1} \leq \frac{|z_i - z_j|}{|y_i - y_j|} \leq (1+\delta),\, 1 \leq i, j \leq n,\, i \neq j$$

Then there exists an "ε-distorted" diffeomorphism $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ satisfying

$$\Phi(y_i) = z_i,\, 1 \leq i \leq n.$$

4 Unlabeled Point Sets

In [2] and [3], Boutin and Kemper investigate how to match point sets where only equal distribution of distances is known. This is similar to the Procrustes Problem, but without knowledge of a labeling between the points.

4.1 Determining if a permutation of edges is a relabeling of points

Theorem 3. If the distribution of pairwise distances of a generic $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$ is equal to the distribution of pairwise distances in $Q = \{q_1, \ldots, q_n\} \subset \mathbb{R}^2$, then Q is conguent to P.

Proof. (see [3] for a full argument)

Let $dist(P) = \{|p_i - p_j|, \forall p_i, p_j \in P\}$ be the set of pairwise distances in P, and similarly, $dist(Q) = \{|q_i - q_j|, \forall q_i, q_j \in Q\}$. Define $E = \{(p_i, p_j)|p_i, p_j \in P\}$ and $F = \{(q_i, q_j)|q_i, q_j \in Q\}$. Let $d(e_r) = |p_i - p_j|$ for some $e \in$

E. Define a function G (defined by Boutin and Kemper in [3]) such that $G(U, V, W, X, Y, Z) = 0$ if $U, V, W, X, Y,$ and Z are the side and diagonal lengths of a quadrilateral.

Remark 1. What is meant by "a generic P", is a configuration P with the following properties:

- All $\binom{n}{2}$ distances $d(p_i, p_j)$ are distinct.
- If $e_1, e_2, e_3, e_4, e_5, e_6 \in E$ are not the lengths of a quadrilateral, then $G(e_1, e_2, \ldots, e_6) \neq 0$

From the first property and knowing that $dist(P) = dist(Q)$, we know that there is a unique permutation Φ that pairs E and F.

Does a pairing of E and F pair P and Q?

Lemma 1. $\Phi^{-1}(q_1, q_2) \cap \Phi^{-1}(q_1, q_3) \neq 0$. for $(q_1, q_2), (q_1, q_3) \in Q$.

Proof. Choose any $q_4 \in Q$. So F consists of the 6 lengths between these four points. We know that $G(f_1, \ldots, f_6) = 0$, so $G(\Phi^{-1} f_1, \ldots, \Phi^{-1} f_6) = 0$. Thus

$$\Phi^{-1}(f_1), \ldots, \Phi^{-1}(f_6) = e_1, \ldots, e_6$$

form a quadrilateral, sides and diagonals. □

There are some cases where this proof fails, cases where equals distributions of distances between two sets does not assure a congruency between the two, but these cases are few and easily avoidable. These are discussed in depth in [3], as well as by Michael Lu and Neo Charalambides. They hope to expand this proof to exclude such cases where the proof fails by creating algorithms that check congruency of sets using the area of polygons made by edges, as well as other methods. □

4.2 Unlabeled, Almost Equal Distance Distributions

To understand more about ε-distortions, consider the Euclidean distance between two sets. Given sets $P = \{p_1, \ldots, p_n\}, Q = \{q_1, \ldots, q_n\}$ let S be a set of tuples $S = (d(p_i), d(q_i))$ (where $d(p_i)$ is the orthogonal distance of a point p_i from the origin) for all $i \in \{1, \ldots, n\}$. The *Euclidean distance* L of the set S can be defined as

$$L = \sum_{i=1}^{n} (d(p_i) - d(q_i))$$

Theorem 4. If there exists a diffeomorphism between two sets, then the Euclidean distance L of the ordered set S must obey the following relation:

$$n(1+\varepsilon) \leq L \leq n(1+\varepsilon)$$

Proof. Consider the sets P and Q, consisting of n points in $\mathbb{R}^D$. Form $P^{'} = \{p_1^{'}, \ldots, p_n^{'}\}$ and $Q^{'} = \{q_1^{'}, \ldots, q_n^{'}\}$ by reordering the sets P and Q in a monotonically increasing manner, meaning each p_i must be be less than or equal to p_j for all $j \leq i$. Take the Euclidean distance for the set S of tuples formed by the lengths of the sets $P^{'}$ and $Q^{'}$. The maximum distance between two points in the ordered sets $P^{'}$ and $Q^{'}$ is $(1 \pm \varepsilon)$. Consider the extreme case, such that for all $(d(p_i^{'}), d(q_i^{'})) \in S$,

$$d(p_i^{'}) - d(q_i^{'}) = 1 + \varepsilon$$

Since there are n points in these sets, the greatest that the Euclidean distance can be for the set S is $n(1 \pm \varepsilon)$ □

References

[1] S.B.Damelin, C.Fefferman, *Extensions, interpolation and matching in* $\mathbb{R}^D$

[2] M.Boutin, G.Kemper, *On Reconstructing n-Point Configurations from the Distributions of Distances or Areas*

[3] M.Boutin, G.Kemper, *Which Point Configurations are Determined by the Distribution of their Pairwise Distances?*

[4] M.Werman, D.Weinshall *Similarity and Affine Invariant Distances Between* 2*D Point Sets*

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