Distributions for TASEP models

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Abstract

We use bethe ansatz method to analyze the TASEP models with focus on particle dependent jump rate case and obtain a formula for marginal distribution. We also consider discrete time version of the TASEP on a ring.

1 Introduction

Everyone has some experience with traffic on the road where each car is moving relatively free but also need to consider the position of cars in front of it. It is quite tempting to think of how to analyze the traffic in a mathematical way. If we abstract the process as a random process and treat each car as a particle, then the model is an example of what we call interacting particle system. TASEP is such a model and is intensively investigated recently. It is the acronym for Tototally Asymmetric Simple Exclusion Process where totally asymmetric means that just like the car, the particle can only move in one direction and simple exclusion means that each particle only interacts with the particle just before it. The interesting point is that even we only have slight modifications on the easiest model when any particle can jump without limitation after a random time(in this situation classical probability theorems tell us almost everything), the analysis becomes much harder. It is not until 1997 that Schütz obtain the trasition probability of the simplest specification of the model, which is just the first step of the analysis.

2 Model details and recent development

There are many kinds of specifications of the model. In terms of the time variable, there are continuous time version and discrete time version. In terms of the sites particle occupying, there are infinite TASEP where particles are at integer sites and periodic TASEP or equivalently TASEP on a ring (for detailed model specification see section 1 of [3]). Schütz [8] first used bethe ansatz method to obtain the transition probability for the continuous infinite TASEP, later they continue their analysis to obtain the marginal distribution of the model for the step initial condition [2]. Johansson derived the result before them using a different argument [9].

For the discrete time version there are two commonly used updating rules, one is sequential update and the other is parallel update. For sequential update, starting from the rightmost one, one at a time, the particle jumps to its right with probability p provided the site is empty. For parallel update, the particles whose right neighbor is not occupied will jump independently at the same time with known probability.

Sequential update is easier to analyze and there are more literatures on this. Rákos and Schütz also presented the similar formula as continuous time case in [2]. They refered the details to [7] where the authors used trajectory counting and generating funcions to derive the result. In [2], Rákos and Sch$\ddot{u}$tz also derived the marginal distribution for discrete time version infinite TASEP using exactly the same method as continuous case. Borodin, Ferrari and Pr$\ddot{a}$hofer [4] presented an alternative formula and justify that by proving it satisfies master equation and boundary condition. They also gave the joint distribution as well as the asymptotic analysis in the paper.

For the parallel update, a different way is used in [11] where they counted the total weight of all possible trajectories. Borodin and others [5] derived the transition probability and proved that by induction. Joint distribution was also discussed in their paper. However traditional method is not used in these papers as the master equation is hard to write down.

For the TASEP on a ring, recently Baik and Liu [3] fully analyzed the continuous time case where they used the bethe ansatz method. There are papers that got transition probability but nothing further has obtained yet.

The immediate geenralization of all the models above is when the jump rates or jump probabilities in discrete case are particle dependent. This generalization is of much interest especially when we consider the TASEP model as directed last passage percolation model. Rákos and Sch$\ddot{u}$tz [1] gave the formula for the infinite continuous time case.

The organization of this report is as follows, in section 3 I will briefly talk about the traditional coordinate bethe method. In section 4 I basically follow [1] and rederive their results using the traditional method instead of mentioning the operator they discussed in their paper. Based on the preliminary result in [1] I obtain a marginal distribution formula for step initial condition which is already obtained in some other literatures such as Okounkov, Reshetikhin[10] and Borodin, Péché[6] where they use different methods. In section 5, I generalize the TASEP on a ring in [3] to discrete time version, sequential update. In the appendix I will briefly mention how to use traditional method to get the transition probavility obtained in [1].

3 Bethe Ansatz

In the model there are integer sites which are either occupied by the particle or empty. Particle jumps to its right at certain rate (with certain probability) provided the neighboring site is vacant. There are two common initial conditions: step and flat. In step condition, particles occupy the negative half initially. In flat condition, particles occupy the sites which are multiples of a number initially. Note that all the states in this particle system should be in the chamber $W = X \in \mathbb{Z}^N : x_1 < \cdots < x_N$. The general bethe ansatz method is as follows, denote the transition probability by $P_Y(X;t)$, then it should satisfy the master equation with nonlinear coefficient given by the indicator of the state. Instead solving this we denote $P_Y(X;t)$ by $U(X;t)$ and solve a free evolution equation with extra boundary condition and initial condition. The main advantage of this way is that the free evolution equation is often the time easy to solve. Now we are ready to use this set of reasoning in the following section.

4 Continuous time infinite TASEP with particle dependent jump rates

4.1 Transition probability

Now we consider the case when jump rates are particle-dependent. Suppose the jump rates are $r_1, ..., r_N$, we denote the state $(..., x_i - 1, x_{i+1}, ...)$ as $X^{(i)}$ and the chamber $\{x_1, ..., x_N | x_1 < \cdots < x_N\}$ by W then the master equation is the following

$$\begin{aligned}\frac{d}{dt}P_Y(X;t) &= r_1\delta_{X^{(1)}\in W}P_Y(X^{(1)};t) + \cdots + r_N\delta_{X^{(N)}\in W}P_Y(X^{(N)};t)\\ &\quad - r_1\delta_{X^{(2)}\in W}P_Y(X;t) - \cdots - r_{N-1}\delta_{X^{(N)}\in W}P_Y(X;t) - r_N P_Y(X;t)\end{aligned}$$

Where the plus part corresponds to particles jumping to this state and minus part corresponds to particles jumping out of this state. The transition probability also satisfies the initial condition which is

$$P_Y(X;0) = \delta_{X=Y}$$

Now we use the idea described in the last section. Denote $P_Y(X;t)$ as $U(X;t)$, we write the master equation as a free evolution equation and a boundary condition, together with the initial condition we have the following three requirements

$$\frac{d}{dt}U(X;t) = \sum_{i=1}^{N} r_i(U(X^{(i)};t) - U(X;t) \tag{3.1}$$

$$r_k U(..., x_{k-1}, x_{k-1}, x_{k+1}, ...) = r_{k-1}U(..., x_{k-1}, x_{k-1}+1, x_{k+1}, ...) \quad k = 2, ..., N \tag{3.2}$$

$$U(X;0) = \delta_{X=Y} \tag{3.3}$$

The boundary condition is chosen to cancel the coefficient before the term $\delta_{X^{(i)}\in W}$ when its value is zero so that the free evolution equation and the master equation match in this case. We define the function

$$F_{k,l}(x;t) = \oint e^{z^{-1}t} z^x \prod_{j=1}^{l-1}(1-r_j z)^{-1} \prod_{j=1}^{k-1}(1-r_j z) \text{đ}z$$

Where the contour is taken to be a circe around origin with radius less than $\frac{1}{r_{max}}$ and $\text{đ}z = \frac{dz}{2\pi i z}$. Then we have the following theorem obtained in [1], here the proof is a bit different from [1]

Theorem 4.1 (**Transition Probability**). *Define $P_Y(X;t)$ as above then suppose the jumprates for particles starting at $y_1,...,y_N$ are $r_1,...,r_N$ respectively, then*

$$P_Y(X;t) = \prod_{i=1}^{N} r_i^{x_i-y_i} e^{-tr_i} det[F_{m,n}(x_n - y_m;t)]_{m,n=1}^{N}$$

Proof. We need to prove that the formula in the theorem satisfies the equations (3.1), (3.2) and (3.3). Note that by product rule the differentiation of the exponential term will give us the minus part in (3.1) so we only need to show that

$$\prod_{i=1}^{N} r_i^{x_i-y_i} e^{-tr_i} \frac{d}{dt} det[F_{m,n}(x_n - y_m;t)]_{m,n=1}^{N} = \sum_{i=1}^{N} r_i P_Y(X^{(i)};t)$$

Note that the differential of the determinant is the sum of N determinant of matrices with each one of them obtained by taking the derivative of one column and we have

$$\begin{aligned}\prod_{i=1}^{N} r_i^{x_i-y_i} e^{-tr_i} \frac{d}{dt} F_{m,n}(x_n - y_m;t) &= \prod_{i=1}^{N} r_i^{x_i-y_i} e^{-tr_i} F_{m,n}((x_n-1) - y_m;t) \\ &= r_n^{x_n-1-y_n} \prod_{i\neq n} r_i^{x_i-y_i} e^{-tr_i} F_{m,n}((x_n-1) - y_m;t)\end{aligned}$$

We see that each of the term in the sum of left hand side corresponds to one term in the sum of right hand side, hence (3.1) holds.
To prove (3.2) we fix k, and we want to show that the difference of left hand side and right hand side is 0. Cancelling the common factors we want to show that

$$\begin{vmatrix} \cdots & F_{1,k-1}(x_{k-1}-y_1) & F_{1,k}(x_{k-1}-y_1) & \cdots \\ \cdots & F_{2,k-1}(x_{k-1}-y_2) & F_{2,k}(x_{k-1}-y_2) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & F_{N,k-1}(x_{k-1}-y_N) & F_{N,k}(x_{k-1}-y_N) & \cdots \end{vmatrix} - r_{k-1} \begin{vmatrix} \cdots & F_{1,k-1}(x_{k-1}-y_1) & F_{1,k}(x_{k-1}+1-y_1) & \cdots \\ \cdots & F_{2,k-1}(x_{k-1}-y_2) & F_{2,k}(x_{k-1}+1-y_2) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & F_{N,k-1}(x_{k-1}-y_N) & F_{N,k}(x_{k-1}+1-y_N) & \cdots \end{vmatrix} = 0$$

By special case ($x_1 = x_2$) of lemma 4.2 in section below we have

$$F_{m,k}(x_{k-1}-y_m) - r_{k-1}F_{m,k}(x_{k-1}+1-y_m) = F_{m,k-1}(x_{k-1}-y_m)$$

Hence we see that the kth column and $(k-1)$th column of the above determinant is the multiple of each other hence the equation holds, which proves (3.2).
For (3.3) if $t = 0$, then we have

$$F_{k,l}(x;0) = \oint z^x \prod_{j=1}^{l-1}(1-r_j z)^{-1} \prod_{j=1}^{k-1}(1-r_j z) \text{đ}z$$

Note that by the way we choose the contour if $x \geq 1$ then $F_{k,l}(x;0) = 0$. Now if $x_1 > y_1$ then by definition we have $x_i > y_1$ for all i, hence the first row will be all zero and the transition probability is 0. If $x_1 = y_1$, then by residue theorem we have $F_{1,1}(0;0) = 1$ and all other entries in the first row are 0. We keep doing this procedure for x_i and y_i and we see that the formula in the theorem when $t = 0$ is $\prod_{i=1}^{N} \delta_{x_i = y_i}$ which is exactly $\delta_{X=Y}$ as expected. Hence the theorem is proved. $\square$

4.2 Marginal Distribution

We first explore some properties of the

$$F_{k,l}(x;t) = \oint e^{z^{-1}t} z^x \prod_{j=1}^{l-1}(1 - r_j z)^{-1} \prod_{j=1}^{k-1}(1 - r_j z) đz$$

function defined in last section, here and below we write $F_{k,l}(x)$ instead of $F_{k,l}(x;t)$ for simplicity, see [1]

Lemma 4.2.

$$\sum_{x=x_1}^{x_2} r_l^x F_{k,l}(x) = r_l^{x_1} F_{k,l+1}(x_1) - r_l^{x_2+1} F_{k,l+1}(x_2 + 1) \tag{1}$$

$$\sum_{x=x_1}^{x_2} r_{k-1}^x F_{k,l}(x) = r_{k-1}^{x_1} F_{k-1,l}(x_1) - r_{k-1}^{x_2+1} F_{k-1,l}(x_2 + 1) \tag{2}$$

Proof. We fisrt note that $\sum_{x=x_1}^{x_2} (r_l z)^x = \frac{(r_l z)^{x_1} - (r_l z)^{x_2+1}}{1 - r_l z}$, so we have

$$\begin{aligned}
\sum_{x=x_1}^{x_2} r_l^x F_{k,l}(x) &= \sum_{x=x_1}^{x_2} \oint e^{z^{-1}t} (r_l z)^x \prod_{j=1}^{l-1}(1 - r_j z)^{-1} \prod_{j=1}^{k-1}(1 - r_j z) đz \\
&= \oint e^{z^{-1}t} \frac{(r_l z)^{x_1} - (r_l z)^{x_2+1}}{1 - r_l z} \prod_{j=1}^{l-1}(1 - r_j z)^{-1} \prod_{j=1}^{k-1}(1 - r_j z) đz \\
&= \oint e^{z^{-1}t} [(r_l z)^{x_1} - (r_l z)^{x_2+1}] \prod_{j=1}^{l}(1 - r_j z)^{-1} \prod_{j=1}^{k-1}(1 - r_j z) đz \\
&= r_l^{x_1} F_{k,l+1}(x_1) - r_l^{x_2+1} F_{k,l+1}(x_2 + 1)
\end{aligned}$$

By linearity of the integral and definition of F. Similarly note that $\sum_{x=x_1}^{x_2} (r_{k-1} z)^x = \frac{(r_{k-1} z)^{x_1} - (r_{k-1} z)^{x_2+1}}{1 - r_{k-1} z}$, procede as above we see that equation (2) holds. $\square$

Now we have the following preliminaary result obtained in [1]

Theorem 4.3 (**Marginal Distribution**). *Suppose initially the particles are at $Y = (y_1, ..., y_N)$ where $y_1 < \cdots < y_N$ and now* *are at* $X = (x_1, ..., x_N)$, *then*

$$P_Y(x_1 \geq a) = \prod_{i=1}^{N} e^{-tr_i} r_i^{a+i-1-y_i}$$
$$\begin{vmatrix} F_{1,2}(a-y_1) & F_{1,3}(a+1-y_1) & \cdots & F_{1,N+1}(a+N-1-y_1) \\ F_{2,2}(a-y_2) & F_{2,3}(a+1-y_2) & \cdots & F_{2,N+1}(a+N-1-y_2) \\ \vdots & \vdots & \cdots & \vdots \\ F_{N,2}(a-y_N) & F_{N,3}(a+1-y_N) & \cdots & F_{N,N+1}(a+N-1-y_N) \end{vmatrix}$$
$$= \prod_{i=1}^{N} e^{-tr_i} r_i^{a+i-1-y_i} det[F_{m,n+1}(a+n-1-y_m)]_{m,n}$$

Proof.

$$\begin{aligned} P_Y(x_1 \geq a) &= \sum_{a \leq x_1 < x_2 \cdots < x_N} P_Y(X) \\ &= \sum_{a \leq x_1 < x_2 \cdots < x_N} \prod_{i=1}^{N} e^{-tr_i} r_i^{x_i - y_i} det[F_{m,n}(x_n - y_m)]_{m,n} \\ &= \prod_{i=1}^{N} e^{-tr_i} r_i^{-y_i} \sum_{a \leq x_1 < x_2 \cdots < x_N} det[r_n^{x_n} F_{m,n}(x_n - y_m)]_{m,n} \\ &= \prod_{i=1}^{N} e^{-tr_i} r_i^{-y_i} \sum_{x_N = a+N-1}^{\infty} \sum_{x_{N-1} = a+N-2}^{x_N - 1} \cdots \sum_{x_1 = a}^{x_2 - 1} det[r_n^{x_n} F_{m,n}(x_n - y_m)]_{m,n} \end{aligned}$$

Note that only the first column involves x_1 and by lemma 4.2, we have

$$\begin{aligned} \sum_{x_1 = a}^{x_2 - 1} r_1^{x_1} F_{m,1}(x_1 - y_m) &= r_1^{y_m} \sum_{x_1 - y_m = a - y_m}^{x_2 - y_m - 1} r_1^{x_1 - y_m} F_{m,1}(x_1 - y_m) \\ &= r_1^{y_m} \sum_{u = a - y_m}^{x_2 - y_m - 1} r_1^{u} F_{m,1}(u) \\ &= r_1^{y_m} [r_1^{a - y_m} F_{m,2}(a - y_m) - r_1^{x_2 - y_m} F_{m,2}(x_2 - y_m)] \\ &= r_1^{a} F_{m,2}(a - y_m) - r_1^{x_2} F_{m,2}(x_2 - y_m) \end{aligned}$$

Note that the second term is just a multiple of the second column so we can add use the column operation an the first column becomes $r_1^a F_{m,2}(a - y_m)$. Similarly we can use lemma 4.2 to simplify the columns as $r_n^{a+n-1} F_{m,n+1}(a+n-1-y_m)$ except the last column. However, note that radius of the contour we take is less than $\frac{1}{r_{max}}$ so similar to lemma 4.2, we have

$$\sum_{x_N = a+N-1}^{\infty} r_N^{x_N} F_{m,N}(x_N - y_m) = r_N^{a+N-1} F_{m,N+1}(a+N-1-y_m)$$

Hence we get the right hand side of the 4.3. $\square$

Now, if we have step initial condition, then $y_i = -N + i$, for $a \geq 1$

$$P_Y(x_1 \geq a) = \prod_{i=1}^{N} e^{-tr_i} r_i^{a+N-1} \begin{vmatrix} F_{1,2}(a+N-1) & F_{1,3}(a+N) & \cdots & F_{1,N+1}(a+2N-2) \\ F_{2,2}(a+N-2) & F_{2,3}(a+N-1) & \cdots & F_{2,N+1}(a+2N-3) \\ \vdots & \vdots & \cdots & \vdots \\ F_{N,2}(a) & F_{N,3}(a+1) & \cdots & F_{N,N+1}(a+N-1) \end{vmatrix}$$

By lemma 4.2 we have $F_{2,2}(a+N-2) = F_{1,2}(a+N-2) - r_1 F_{1,2}(a+N-1)$, similar for subsequent rows. So we start from last row, for the kth row add r_{k-1} times the previous row to the current row and the above determinant becomes

$$\begin{vmatrix} F_{1,2}(a+N-1) & F_{1,3}(a+N) & \cdots & F_{1,N+1}(a+2N-2) \\ F_{1,2}(a+N-2) & F_{1,3}(a+N-1) & \cdots & F_{1,N+1}(a+2N-3) \\ \vdots & \vdots & \cdots & \vdots \\ F_{1,2}(a) & F_{1,3}(a+1) & \cdots & F_{1,N+1}(a+N-1) \end{vmatrix}$$

and we take the transpose of this. Consider a new function

$$\tilde{F}_{k,l} = \oint e^{(z^{-1}-r_{l-1})t} z^x \prod_{j=1}^{l-1} (1-r_j z)^{-1} \prod_{j=1}^{k-1} (1-r_j z) đz$$

Then we rewrite the formula above as

$$P_Y(x_1 \geq a) = \prod_{i=1}^{N} r_i^{a+N-1} \begin{vmatrix} \tilde{F}_{1,2}(a+N-1) & \tilde{F}_{1,2}(a+N-2) & \cdots & \tilde{F}_{1,2}(a) \\ \tilde{F}_{1,3}(a+N) & \tilde{F}_{1,3}(a+N-1) & \cdots & \tilde{F}_{1,3}(a+1) \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{F}_{1,N+1}(a+2N-2) & \tilde{F}_{1,N+1}(a+1) & \cdots & \tilde{F}_{1,N+1}(a+N-1) \end{vmatrix}$$

We explore some of the properties of the $\tilde{F}$ function. We make the definition that $r_0 = 0$.

Lemma 4.4.

$$\frac{d}{dt} \tilde{F}_{k,l}(x;t) = e^{(r_{l-2}-r_{l-1})t} \tilde{F}_{k,l-1}(x-1;t) \quad l \geq 2 \tag{1}$$

$$\tilde{F}_{k,l}(x;t) = 0 \quad x \geq 1 \tag{2}$$

Proof. (1) is proved by straightforward differentiation and (2) is because of the analyticity. $\square$

Now we write the entry in determinant as integral by 4.4 and note that $r_0 = 0$. We have

$$P_Y(x_1 \geq a) = \prod_{i=1}^{N} r_i^{a+N-1}$$
$$\begin{vmatrix} \int_0^t e^{-r_1\tau}\tilde{F}_{1,1}(a+N-2;\tau)d\tau & \int_0^t e^{-r_1\tau}\tilde{F}_{1,1}(a+N-3;\tau)d\tau & \cdots & \int_0^t e^{-r_1\tau}\tilde{F}_{1,1}(a-1;\tau)d\tau \\ \int_0^t e^{(r_1-r_2)\tau}\tilde{F}_{1,2}(a+N-1;\tau)d\tau & \int_0^t e^{(r_1-r_2)\tau}\tilde{F}_{1,2}(a+N-2;\tau)d\tau & \cdots & \int_0^t e^{(r_1-r_2)\tau}\tilde{F}_{1,2}(a;\tau)d\tau \\ \vdots & \vdots & \cdots & \vdots \\ \int_0^t e^{(r_{N-1}-r_N)\tau}\tilde{F}_{1,N}(a+2N-3;\tau)d\tau & \int_0^t e^{(r_{N-1}-r_N)\tau}\tilde{F}_{1,N}(a+2N-4;\tau)d\tau & \cdots & \int_0^t e^{(r_{N-1}-r_N)\tau}\tilde{F}_{1,N}(a+N-2;\tau)d\tau \end{vmatrix}$$

For the second row the nth column , we perform integration by parts 1 time

$$\begin{aligned} \int_0^t e^{(r_1-r_2)\tau}\tilde{F}_{1,2}(a+N-n;\tau)d\tau &= \frac{1}{r_1-r_2}\int_0^t \tilde{F}_{1,2}(a+N-n;\tau)de^{(r_1-r_2)\tau} \\ &= \frac{1}{r_1-r_2}(e^{(r_1-r_2)t}\tilde{F}_{1,2}(a+N-n;t) - \int_0^t e^{(r_1-r_2)\tau}e^{-r_1\tau}\tilde{F}_{1,1}(a+N-n-1;\tau)d\tau) \\ &= \frac{e^{(r_1-r_2)t}}{r_1-r_2}\tilde{F}_{1,2}(a+N-n;t) - \frac{1}{r_1-r_2}\int_0^t e^{-r_2\tau}\tilde{F}_{1,1}(a+N-n-1;\tau)d\tau \end{aligned}$$

We see that the the first term above is just a multiple of the corresponding entry in the first row, so the second column becomes

$$-\frac{1}{r_1-r_2}\int_0^t e^{-r_2\tau}\tilde{F}_{1,1}(a+N-n-1;\tau)d\tau$$

In general for the mth row, we perform $m-1$ times integration by parts and get a linear combination of the previous rows and an integral term. The coefficient in this linear combination before the ith row term is $(\pm 1)\prod_{j=i}^{m-1}\frac{1}{r_j-r_m}e^{(r_i-r_m)t}$ and the integral term is $(\pm 1)\prod_{i=1}^{m-1}\frac{1}{r_i-r_m}\int_0^t e^{-r_m\tau}\tilde{F}_{1,1}(a+N-n-1;\tau)d\tau$, hence

$$P_Y(x_1 \geq a) = \prod_{i=1}^{N} r_i^{a+N-1}$$
$$\begin{vmatrix} \int_0^t e^{-r_1\tau}\tilde{F}_{1,1}(a+N-2;\tau)d\tau & \int_0^t e^{-r_1\tau}\tilde{F}_{1,1}(a+N-3;\tau)d\tau & \cdots & \int_0^t e^{-r_1\tau}\tilde{F}_{1,1}(a-1;\tau)d\tau \\ -\frac{1}{r_1-r_2}\int_0^t e^{r_2\tau}\tilde{F}_{1,1}(a+N-2;\tau)d\tau & -\frac{1}{r_1-r_2}\int_0^t e^{-r_2\tau}\tilde{F}_{1,1}(a+N-3;\tau)d\tau & \cdots & -\frac{1}{r_1-r_2}\int_0^t e^{-r_2\tau}\tilde{F}_{1,1}(a-1;\tau)d\tau \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\pm 1}{\prod_{i=1}^{N-1}(r_i-r_N)}\int_0^t e^{-r_N\tau}\tilde{F}_{1,1}(a+N-2;\tau)d\tau & \cdots & \cdots & \cdots \end{vmatrix}$$
$$= (-1)^{\left[\frac{N}{2}\right]}\prod_{i=1}^{N} r_i^{a+N-1}\frac{1}{\prod\limits_{1\leq i<j\leq N}(r_i-r_j)}$$
$$\begin{vmatrix} \int_0^t e^{-r_1\tau}\tilde{F}_{1,1}(a+N-2;\tau)d\tau & \int_0^t e^{-r_1\tau}\tilde{F}_{1,1}(a+N-3;\tau)d\tau & \cdots & \int_0^t e^{-r_1\tau}\tilde{F}_{1,1}(a-1;\tau)d\tau \\ \int_0^t e^{r_2\tau}\tilde{F}_{1,1}(a+N-2;\tau)d\tau & \int_0^t e^{-r_2\tau}\tilde{F}_{1,1}(a+N-3;\tau)d\tau & \cdots & \int_0^t e^{-r_2\tau}\tilde{F}_{1,1}(a-1;\tau)d\tau \\ \vdots & \vdots & \cdots & \vdots \\ \int_0^t e^{-r_N\tau}\tilde{F}_{1,1}(a+N-2;\tau)d\tau & \int_0^t e^{-r_N\tau}\tilde{F}_{1,1}(a+N-3;\tau)d\tau & \cdots & \int_0^t e^{-r_N\tau}\tilde{F}_{1,1}(a-1;\tau)d\tau \end{vmatrix}$$

$$= (-1)^{\left[\frac{N}{2}\right]} \prod_{i=1}^{N} r_i^{a+N-1} \frac{1}{\prod\limits_{1\le i<j\le N}(r_i-r_j)} \int_{[0,t]^N} \prod_{i=1}^{N} e^{-r_i\tau_i} \begin{vmatrix} \tilde{F}_{1,1}(a+N-2;\tau_1) & \tilde{F}_{1,1}(a+N-3;\tau_1) & \cdots & \tilde{F}_{1,1}(a-1;\tau_1) \\ \tilde{F}_{1,1}(a+N-2;\tau_2) & \tilde{F}_{1,1}(a+N-3;\tau_2) & \cdots & \tilde{F}_{1,1}(a-1;\tau_2) \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{F}_{1,1}(a+N-2;\tau_N) & \tilde{F}_{1,1}(a+N-3;\tau_N) & \cdots & \tilde{F}_{1,1}(a-1;\tau_N) \end{vmatrix} d\tau_1\cdots d\tau_N$$

We look at the general (m,n)th entry in the above determinant, it is by definition, (note that $r_0 = 0$)

$$\begin{aligned} \tilde{F}_{1,1}(a+N-n-1;\tau_m) &= \oint e^{z^{-1}}\tau_m z^{a+N-n-1} đz \\ &= \frac{\tau_m^{a+N-n-1}}{(a+N-n-1)!} \end{aligned}$$

The last step follows from expanding the first factor. Hence, we have

$$\begin{aligned} P_Y(x_1\ge a) = (-1)^{\left[\frac{N}{2}\right]} &\frac{1}{\prod\limits_{1\le i<j\le N}(r_i-r_j)} \prod_{i=1}^{N} r_i^{a+N-1} \frac{1}{(a+N-i-1)!} \\ &\int_{[0,t]^N} \prod_{i=1}^{N} e^{-r_i\tau_i}\tau_i^{a-1} \begin{vmatrix} \tau_1^{N-1} & \tau_1^{N-2} & \cdots & 1 \\ \tau_2^{N-1} & \tau_2^{N-2} & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ \tau_N^{N-1} & \tau_N^{N-1} & \cdots & 1 \end{vmatrix} d\tau_1\cdots d\tau_N \\ &= (-1)^{\left[\frac{N}{2}\right]} \frac{1}{\prod\limits_{1\le i<j\le N}(r_i-r_j)} \prod_{i=1}^{N} r_i^{a+N-1} \frac{1}{(a+N-i-1)!} \int_{[0,t]^N} \prod_{i=1}^{N} e^{-r_i\tau_i} \prod_{1\le i<j\le N}(\tau_i-\tau_j) d\tau_1\cdots d\tau_N \end{aligned}$$

5 Discrete TASEP on the ring

In this section we consider the extension from continuous time to discrete time sequential update of the model and formula obtained in [3]. In this case, the free evolution equation, boundary condition and initial condition are the following [4]

$$U(x_1,...,x_N;t+1) = \sum_{b_1,...,b_N\in\{0,1\}} (1-p)^N \left(\frac{p}{1-p}\right)^{b_1+\cdots b_N} U(x_1-b_1,...,x_N-b_N;t) \tag{1}$$

$$U(...,x_{i-1},x_{i-1},x_{i+1},...;t) = U(...,x_{i-1},x_{i-1}+1,x_{i+1},...;t) \quad i=2,...,N \tag{2}$$

$$U(X;0) = \delta_{X=Y} \tag{3}$$

As suggested in [3], we should add one more boundary for the ring case, namely the following

$$U(x_1,...,x_{N-1},x_1+L-1;t) = U(x_1-1,x_2,...,x_{N-1},x_1+L-1) \tag{4}$$

Theorem 5.1. *(Transition probability) The transition probability for the discrete time sequential update TASEP on a ring is given by*

$$P_Y(X;t) = \oint_{|z|=r} det\left[\frac{1}{L}\sum_{w\in R_z}\frac{f_{ij}(x_i)}{w+\rho}\right]_{i,j=1}^{N} đz \tag{1}$$

Where $\rho = \frac{N}{L}$ with N the number of particles in one period, L is the length of the period, the definition of R_z is in [3]. And

$$f_{ij}(x_i) := w^{j-i+1}(w+1)^{-x_i+y_j+i-j}(1+pw)^t$$

Proof. We first check that the equation (5) satisfies the master equation. Note that we only need to check the determinant. Denote the determinant by $\tilde{U}(X;t)$. We have the right hand side is

$$\begin{aligned}
&\sum_{b_1,\ldots,b_N\in\{0,1\}}(1-p)^N\left(\frac{p}{1-p}\right)^{b_1+\cdots+b_N} det\left[\frac{1}{L}\sum_{w\in R_z}\frac{w^{j-i+1}(w+1)^{-x_i+b_i+y_j+i-j}(1+pw)^t}{w+\rho}\right]_{i,j=1}^N\\
&= det\left[\frac{1}{L}(1-p)\sum_{b_i=0}^{1}\left(\frac{p}{1-p}\right)^{b_i}\sum_{w\in R_z}\frac{w^{j-i+1}(w+1)^{-x_i+b_i+y_j+i-j}(1+pw)^t}{w+\rho}\right]_{i,j=1}^N\\
&= det\left[\frac{1}{L}(1-p)\sum_{w\in R_z}\frac{w^{j-i+1}(w+1)^{-x_i+y_j+i-j}(1+pw)^t}{w+\rho}\right]_{i,j=1}^N\\
&+ det\left[\frac{p}{L}\sum_{w\in R_z}\frac{w^{j-i+1}(w+1)^{-x_i+y_j+i-j}(w+1)(1+pw)^t}{w+\rho}\right]_{i,j=1}^N\\
&= det\left[\frac{1}{L}(pw+p+1-p)\sum_{w\in R_z}\frac{w^{j-i+1}(w+1)^{-x_i+y_j+i-j}(1+pw)^t}{w+\rho}\right]_{i,j=1}^N\\
&= det\left[\frac{1}{L}\sum_{w\in R_z}\frac{w^{j-i+1}(w+1)^{-x_i+y_j+i-j}(1+pw)^{t+1}}{w+\rho}\right]\\
&= \tilde{U}(X;t+1)
\end{aligned}$$

Hence we see that the formula in the theorem indeed satisfies the master equation.The proof of the formula satisfying two boundary conditions and initial condition is exactly the same as in [3] with all the factors e^{tw} changing to $(1+pw)^t$. For the detailed proof, see [3]. $\square$

Appendix

In this appendix, we briefly talk about how to derive the formula for transition probability in section 3 using the traditional bethe ansatz method, some of the formulas are already obtained in [1]. First, we find a fundamental solution of the free evolution equation. The solution is

$$\prod_{i=1}^{N} r_j^{x_j} z_j^{x_j} e^{(z_j^{-1}-r_j)t}$$

It turns out that for any permutation σ the following is also a solution of the free evolution euqation

$$\tilde{U}_\sigma(X;t)=\prod_{i=1}^{N} r_j^{x_j} z_{\sigma(j)}^{x_j} e^{(z_j^{-1}-r_j)t}$$

By the linearity of integral and the structure of the free evolution equation we have

$$\oint\cdots\oint\sum_{\sigma}C_\sigma(z)\tilde{U}_\sigma(X;t)đz_1\cdots đz_N$$

is also a solution of the free evolution eqaution, where $C_\sigma(z)$ is a function only depends on z and is yet to be decided. Here the contour is taken to be a circle centered at origin with radius less than $\frac{1}{r_{max}}$. Now we use the boundary condition and initial condition to determine the C function in the formula. The boundary condition gives us that

$$\sum_\sigma C_\sigma(z)(1-r_k z_{\sigma(k+1)}) \prod_{j\neq k,k+1} r_j^{x_j} z_{\sigma(j)}^{x_j} \prod_{j=1}^{N} e^{(z_j^{-1}-r_j)t} = 0$$

Which gives us

$$C_{T_k\sigma}(z) = -\frac{1-r_k z_{\sigma(k+1)}}{1-r_k z_{\sigma(k)}} C_\sigma(z) \quad k=1,...,N-1$$

Where T_k operator means exchanging the kth and $(k+1)$th position. It can be shown by looking at the specific contribution for one $z_{\sigma(k)}$ during the process of changing the permutation σ to identity that

$$C_\sigma(z) = C_{id}(z)(-1)^\sigma \prod_{k=1}^{N} \prod_{j=1}^{k-1} (1-r_j z_{\sigma(k)})^{-1} \prod_{j=1}^{\sigma(k)-1} (1-r_j z_{\sigma(k)})$$

Plug into the formual for U and together with initial condition we can show that $C_{id} = \prod_{j=1}^{N} r_j^{-y_j} z_j^{x_j}$. Plug this in and writing the final formula as determinant we get the final deternimantal solution in section 3.

References

[1] G.M.Schütz A.Rákos. "Bethe Ansatz and current distribution for the TASEP with Particle-Dependent Hopping Rates". In: *Markov Processes Relat. Fields* 12 (2006), pp. 323–334.

[2] G.M.Schütz A.Rákos. "Current distribution and random matrix ensembles for an integrable asymmetric fragmentation process". In: *J. Statist. Phys* 118.(3-4) (2005), pp. 511–530.

[3] J. Baik and Z. Liu. "Fluctuations of TASEP on a ring in relaxation time scale". In: *ArXiv e-prints* (May 2016). arXiv: 1605.07102 [math-ph].

[4] Alexei Borodin, Patrik L. Ferrari, and Prähofer Michael. "Fluctuations in the Discrete TASEP with Periodic Initial Configurations and the Airy1 Process". In: *International Mathematics Research Papers* 2007 (2007), rpm002. DOI: 10.1093/imrp/rpm002.

[5] Alexei Borodin, Patrik L. Ferrari, and Tomohiro Sasamoto. "Large Time Asymptotics of Growth Models on Space-like Paths II: PNG and Parallel TASEP". In: *Communications in Mathematical Physics* 283.2 (2008), pp. 417–449. ISSN: 1432-0916. DOI: 10.1007/s00220-008-0515-4. URL: http://dx.doi.org/10.1007/s00220-008-0515-4.

[6] Alexei Borodin and Sandrine Péché. "Airy Kernel with Two Sets of Parameters in Directed Percolation and Random Matrix Theory". In: *Journal of Statistical Physics* 132.2 (2008), pp. 275–290. DOI: 10.1007/s10955-008-9553-8. URL: https://doi.org/10.1007/s10955-008-9553-8.

[7] J. G. Brankov, V. B. Priezzhev, and R. V. Shelest. "Generalized determinant solution of the discrete-time totally asymmetric exclusion process and zero-range process". In: *Phys. Rev. E* 69 (6 2004), p. 066136. DOI: 10.1103/PhysRevE.69.066136. URL: https://link.aps.org/doi/10.1103/PhysRevE.69.066136.

[8] G.M.Schütz. "Exact solution of the master equation for the asymmetric exclusion process". In: *J. Statist. Phys* 88.(1-2) (1997), pp. 427–445.

[9] K.Johansson. "Shape fluctuations and random matrices". In: *Comm. Math. Phys.* 209.(2) (2000), pp. 437–476.

[10] Nikolai Okounkov Andrei; Reshetikhin. "Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram". In: *J. Amer. Math. Soc.* 16.3 (2003), pp. 581–603.

[11] A M Povolotsky and V B Priezzhev. "Determinant solution for the totally asymmetric exclusion process with parallel update". In: *Journal of Statistical Mechanics: Theory and Experiment* 2006.07 (2006), P07002. URL: http://stacks.iop.org/1742-5468/2006/i=07/a=P07002.