

# A Permutation Model with Finite Partitions of the Set of Atoms as Supports

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## Abstract

The method of permutation models was introduced by Fraenkel [1] in 1922 to prove the independence of the axiom of choice in set theory with atoms. We present a variant of the basic Fraenkel model, in which supports are finite partitions of the set of atoms, rather than finite sets of atoms. Among our results are that, in this model, every well-ordered family of well-orderable sets has a choice function and that the union of such a family is well-orderable.

## 1 Introduction

We begin with an overview of permutation models, borrowed largely from Jech [4, Ch. 4] and Halbeisen [2, Ch. 7].

### 1.1 Set theory with atoms

The version of set theory in which permutation models are used is *set theory with atoms* (ZFA). Unlike standard Zermelo-Fraenkel (ZF) set theory, ZFA admits the existence of a set of *atoms*, or objects that are not sets. An atom has no elements and is distinct from the empty set, but may be an element of a set. The language of ZFA adds to the language of ZF the constant symbols  $0$  and  $A$ , which denote the empty set and the set of atoms, respectively. Thus, the primitive symbols of ZFA are  $\in$ ,  $0$  and  $A$ . The axioms of ZFA include the axioms of ZF with the following changes:

- *Extensionality*  $(\forall \text{ set } X)(\forall \text{ set } Y)[\forall u(u \in X \leftrightarrow u \in Y) \leftrightarrow X = Y]$ ,
- *Regularity*  $(\forall \text{ nonempty set } S)(\exists x \in S)[x \cap S = 0]$ .

ZFA also includes the following axioms, which are not in ZF:

- *Empty set*     $\neg\exists x(x \in 0)$ ,
- *Atoms*     $\forall z[z \in A \leftrightarrow z \neq 0 \wedge \neg\exists x(x \in z)]$ .

The set  $A$  may be of any size, though we are interested in the case that  $A \neq 0$ . If ZFA were to include the axiom  $A = 0$ , then ZFA would be ZF.

The development of ZFA is analogous to the cumulative hierarchy ZF. The ordinals can be defined, in ZF, as transitive sets that are well-ordered. In ZFA, we will further require that no ordinal have an atom among its elements. Then for any set  $S$ , define  $\mathcal{P}^\alpha(S)$  by<sup>1</sup>

$$\begin{aligned}\mathcal{P}^0(S) &= S, \\ \mathcal{P}^{\alpha+1}(S) &= \mathcal{P}^\alpha(S) \cup \mathcal{P}(\mathcal{P}^\alpha(S)), \\ \mathcal{P}^\lambda(S) &= \bigcup_{\alpha < \lambda} \mathcal{P}^\alpha(S) \quad (\lambda \text{ limit ordinal}).\end{aligned}$$

Now define

$$\mathcal{P}^\infty(S) = \bigcup_{\alpha \in \text{Ord}} \mathcal{P}^\alpha(S),$$

where Ord is the class of all ordinals. Then the universe of ZFA is the class  $\mathcal{P}^\infty(A)$ .

**Definition 1.1.** The class  $\mathcal{P}^\infty(0)$  of all pure sets is called the *kernel*.

Let us make two observations: First, notice that the kernel, which is a subclass of the universe, is a model of ZF and contains all of the ordinals. Second, at each stage of the development of  $\mathcal{P}^\infty(A)$ , *all* possible sets of previously available objects (including choice functions) are added. Thus,  $\mathcal{P}^\infty(A)$  satisfies the axiom of choice. When building permutation models, we will always work in the theory ZFA + axiom of choice.

## 1.2 Permutation models

In this section, we will give the definition of ‘permutation model’. The crucial idea behind permutation models is that the axioms of ZFA do not distinguish between the atoms. Any permutation of the atoms may be extended to a permutation of the universe: Let  $\pi$  be any permutation of  $A$ . For every set

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<sup>1</sup> $\mathcal{P}(x)$  denotes the power set of  $x$ .

$x$ , there is a least ordinal  $\alpha$  such that  $x \in \mathcal{P}^\alpha(A)$ . So, by induction on the ordinals, we can define  $\pi x$  by

$$\pi x = \{\pi y : y \in x\}.$$

It is easily shown that  $x \in y$  if and only if  $\pi x \in \pi y$  for any two objects  $x$  and  $y$  in the universe. Thus,  $\pi$  is an  $\epsilon$ -automorphism of the universe.

Now, let  $\mathcal{G}$  be a group of permutations of  $A$ . Very roughly, the elements of a permutation model are the objects that are “stable” under certain subgroups of  $\mathcal{G}$ . The following two definitions will make this characterization more precise:

**Definition 1.2.** A set  $\mathcal{F}$  of subgroups of  $\mathcal{G}$  is a *normal filter* on  $\mathcal{G}$  if it satisfies the following:

- i.  $\mathcal{G} \in \mathcal{F}$ ;
- ii. if  $H \in \mathcal{F}$  and  $K$  is a subgroup of  $\mathcal{G}$  such that  $H \subseteq K$ , then  $K \in \mathcal{F}$ ;
- iii. if  $H \in \mathcal{F}$  and  $K \in \mathcal{F}$ , then  $H \cap K \in \mathcal{F}$ ;
- iv. if  $\pi \in \mathcal{G}$  and  $H \in \mathcal{F}$ , then  $\pi H \pi^{-1} \in \mathcal{F}$ ;
- v. for each  $a \in A$ ,  $\{\pi \in \mathcal{G} : \pi a = a\} \in \mathcal{F}$ .

**Convention 1.3.** The symbols  $\mathcal{G}$  and  $\mathcal{F}$  will always denote, respectively, a fixed group of permutations of  $A$  and a fixed normal filter on  $\mathcal{G}$ .

**Definition 1.4.** For each  $x$ , let  $\text{sym}(x) = \{\pi \in \mathcal{G} : \pi x = x\}$ . Then  $x$  is said to be *symmetric* if  $\text{sym}(x) \in \mathcal{F}$ .

This brings us to the definition of ‘permutation model’. The class

$$\mathcal{V} = \{x : x \text{ is symmetric and } x \subseteq \mathcal{V}\}$$

of all hereditarily symmetric objects<sup>2</sup> is called a *permutation model*. For proof that  $\mathcal{V}$  is a model of ZFA, the reader is referred to [4, Thm. 4.1]. It is also worth noting explicitly that all elements of the kernel are in  $\mathcal{V}$  and that  $A \in \mathcal{V}$ : Since  $0$  is symmetric, it is easily shown, by induction on the ordinals, that all pure sets are hereditarily symmetric. The set  $A$  is symmetric since  $\text{sym}(A) = \mathcal{G} \in \mathcal{F}$ . Further,  $\text{sym}(a) \in \mathcal{F}$  for each  $a \in A$  by definition of  $\mathcal{F}$ . Thus,  $A$  is hereditarily symmetric.

The permutation models in this paper will be of a special form. To state that form precisely, we need the following definition:

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<sup>2</sup>Note that the definition of  $\mathcal{V}$  is a recursion on the rank of  $x$ . An object  $x$  is hereditarily symmetric if  $x$  is symmetric and all elements of  $x$  are symmetric and all of their elements are symmetric, etc.

**Definition 1.5.** For each  $x$ , let  $\text{fix}(x) = \{\pi \in \mathcal{G} : \pi y = y \text{ for all } y \in x\}$ . We say that  $x$  is a *support* of  $y$  if  $\text{fix}(x) \subseteq \text{sym}(y)$ .

In our permutation models, the normal filter will always be determined by the group  $\mathcal{G}$  and by a set of supports. Here, let  $S$  denote the chosen set of supports. Then the corresponding normal filter is

$$\mathcal{F} = \{H : H \text{ is a subgroup of } \mathcal{G}, H \supseteq \text{fix}(x) \text{ for some } x \in S\}. \quad (1)$$

Note that not every set of supports will generate a normal filter. We will always choose  $S$  so that a normal filter is produced, but this must be verified in each case. Let us make two additional remarks: First, by this construction,  $x$  is symmetric if and only if there exists  $y \in S$  such that  $\text{fix}(y) \subseteq \text{sym}(x)$ . To mean “ $x$  has a support  $y \in S$ ,” we will usually just write “ $x$  has a support  $y$ ,” and it will be understood that  $y$  is of the correct type. Second, when building permutation models in this paper, we will not mention the normal filter explicitly; only the set of supports.

To conclude this section, we list a few tricks (found in [4, Ch. 4]) which we will use regularly.

**Lemma 1.6.**

- (a) For all  $\pi \in \mathcal{G}$  and all  $x$ ,  $\text{sym}(\pi x) = \pi \text{sym}(x) \pi^{-1}$  and  $\text{fix}(\pi x) = \pi \text{fix}(x) \pi^{-1}$ .
- (b) If  $x$  is a support of  $y$ , then  $\pi x$  is a support of  $\pi y$  for all  $\pi \in \mathcal{G}$ .
- (c) A set  $x$  is well-orderable in  $\mathcal{V}$  if and only if  $\text{fix}(x) \in \mathcal{F}$ .

*Proof.*

- (a) Let  $\sigma \in \text{sym}(\pi x)$ . Then  $\sigma \pi x = \pi x$ , so that  $\pi^{-1} \sigma \pi x = x$ . Hence,  $\pi^{-1} \sigma \pi \in \text{sym}(x)$ . It follows that  $\sigma \in \pi \text{sym}(x) \pi^{-1}$ . Conversely, let  $\tau \in \pi \text{sym}(x) \pi^{-1}$ . Writing  $\tau = \pi \gamma \pi^{-1}$ , where  $\gamma \in \text{sym}(x)$ , we have  $\tau \pi x = \pi \gamma \pi^{-1} \pi x = \pi \gamma x = \pi x$ , so that  $\tau \in \text{sym}(\pi x)$ . The other identity is proved similarly.
- (b) Since  $x$  is a support of  $y$ , we have  $\text{fix}(x) \subseteq \text{sym}(y)$ . So by (a), we get  $\text{fix}(\pi x) = \pi \text{fix}(x) \pi^{-1} \subseteq \pi \text{sym}(y) \pi^{-1} = \text{sym}(\pi y)$ . Hence,  $\pi x$  is a support of  $\pi y$ .
- (c) We know that  $x$  is well-orderable in  $\mathcal{V}$  if and only if there exists in  $\mathcal{V}$  a one-to-one mapping  $f : x \rightarrow \alpha$  for some ordinal  $\alpha$ . It is easily shown, by induction on the ordinals, that each element of the kernel is fixed by  $\mathcal{G}$ . Suppose there is such a function. Then  $f(y)$  is in the kernel for

each  $y \in x$ . So, for all  $\pi \in \mathcal{G}$  and  $y \in x$ , we have  $\pi f(y) = f(y)$ , so that  $\pi \langle y, f(y) \rangle = \langle \pi y, f(y) \rangle$ . Since  $f$  is one-to-one, this implies that  $\pi f = f$  only if  $\pi y = y$  for all  $y \in x$ . That is,  $\text{fix}(x) \supseteq \text{sym}(f) \in \mathcal{F}$ , so that  $\text{fix}(x) \in \mathcal{F}$ . Conversely, suppose  $\text{fix}(x) \in \mathcal{F}$ , and use the axiom of choice in the full universe<sup>3</sup>  $\mathcal{P}^\infty(A)$  to find a one-to-one mapping  $f : x \rightarrow \alpha$  for some ordinal  $\alpha$ . We will show that  $f \in \mathcal{V}$ . For all  $\pi \in \text{fix}(x)$  and  $y \in x$ , we have  $\pi \langle y, f(y) \rangle = \langle y, \pi f(y) \rangle$ . This implies that  $\pi f(y) = f(y)$  for all  $y$ , so that  $\pi \in \text{sym}(f)$ . Hence,  $\text{sym}(f) \supseteq \text{fix}(x) \in \mathcal{F}$ , so that  $f$  is symmetric. Since  $x$  and  $\alpha$  are hereditarily symmetric,  $f$  is also hereditarily symmetric. □

## 2 The basic Fraenkel model

In this section, we present a few well-known examples involving the basic Fraenkel model and one of its variants. This is done to acquaint the reader with some common techniques in permutation models and to state a few results to which we can compare the new model introduced in the next section.

Assume  $A$  is countably infinite, let  $\mathcal{G}$  be the group of all permutations of  $A$ , and let the supports be finite subsets of  $A$ . The corresponding permutation model is called the basic Fraenkel model and will be denoted by  $\mathcal{V}_F$ .

**Example 2.1.** *The Well-Ordering Principle is false in  $\mathcal{V}_F$ . [4, §4.3]*

*Proof.* We will show that the set  $A$  cannot be well-ordered. By Lemma 1.6(c),  $A$  is well-orderable in  $\mathcal{V}_F$  only if  $\text{fix}(A) \in \mathcal{F}$ . Further,  $\text{fix}(A) \in \mathcal{F}$  only if there exists a finite subset  $E$  of  $A$  such that  $\text{fix}(E) \subseteq \text{fix}(A)$ . For any such  $E$ , however, we can find  $a, b \in A \setminus E$  and  $\pi \in \text{fix}(E)$  such that  $\pi a = b$ . That is, we can find  $\pi \in \text{fix}(E)$  such that  $\pi \notin \text{fix}(A)$ . Thus,  $A$  is not well-orderable in  $\mathcal{V}_F$ . □

**Example 2.2.** *The statement “Every family of pairs has a choice function” is false in  $\mathcal{V}_F$ . [4, Exercise 4.3]*

*Proof.* We will show that the family  $S = \{\{a, b\} : a, b \in A\}$  has no choice function in  $\mathcal{V}_F$ . Suppose there is such a function  $c$ . Then  $c$  has a support  $E$ . Since  $E$  is finite, we can find  $a, b \in A \setminus E$  and  $\pi \in \text{fix}(E)$  such that  $\pi a = b$  and  $\pi b = a$ . Assume that  $\langle \{a, b\}, a \rangle \in c$  (otherwise  $\langle \{a, b\}, b \rangle \in c$ , and the argument is similar). Then for  $\pi \in \text{fix}(E) \subseteq \text{sym}(c)$ , we have  $\langle \{a, b\}, b \rangle = \pi \langle \{a, b\}, a \rangle \in$

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<sup>3</sup>That is, the “outer” universe; the class of all atoms and sets (hereditarily symmetric or not).

$\pi c = c$ . But the fact that  $\langle \{a, b\}, a \rangle \in c$  and  $\langle \{a, b\}, b \rangle \in c$  shows that  $c$  is not a function, a contradiction.  $\square$

**Corollary 2.3.** *The Ordering Principle is false in  $\mathcal{V}_F$ .*

*Proof.* If  $A$  could be linearly ordered, then  $c(\{a, b\}) = \min(\{a, b\})$  would be a choice function on  $S$ .  $\square$

**Definition 2.4.** A set is *amorphous* if it is not the union of two infinite, disjoint sets.

If the axiom of choice holds, then all amorphous sets are finite (see [5]). In models of ZFA without the axiom of choice, such as the basic Fraenkel model, we can find infinite amorphous sets:

**Example 2.5.** *The set  $A$  is infinite, but amorphous in  $\mathcal{V}_F$ . [4, Exercise 4.7]*

*Proof.* To show that  $A$  is amorphous, assume for contradiction that there exist infinite, disjoint sets  $X$  and  $Y$  such that  $A = X \cup Y$ . Then the set  $X$  has a support  $E$ . Since  $E$  is finite, the sets  $X \setminus E$  and  $Y \setminus E$  are nonempty. So, we can find  $a \in X \setminus E$ ,  $b \in Y \setminus E$  and  $\pi \in \text{fix}(E)$  such that  $\pi a = b$ . Since  $\text{fix}(E) \subseteq \text{sym}(X)$ , we have  $b = \pi a \in \pi X = X$ . This contradicts the fact that  $X$  and  $Y$  are disjoint.  $\square$

Consider the following variant of the basic Fraenkel model:  $A$  is uncountable,  $\mathcal{G}$  is the group of all permutations of  $A$ , and the supports are countable subsets of  $A$ . We will call the corresponding permutation model  $\mathcal{V}_{F+}$ .

**Example 2.6.** *The statement “Every well-ordered family of sets has a choice function” is true in  $\mathcal{V}_{F+}$ . [3,  $\mathcal{N}12(\aleph_1)$ ]*

*Proof.* Let  $W$  be a well-ordered family of sets. By Lemma 1.6(c),  $\text{fix}(W) \in \mathcal{F}$ . Therefore, there exists a countable subset  $E_0$  of  $A$  such that  $\text{fix}(E_0) \subseteq \text{fix}(W)$ . Without loss of generality, assume that  $E_0 = 0$  (the set  $A \setminus E_0$  looks like  $A$ ). Now, let  $S$  be any countable subset of  $A$ . First, we will show that every set in  $W$  contains an element that is fixed by the group  $\text{fix}(S)$ . Let  $X \in W$  and let  $x \in X$ . We know that  $x$  has a support  $E$ ; assume  $E$  is infinite. Since  $E$  and  $S$  are isomorphic to  $\mathbb{N}$ , we can find a permutation  $\pi \in \mathcal{G}$  such that  $\pi E = S$ . By Lemma 1.6(b),  $S$  is a support of  $\pi x$ . Further,  $\pi x \in X$ , since  $X$  is fixed by  $\text{fix}(E_0) = \mathcal{G}$ . Now, let  $W_S = \{X_S : X \in W\}$ , where  $X_S = \{x \in X : \text{fix}(S) \subseteq \text{sym}(x)\}$ . We have shown that  $0 \neq X_S \subseteq X$  for each  $X \in W$ . Therefore, a choice function on  $W_S$  will yield a choice function on  $W$ . The axiom of choice is true in the full universe, so let  $c$  be a choice function on  $W_S$ . We will show

that, in fact,  $c$  is in the permutation model; i.e.  $c$  is hereditarily symmetric. For each  $\langle X_S, x \rangle \in c$  and  $\pi \in \text{fix}(S)$ , we have  $\pi \langle X_S, x \rangle = \langle X_S, x \rangle \in c$ . This shows that  $c$  is symmetric, supported by  $S$ . Since  $X_S$  and  $x$  are elements of  $\mathcal{V}_{F^+}$  and are therefore hereditarily symmetric, it follows that  $c$  is hereditarily symmetric.  $\square$

### 3 The finite partition model

In this section, we introduce a new permutation model and present our main results.

Assume  $A$  is countably infinite, let  $\mathcal{G}$  be the group of all permutations of  $A$ , and let the supports be finite partitions of  $A$  (Lemma 3.1 will show that the supports generate a normal filter). We will refer to the corresponding permutation model as the ‘finite partition model,’ denoted by  $\mathcal{V}_p$ .

**Lemma 3.1.** *Let  $S$  be the set of all finite partitions of  $A$ . Then the set*

$$\mathcal{F} = \{H : H \text{ is a subgroup of } \mathcal{G}, H \supseteq \text{fix}(P) \text{ for some } P \in S\}$$

*is a normal filter on  $\mathcal{G}$ .*

*Proof.* Provided  $\mathcal{F}$  is of the form (1) and the set  $S$  is nonempty, clauses i and ii of Definition 1.2 are trivial. Since that is the case here, we verify clauses iii-v:

- iii. Let  $H, K \in \mathcal{F}$ . Then there exist  $P, Q \in S$  such that  $\text{fix}(P) \subseteq H$  and  $\text{fix}(Q) \subseteq K$ . Let  $P \wedge Q$  denote the coarsest common refinement of  $P$  and  $Q$ , given by

$$P \wedge Q = \{p \cap q : p \in P, q \in Q, p \cap q \neq \emptyset\}.$$

Since  $P \wedge Q$  is a common refinement of  $P$  and  $Q$ , it is clear that  $\text{fix}(P \wedge Q) \subseteq \text{fix}(P)$  and  $\text{fix}(P \wedge Q) \subseteq \text{fix}(Q)$ . Therefore,  $\text{fix}(P \wedge Q) \subseteq \text{fix}(P) \cap \text{fix}(Q) \subseteq H \cap K$ . Since  $P \wedge Q \in S$ , this implies that  $H \cap K \in \mathcal{F}$ .

- iv. Let  $\pi \in \mathcal{G}$  and  $H \in \mathcal{F}$ . Then there exists  $P \in S$  such that  $\text{fix}(P) \subseteq H$ . By Lemma 1.6(a),  $\pi \text{fix}(P) \pi^{-1} = \text{fix}(\pi P)$ . Since  $\pi \text{fix}(P) \pi^{-1} \subseteq \pi H \pi^{-1}$ , it is enough to show that  $\pi P \in S$ . Clearly,  $\pi P$  is finite, since  $P$  is finite. To see that  $\pi P$  is a partition of  $A$ , let  $a \in A$ . Since  $\pi^{-1}a \in p$  for some  $p \in P$ , we have  $a \in \pi p \in \pi P$ . Let  $\pi p' \in \pi P$  be such that  $a \in \pi p'$ . Then  $\pi^{-1}a \in p'$ , so that  $p' \cap p \neq \emptyset$ . Consequently,  $p' = p$  and  $\pi p' = \pi p$ .

v. For each  $a \in A$ ,  $\{\pi \in \mathcal{G} : \pi a = a\} = \text{fix}(\{\{a\}, A \setminus \{a\}\}) \in \mathcal{F}$ .

□

The rest of this paper will address the question, “What is happening in the finite partition model?” It will be beneficial to keep in mind the basic Fraenkel model. For example, the finite partition model includes the basic Fraenkel model:

**Proposition 3.2.**  $\mathcal{V}_F \subset \mathcal{V}_p$ .

*Proof.* Let  $\mathcal{F}_F$  denote the normal filter of  $\mathcal{V}_F$ , generated by finite subsets of  $A$ . Let  $\mathcal{F}_p$  denote the normal filter of  $\mathcal{V}_p$ , generated by finite partitions of  $A$ . If  $x$  is symmetric with respect to  $\mathcal{F}_F$ , then there exists a finite subset  $E$  of  $A$  such that  $\text{fix}(E) \subseteq \text{sym}(x)$ . Let  $P = \{\{a\}\}_{a \in E} \cup \{A \setminus E\}$ . Then  $P$  is a finite partition of  $A$ , and  $\text{fix}(P) = \text{fix}(E)$ . Consequently,  $\text{fix}(P) \subseteq \text{sym}(x)$ , so that  $x$  is symmetric with respect to  $\mathcal{F}_p$ . This shows that any object that is hereditarily symmetric with respect to  $\mathcal{F}_F$  is hereditarily symmetric with respect to  $\mathcal{F}_p$ ; i.e.  $y \in \mathcal{V}_F$  implies  $y \in \mathcal{V}_p$  for all  $y$ . □

Due to Proposition 3.2 and the intensional similarity of  $\mathcal{V}_F$  and  $\mathcal{V}_p$  (same  $A$  and same  $\mathcal{G}$ ), we might expect that most statements which are true in  $\mathcal{V}_F$  will also be true in  $\mathcal{V}_p$ . Though little is known yet about  $\mathcal{V}_p$ , our results are consistent with this intuition. Proposition 3.3 shows that Example 2.5 does not transfer to  $\mathcal{V}_p$ . Further,  $\mathcal{V}_F$  satisfies the statement “Every set is either well-orderable or has an infinite amorphous subset” (see [3,  $\mathcal{N}1$ ]), which is consistent with Examples 2.1 and 2.5, while Propositions 3.3 and 3.4 show that  $\mathcal{V}_p$  does not. However, Proposition 3.4 and all subsequent propositions are true in both models<sup>4</sup>.

**Proposition 3.3.** *The set  $A$  has no infinite amorphous subset in  $\mathcal{V}_p$ .*

*Proof.* Let  $S$  be an infinite subset of  $A$ , and let  $X$  and  $Y$  be infinite, disjoint subsets of  $S$  such that  $S = X \cup Y$ . Then  $\{X, Y, A \setminus S\}$  is a finite partition of  $A$ , and  $\text{fix}(\{X, Y, A \setminus S\}) \subseteq \text{sym}(X)$  and  $\text{fix}(\{X, Y, A \setminus S\}) \subseteq \text{sym}(Y)$ . Hence,  $X, Y \in \mathcal{V}_p$ . □

**Proposition 3.4.** *The Well-Ordering Principle is false in  $\mathcal{V}_p$ .*

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<sup>4</sup>See [3,  $\mathcal{N}1$ ] for a list of consequences of the axiom of choice that are known to be true, or known to be false, in the basic Fraenkel model.



*Proof.* We will show that the set  $A$  cannot be well-ordered. Let  $P$  be any finite partition of  $A$ . Then there exists a block  $p$  in  $P$  such that  $|p| \geq 2$ . Let  $a, b \in p$  and  $\pi \in \text{fix}(P)$  be such that  $\pi a = b$ . Since  $\pi \notin \text{fix}(A)$ , we have that  $\text{fix}(P) \not\subseteq \text{fix}(A)$ . Hence,  $\text{fix}(A) \notin \mathcal{F}$ . But  $A$  can be well-ordered in  $\mathcal{V}_p$  only if  $\text{fix}(A) \in \mathcal{F}$ .  $\square$

**Proposition 3.5.** *The statement “Every family of pairs has a choice function” is false in  $\mathcal{V}_p$ .*

*Proof.* We will show that the set  $S = \{\{a, b\} : a, b \in A\}$  has no choice function in  $\mathcal{V}_p$ . Suppose there is such a function  $c$ , and let  $P$  be a support of  $c$ . Let  $p \in P$  be such that  $|p| \geq 2$ . Then we can find  $a, b \in p$  and  $\pi \in \text{fix}(P)$  such that  $\pi a = b$  and  $\pi b = a$ . Assume that  $\langle \{a, b\}, a \rangle \in c$  (the other case is similar). Then we also have  $\langle \{a, b\}, b \rangle = \langle \pi\{a, b\}, \pi a \rangle \in \pi c = c$ , which shows that  $c$  is not well-defined.  $\square$

**Corollary 3.6.** *The Ordering Principle is false in  $\mathcal{V}_p$ .*

**Definition 3.7.** A set is *Dedekind finite* if it has no countably infinite subset.

Like amorphous sets, all Dedekind finite sets are finite, provided the axiom of choice holds (see [5]). In the finite partition model, we find the following:

**Proposition 3.8.** *The set  $A$  is infinite, but Dedekind finite in  $\mathcal{V}_p$ .*

*Proof.* Assume for contradiction that  $A$  is Dedekind infinite. Then there exists a one-to-one function  $f : \mathbb{N} \rightarrow A$ . Let  $P$  be a support of  $f$ , and let  $\pi \in \text{fix}(P)$  be such that  $\pi$  moves every atom in each non-singleton block of  $P$ . Since  $P$  contains only finitely many singletons,  $\pi$  fixes only finitely many atoms. Now, let  $n \in \mathbb{N}$ . Since  $n$  is in the kernel, we have  $\pi n = n$ . This implies that  $\pi(f(n)) = f(n)$ , for otherwise  $\pi n = n$  while  $f(\pi n) = \pi(f(n)) \neq f(n)$ . But  $f$  is one-to-one, and thus,  $\pi$  fixes  $\omega$  many values of  $f$  in  $A$ , a contradiction.  $\square$

**Proposition 3.9.** *The set  $\mathcal{P}(A)$  is infinite, but Dedekind finite in  $\mathcal{V}_p$ .*

To prove Proposition 3.9, we will need the following lemma:

**Lemma 3.10.** *Let  $P$  be a finite partition of  $A$ . Then  $E \subset A$  is supported by  $P$  if and only if  $E$  is the union of a subset of  $P$ . Therefore, only finitely many subsets of  $A$  are supported by  $P$ .*

*Proof.* Let  $E$  be the union of a subset of  $P$ , and assume  $E \neq \emptyset$ . Then for each  $a \in E$ , there exists a block  $p \in P$  such that  $a \in p \subset E$ . Let  $\pi \in \text{fix}(P)$ . Then we have  $\pi a \in \pi p = p \subset E$ . Thus,  $P$  supports  $E$ . Conversely, suppose  $E$  is not the union of a subset of  $P$ . Then there exists a block  $p \in P$  such that  $p \cap E$  and  $p \setminus E$  are nonempty. For  $a \in p \cap E$  and  $b \in p \setminus E$ , let  $\pi \in \text{fix}(P)$  be such that  $\pi a = b$ . Then  $\pi a \notin E$ , which shows that  $P$  does not support  $E$ .  $\square$

*Proof of Proposition 3.9.* Suppose for contradiction that  $\mathcal{P}(A)$  is Dedekind infinite. Then there exists a one-to-one function  $f : \mathbb{N} \rightarrow \mathcal{P}(A)$ . Let  $P$  be a support of  $f$ , and let  $\pi$  be an arbitrary permutation in  $\text{fix}(P)$ . Since each  $n \in \mathbb{N}$  is in the kernel and  $f$  is one-to-one,  $\pi$  fixes  $\omega$  many values of  $f$  in  $\mathcal{P}(A)$ . This shows that  $P$  supports infinitely many subsets of  $A$ , contradicting Lemma 3.10.  $\square$

**Proposition 3.11.** *The statement “Every well-ordered family of pairs has a choice function” is true in  $\mathcal{V}_p$ .*

To prove Proposition 3.11, we will need a definition and a few lemmas:

**Definition 3.12.** Let  $P$  and  $Q$  be partitions of a set  $S$ . Then  $P$  and  $Q$  are said to be *independent* if  $p \cap q$  is infinite for all  $p \in P$  and  $q \in Q$ .

**Lemma 3.13.** *Let  $P = \{X_1, X_2\}$  and  $Q = \{Y_1, Y_2\}$  be independent partitions of a subset  $S$  of  $A$ . Let  $H$  be the group of all permutations of  $S$ , and let  $\pi \in H$ . Then there exists a finite sequence of permutations  $\pi_1, \dots, \pi_n \in \text{fix}_H(P) \cup \text{fix}_H(Q)$  such that  $\pi = \pi_n \cdots \pi_1$ .*

*Proof.* For convenience, let  $B_1 = X_1 \cap Y_1$ ,  $B_2 = X_1 \cap Y_2$ ,  $B_3 = X_2 \cap Y_2$  and  $B_4 = X_2 \cap Y_1$ . First, divide each block  $B_i$  into four sets  $S_{ij}$  defined by  $S_{ij} = \{a \in B_i : \pi a \in B_j\}$ . Further, for each  $S_{ij}$ , let  $R_{ij}$  (“red atoms”) and  $G_{ij}$  (“green atoms”) be such that  $S_{ij} = R_{ij} \dot{\cup} G_{ij}$ , and such that  $R_{ij}$  and  $G_{ij}$  are infinite if  $S_{ij}$  is infinite. The proof will be done in three steps: First, we will find permutations  $\pi_1, \dots, \pi_m \in \text{fix}_H(P) \cup \text{fix}_H(Q)$  that move each red atom to its correct block, while keeping each green atom in its original block. That is,  $\pi_1, \dots, \pi_m$  will be such that, for  $a \in S_{ij}$ , if  $a \in R_{ij}$ , then  $\pi_m \cdots \pi_1 a \in B_j$ ; otherwise,  $\pi_m \cdots \pi_1 a \in B_i$ . Second, we will find permutations  $\pi_{m+1}, \dots, \pi_{n-1} \in \text{fix}_H(P) \cup \text{fix}_H(Q)$  that move each green atom to its correct block, while keeping the image of each red atom under  $\pi_m \cdots \pi_1$  in the same block. Finally, we will find a permutation  $\pi_n \in \text{fix}_H(P) \cup \text{fix}_H(Q)$  that moves the image of each atom under  $\pi_{n-1} \cdots \pi_1$ , now already in the correct block, to its image under  $\pi$ .

To begin, one observation will be useful: For each  $i$ , the set  $S_{ij}$  is infinite for at least one  $j$ , and similarly for each  $j$ , the set  $S_{ij}$  is infinite for at least one  $i$ . Therefore, the sets  $\bigcup_j G_{ij}$  and  $\bigcup_j R_{ji}$  are infinite for each  $i$ .

Now, let  $E_i \subset B_i$  be such that

- $|E_1| = |R_{21}|$  and  $B_1 \setminus E_1$  is infinite;
- $|E_2| = |R_{12} \cup R_{13}|$  and  $B_2 \setminus E_2$  is infinite;
- $|E_3| = |R_{43}|$  and  $B_3 \setminus E_3$  is infinite;

- $|E_4| = |R_{31} \cup R_{34}|$  and  $B_4 \setminus E_4$  is infinite.

Since  $\cup_j G_{1j} \subset B_1 \setminus (R_{12} \cup R_{13})$ ,  $\cup_j G_{2j} \subset B_2 \setminus R_{21}$ ,  $\cup_j G_{3j} \subset B_3 \setminus (R_{31} \cup R_{34})$  and  $\cup_j G_{4j} \subset B_4 \setminus R_{43}$ , we know that  $B_1 \setminus (R_{12} \cup R_{13})$ ,  $B_2 \setminus R_{21}$ ,  $B_3 \setminus (R_{31} \cup R_{34})$  and  $B_4 \setminus R_{43}$  are infinite. Therefore, we can find a permutation  $\pi_1 \in \text{fix}_H(P)$  such that

$$\left\{ \begin{array}{l} \pi_1(R_{12} \cup R_{13}) = E_2 \\ \pi_1(B_1 \setminus (R_{12} \cup R_{13})) = B_1 \setminus E_1 \\ \pi_1 R_{21} = E_1 \\ \pi_1(B_2 \setminus R_{21}) = B_2 \setminus E_2 \\ \pi_1(R_{31} \cup R_{34}) = E_4 \\ \pi_1(B_3 \setminus (R_{31} \cup R_{34})) = B_3 \setminus E_3 \\ \pi_1 R_{43} = E_3 \\ \pi_1(B_4 \setminus R_{43}) = B_4 \setminus E_4 \end{array} \right.$$

Similarly, let  $E'_i \subset B_i$  be such that

- $|E'_1| = |\pi_1(R_{31} \cup R_{41} \cup R_{42})|$  and  $B_1 \setminus E'_1$  is infinite;
- $|E'_2| = |\pi_1 R_{32}|$  and  $B_2 \setminus E'_2$  is infinite;
- $|E'_3| = |\pi_1(R_{13} \cup R_{23} \cup R_{24})|$  and  $B_3 \setminus E'_3$  is infinite;
- $|E'_4| = |\pi_1 R_{14}|$  and  $B_4 \setminus E'_4$  is infinite.

Since  $B_1 \setminus \pi_1 R_{14}$ ,  $B_2 \setminus \pi_1(R_{13} \cup R_{23} \cup R_{24})$ ,  $B_3 \setminus \pi_1 R_{32}$  and  $B_4 \setminus \pi_1(R_{31} \cup R_{41} \cup R_{42})$  are infinite, we can find a permutation  $\pi_2 \in \text{fix}_H(Q)$  such that

$$\left\{ \begin{array}{l} \pi_2(\pi_1 R_{14}) = E'_4 \\ \pi_2(B_1 \setminus \pi_1 R_{14}) = B_1 \setminus E'_1 \\ \pi_2(\pi_1(R_{13} \cup R_{23} \cup R_{24})) = E'_3 \\ \pi_2(B_2 \setminus \pi_1(R_{13} \cup R_{23} \cup R_{24})) = B_2 \setminus E'_2 \\ \pi_2(\pi_1 R_{32}) = E'_2 \\ \pi_2(B_3 \setminus \pi_1 R_{32}) = B_3 \setminus E'_3 \\ \pi_2(\pi_1(R_{31} \cup R_{41} \cup R_{42})) = E'_1 \\ \pi_2(B_4 \setminus \pi_1(R_{31} \cup R_{41} \cup R_{42})) = B_4 \setminus E'_4 \end{array} \right.$$

By the same method, we can find a permutation  $\pi_3 \in \text{fix}_H(P)$  such that  $\pi_3$  moves  $\pi_2 \pi_1 R_{42}$  from  $B_1$  to  $B_2$  and  $\pi_2 \pi_1 R_{24}$  from  $B_3$  to  $B_4$ , while keeping every

other  $\pi_2\pi_1R_{ij}$  in the same block. Now we have  $\pi_3\pi_2\pi_1R_{ij} \subset B_j$  for each  $i$ , as desired in the first step.

Using the fact that the set  $\pi_3\pi_2\pi_1 \cup_j R_{ji}$  is infinite for each  $i$ , the second stage follows by a method similar to that of the first stage. This will give us permutations  $\pi_4, \pi_5, \pi_6 \in \text{fix}_H(P) \cup \text{fix}_H(Q)$  such that  $\pi_6 \cdots \pi_1 S_{ij} = \pi_6 \cdots \pi_1 (R_{ij} \cup G_{ij}) \subset B_j$  for each  $i$ . In other words, if  $\pi a \in B_j$ , then  $\pi_6 \cdots \pi_1 a \in B_j$  for all  $a \in S$ . This makes the final step clear: We can find a permutation  $\pi_7 \in \text{fix}_H(P) \cup \text{fix}_H(Q)$  such that  $\pi_7 \cdots \pi_1 a = \pi a$  for all  $a$ . Hence,  $\pi = \pi_7 \cdots \pi_1$ .  $\square$

The following lemma is a version of Mostowski's intersection lemma (88 in Mostowski [6]). Part (a) is a known result, implied, for example, by IIA4 in Pincus [7]. For the convenience of the reader, we prove this special case. With only minor adjustments, part (b) is then obtained from (a).

**Lemma 3.14.**

- (a) *Let  $E_1$  and  $E_2$  be finite subsets of an infinite subset  $S$  of  $A$ . Let  $H$  be the group of all permutations of  $S$ , and let  $\pi \in \text{fix}_H(E_1 \cap E_2)$ . Then there exists a finite sequence of permutations  $\pi_1, \dots, \pi_n \in \text{fix}_H(E_1) \cup \text{fix}_H(E_2)$  such that  $\pi = \pi_n \cdots \pi_1$ .*
- (b) *Let  $P$  and  $Q$  be finite partitions of a subset  $S$  of  $A$ , each consisting of singletons and one infinite block. Let  $P_1 = \{p \in P : |p| = 1\}$  and  $Q_1 = \{q \in Q : |q| = 1\}$ . Let  $H$  be the group of all permutations of  $S$ , and let  $\pi \in \text{fix}_H(P_1 \cap Q_1)$ . Then there exists a finite sequence of permutations  $\pi_1, \dots, \pi_n \in \text{fix}_H(P_1) \cup \text{fix}_H(Q_1)$  such that  $\pi = \pi_n \cdots \pi_1$ .*

*Proof.*

- (a) Let  $B_1 = E_1 \setminus E_2$ ,  $B_2 = E_2 \setminus E_1$  and  $B_3 = S \setminus (E_1 \cup E_2)$ . Since  $\pi \in \text{fix}_H(E_1 \cap E_2)$  and  $\cup_i B_i = S \setminus (E_1 \cap E_2)$ , we know that  $\pi \cup_i B_i = \cup_i B_i$ . So, divide each  $B_i$  into three sets  $S_{ij}$  defined by  $S_{ij} = \{a \in B_i : \pi a \in B_j\}$ . Two observations will be useful: First,  $B_3$  is infinite, while  $B_1$  and  $B_2$  are finite. Therefore,  $S_{33}$  is infinite. Second, since  $B_1$  is finite, the number of elements in  $B_1$  moved outside  $B_1$  by  $\pi$  is equal to the number of elements outside  $B_1$  moved into  $B_1$  by  $\pi$ ; i.e.  $|S_{12} \cup S_{13}| = |S_{21} \cup S_{31}|$ . For the same reason,  $|S_{21} \cup S_{23}| = |S_{12} \cup S_{32}|$ .

By the first observation, we can find a subset  $T$  of  $S_{33}$  such that  $|T| = |S_{12}|$ . By the second observation, we have  $|B_2| \geq |S_{21} \cup S_{23}| = |S_{12} \cup S_{32}| = |S_{32} \cup T|$ . So, we can find a subset  $X$  of  $B_2$  such that  $|X| = |S_{32} \cup T|$ . Let  $Y \subset B_3$  be such that  $|Y| = |S_{21} \cup S_{23}|$ . It follows that  $|X| = |Y|$  and that

$B_3 \setminus (S_{32} \cup T)$  is infinite. Therefore, we can find a permutation  $\pi_1 \in \text{fix}_H(E_1)$  such that

$$\begin{cases} \pi_1(S_{21} \cup S_{23}) = Y \\ \pi_1(B_2 \setminus (S_{21} \cup S_{23})) = B_2 \setminus X \\ \pi_1(S_{32} \cup T) = X \\ \pi_1(B_3 \setminus (S_{32} \cup T)) = B_3 \setminus Y \end{cases} .$$

Similarly, we can find  $X' \subseteq B_1$  and  $Y' \subseteq B_3$  such that  $|X'| = |\pi_1(S_{21} \cup S_{31})|$  and  $|Y'| = |\pi_1(S_{12} \cup S_{13})|$ . Observing that  $|X'| = |Y'|$  and that  $B_3 \setminus \pi_1(S_{21} \cup S_{31})$  is infinite, we can find a permutation  $\pi_2 \in \text{fix}_H(E_2)$  such that

$$\begin{cases} \pi_2(\pi_1(S_{12} \cup S_{13})) = Y' \\ \pi_2(B_1 \setminus \pi_1(S_{12} \cup S_{13})) = B_1 \setminus X' \\ \pi_2(\pi_1(S_{21} \cup S_{31})) = X' \\ \pi_2(B_3 \setminus \pi_1(S_{21} \cup S_{31})) = B_3 \setminus Y' \end{cases} .$$

Finally, recalling that  $|T| = |S_{12}|$ , we can find a permutation  $\pi_3 \in \text{fix}_H(E_1)$  such that  $\pi_3(\pi_2\pi_1T) = \pi_2\pi_1S_{12} \subseteq B_3$  and  $\pi_3(\pi_2\pi_1S_{12}) = \pi_2\pi_1T \subseteq B_2$ .

Now we have  $\pi_3\pi_2\pi_1S_{ij} \subseteq B_j$  for all  $i$ . In other words, if  $\pi a \in B_j$ , then  $\pi_3\pi_2\pi_1a \in B_j$  for all  $a \in S$ . Therefore, we can find a permutation  $\pi_4 \in \text{fix}_H(E_1) \cup \text{fix}_H(E_2)$  such that  $\pi_4 \cdots \pi_1a = \pi a$  for all  $a \in S$ . Hence,  $\pi = \pi_4 \cdots \pi_1$ .

- (b) Let  $E_1 = \bigcup P_1$  and  $E_2 = \bigcup Q_1$ . Observe that  $E_1$  and  $E_2$  are finite subsets of  $S$  and that  $\text{fix}_H(P_1) = \text{fix}_H(E_1)$ ,  $\text{fix}_H(Q_1) = \text{fix}_H(E_2)$  and  $\text{fix}_H(P_1 \cap Q_1) = \text{fix}_H(E_1 \cap E_2)$ . The result follows from (a).

□

**Lemma 3.15.** *Let  $P$  be a support of a set  $\{x, y\}$ , and let  $Q$  be a refinement of  $P$  such that  $Q$  supports  $x$ . Let  $p$  be an infinite block of  $P$ , and let  $Q_{p,\infty} = \{q \in Q : q \subseteq p, q \text{ is infinite}\}$ . Then  $(Q \setminus Q_{p,\infty}) \cup \{\bigcup Q_{p,\infty}\}$  is a support of  $x$ .*

*Proof.* We proceed by induction on  $|Q_{p,\infty}|$ . The case that  $|Q_{p,\infty}| = 1$  is clear. So, for some  $n \in \mathbb{N}$ , assume that if  $Q$  is a refinement of  $P$  such that  $Q$  supports  $x$  and  $|Q_{p,\infty}| = n$ , then  $(Q \setminus Q_{p,\infty}) \cup \{\bigcup Q_{p,\infty}\}$  is a support of  $x$ . Suppose  $Q$  is such that  $|Q_{p,\infty}| = n+1$ , and let  $q_1, q_2 \in Q_{p,\infty}$ . Let  $Q' = (Q \setminus \{q_1, q_2\}) \cup \{q_1 \cup q_2\}$ . We want to show that  $Q'$  is a support of  $x$ : Once this is done, we observe that  $Q'_{p,\infty} = (Q_{p,\infty} \setminus \{q_1, q_2\}) \cup \{q_1 \cup q_2\}$  is of size  $n$ . The induction hypothesis will then allow us to conclude that  $(Q' \setminus Q'_{p,\infty}) \cup \{\bigcup Q'_{p,\infty}\} = (Q \setminus Q_{p,\infty}) \cup \{\bigcup Q_{p,\infty}\}$

is a support of  $x$ . So, to show that  $Q'$  is a support of  $x$ , let  $\pi \in \text{fix}(Q')$ . First, write  $\pi = \pi_2\pi_1$ , where

$$\begin{cases} \pi_1 \upharpoonright_{q_1 \cup q_2} = \pi \upharpoonright_{q_1 \cup q_2} \\ \pi_1 \upharpoonright_{A \setminus (q_1 \cup q_2)} = \text{Id} \upharpoonright_{A \setminus (q_1 \cup q_2)} \end{cases} \quad \text{and} \quad \begin{cases} \pi_2 \upharpoonright_{q_1 \cup q_2} = \text{Id} \upharpoonright_{q_1 \cup q_2} \\ \pi_2 \upharpoonright_{A \setminus (q_1 \cup q_2)} = \pi \upharpoonright_{A \setminus (q_1 \cup q_2)} \end{cases}.$$

We know that  $\pi_2 x = x$ , since  $\pi_2 \in \text{fix}(Q)$ . Therefore, if we can show that  $\pi_1 x = x$ , then we will have  $\pi x = x$ , as desired.

To show that  $\pi_1 x = x$ , choose a permutation  $\sigma \in \text{fix}(P)$  such that  $\sigma \upharpoonright_{A \setminus (q_1 \cup q_2)} = \text{Id} \upharpoonright_{A \setminus (q_1 \cup q_2)}$ , and such that  $\{q_1, q_2\}$  and  $\sigma\{q_1, q_2\}$  are independent partitions of  $q_1 \cup q_2$ . For example, let  $B_1, B_2, B_3$  and  $B_4$  be infinite, pairwise disjoint sets such that  $q_1 = B_1 \cup B_2$  and  $q_2 = B_3 \cup B_4$ , and let  $\sigma$  be such that  $\sigma B_1 = B_1$ ,  $\sigma B_2 = B_4$ ,  $\sigma B_3 = B_3$  and  $\sigma B_4 = B_2$ . Let  $H$  be the group of all permutations of  $q_1 \cup q_2$ . Then  $\pi_1 \upharpoonright_{q_1 \cup q_2} \in H$ . So, by Lemma 3.13, there exists a finite sequence of permutations  $\tau_1, \dots, \tau_k \in \text{fix}_H(\{q_1, q_2\}) \cup \text{fix}_H(\sigma\{q_1, q_2\})$  such that  $\pi_1 \upharpoonright_{q_1 \cup q_2} = \tau_k \cdots \tau_1$ . For each  $\tau_i$ , define  $\gamma_i$  by  $\gamma_i \upharpoonright_{q_1 \cup q_2} = \tau_i$  and  $\gamma_i \upharpoonright_{A \setminus (q_1 \cup q_2)} = \text{Id} \upharpoonright_{A \setminus (q_1 \cup q_2)}$ . Then we have  $\pi_1 = \gamma_k \cdots \gamma_1$ . Now, since  $Q$  is a support of  $x$ , we know that  $\sigma Q$  is a support of  $\sigma x$ . Further,  $\sigma \in \text{fix}(P)$ , which implies that  $\sigma x = x$  or  $\sigma x = y$ . It follows that  $\sigma Q$  is a support of  $x$ : If  $\sigma' \in \text{fix}(\sigma Q)$  and  $\sigma' y = y$ , then  $\sigma' x = x$ , since  $\text{fix}(\sigma Q) \subseteq \text{fix}(P)$ . Thus,  $\gamma_i x = x$  for all  $i$ . It follows that  $\pi_1 x = \gamma_k \cdots \gamma_1 x = x$ , which completes the proof.  $\square$

**Lemma 3.16.** *Let  $P$  be a support of a set  $\{x, y\}$ , and let  $p$  be an infinite block of  $P$ . Let  $Q$  be a refinement of  $P$  such that  $Q$  supports  $x$ , and such that the set  $Q_p = \{q \in Q : q \subseteq p\}$  consists of singletons and one infinite block. Then  $(Q \setminus Q_p) \cup \{p\}$  is a support of  $x$ .*

*Proof.* Let  $Q' = (Q \setminus Q_p) \cup \{p\}$ , and let  $\pi \in \text{fix}(Q')$ . First, write  $\pi = \pi_2\pi_1$ , where

$$\begin{cases} \pi_1 \upharpoonright_p = \pi \upharpoonright_p \\ \pi_1 \upharpoonright_{A \setminus p} = \text{Id} \upharpoonright_{A \setminus p} \end{cases} \quad \text{and} \quad \begin{cases} \pi_2 \upharpoonright_p = \text{Id} \upharpoonright_p \\ \pi_2 \upharpoonright_{A \setminus p} = \pi \upharpoonright_{A \setminus p} \end{cases}.$$

Now, let  $Q_{p,1} = \{q \in Q_p : |q| = 1\}$ , and choose a permutation  $\sigma \in \text{fix}(P)$  such that  $\sigma \upharpoonright_{A \setminus p} = \text{Id} \upharpoonright_{A \setminus p}$  and  $Q_{p,1} \cap \sigma Q_{p,1} = \emptyset$ . Let  $H$  be the group of all permutations of  $p$ . Then  $\pi_1 \upharpoonright_p \in H = \text{fix}_H(0) = \text{fix}_H(Q_{p,1} \cap \sigma Q_{p,1})$ . So, by Lemma 3.14(b), there exists a finite sequence of permutations  $\tau_1, \dots, \tau_n \in \text{fix}_H(Q_{p,1}) \cup \text{fix}_H(\sigma Q_{p,1})$  such that  $\pi_1 \upharpoonright_p = \tau_n \cdots \tau_1$ . For each  $\tau_i$ , define  $\gamma_i$  by  $\gamma_i \upharpoonright_p = \tau_i$  and  $\gamma_i \upharpoonright_{A \setminus p} = \text{Id} \upharpoonright_{A \setminus p}$ . Then we have  $\pi = \pi_2 \gamma_n \cdots \gamma_1$  and  $\pi_2, \gamma_n, \dots, \gamma_1 \in \text{fix}(Q) \cup \text{fix}(\sigma Q)$ . As in the proof of Lemma 3.15, that  $Q$  is a support of  $x$  and  $\sigma \in \text{fix}(P)$  implies that  $\sigma Q$  also supports  $x$ . Therefore,  $\pi x = \pi_2 \gamma_n \cdots \gamma_1 x = x$ , which shows that  $Q'$  supports  $x$ .  $\square$

*Proof of Proposition 3.11.* Let  $W$  be a well-ordered family of pairs. Then  $\text{fix}(W) \in \mathcal{F}$ , so that  $W$  has a support  $P$  such that  $\text{fix}(P) \subseteq \text{fix}(W)$ . Suppose  $\{x, y\} \in W$ , and let  $Q$  be a support of  $x$ . Assume, without loss of generality, that all finite blocks of  $P$  and  $Q$  are singletons and that  $Q$  is a refinement of  $P$ . We will show that  $P$  supports  $x$ . Let  $p_1, \dots, p_n$  be the infinite blocks of  $P$ . First, let  $Q_{p_i} = \{q \in Q : q \subseteq p_i\}$ , and let  $Q_{p_i, \infty} = \{q \in Q_{p_i} : q \text{ is infinite}\}$ . By Lemma 3.15,  $Q' = (Q \setminus Q_{p_1, \infty}) \cup \{\bigcup Q_{p_1, \infty}\}$  is a support of  $x$ . Observe that the set  $Q'_{p_1} = \{q \in Q' : q \subseteq p_1\}$  is equal to  $(Q_{p_1} \setminus Q_{p_1, \infty}) \cup \{\bigcup Q_{p_1, \infty}\}$ ; that is,  $Q'_{p_1}$  consists of the set  $Q_{p_1} \setminus Q_{p_1, \infty}$  of singletons and one infinite block  $\bigcup Q_{p_1, \infty}$ . Therefore, Lemma 3.16 applies, and we get that  $Q_1 = (Q' \setminus Q'_{p_1}) \cup \{p_1\} = (Q \setminus Q_{p_1}) \cup \{p_1\}$  is a support of  $x$ . Applying this argument to  $Q_1$  and  $p_2$ , we obtain a new support  $Q_2 = (Q \setminus (Q_{p_1} \cup Q_{p_2})) \cup \{p_1, p_2\}$  of  $x$ . Continuing this process through  $p_n$ , we get that  $Q_n = (Q \setminus \bigcup_{i \leq n} Q_{p_i}) \cup \{p_i\}_{i \leq n}$  is a support of  $x$ . But  $Q_n = P$ : The infinite blocks of  $Q_n$  are precisely the infinite blocks of  $P$ . Each singleton in  $Q_n$  is not in  $\bigcup_{i \leq n} Q_{p_i}$ , and is thus a subset of a singleton in  $P$ . Therefore,  $P$  is a support of  $x$ .

Now, let  $c$  be a choice function on  $W$  in the full universe, and let  $\pi \in \text{fix}(P)$ . Then for each  $\langle X, x \rangle \in c$  and  $\pi \in \text{fix}(P)$ , we have  $\pi \langle X, x \rangle = \langle X, x \rangle \in c$ . Thus,  $c$  is symmetric. It follows that  $c$  is hereditarily symmetric since  $X$  and  $x$  are in  $\mathcal{V}_p$ . Therefore,  $c \in \mathcal{V}_p$ .  $\square$

Without much trouble, we can extend Proposition 3.11 to the following:

**Proposition 3.17.** *The statement “Every well-ordered family of well-orderable sets has a choice function” is true in  $\mathcal{V}_p$ .*

To prove Proposition 3.17, we only need to make small extensions of Lemmas 3.15 and 3.16:

- Lemma 3.15 (extended): *Let  $P$  be a support of a well-orderable set  $X$ , and let  $Q$  be a refinement of  $P$  such that  $Q$  supports each element of  $X$ . Let  $p$  be an infinite block of  $P$ , and let  $Q_{p, \infty} = \{q \in Q : q \subseteq p, q \text{ is infinite}\}$ . Then  $(Q \setminus Q_{p, \infty}) \cup \{\bigcup Q_{p, \infty}\}$  is a support of every element of  $X$ .*

*Proof.* The proof is similar to that of Lemma 3.15: The induction hypothesis is “For some  $n \in \mathbb{N}$ , assume that if  $Q$  is a refinement of  $P$  such that  $Q$  supports each element of  $X$  and  $|Q_{p, \infty}| = n$ , then  $(Q \setminus Q_{p, \infty}) \cup \{\bigcup Q_{p, \infty}\}$  is a support of every element of  $X$ .” For the case that  $|Q_{p, \infty}| = n + 1$ , let  $q_1, q_2 \in Q_{p, \infty}$ . Let  $Q' = (Q \setminus \{q_1, q_2\}) \cup \{q_1 \cup q_2\}$ , and let  $\pi \in \text{fix}(Q')$ . Choose  $\sigma \in \text{fix}(P)$  such that  $\{q_1, q_2\}$  and  $\sigma\{q_1, q_2\}$  are independent partitions of  $q_1 \cup q_2$ , and use Lemma 3.13 to write  $\pi$  as

$\pi = \pi_2 \gamma_k \cdots \gamma_1$ , where  $\pi_2, \gamma_k, \dots, \gamma_1 \in \text{fix}(Q) \cup \text{fix}(\sigma Q)$ . Now, we want to show that  $\sigma Q$  supports each element of  $X$ . This is easy: Since  $Q$  supports each element of  $X$ , we know by Lemma 1.6(b) that  $\sigma Q$  supports each element of  $\sigma X = X$ . Consequently,  $\pi x = \pi_2 \gamma_k \cdots \gamma_1 x = x$  for all  $x \in X$ , which shows that  $Q'$  is also a support of every element of  $X$ . As in the proof of Lemma 3.15, the rest follows by induction hypothesis.  $\square$

- Lemma 3.16 (extended): *Let  $P$  be a support of a well-orderable set  $X$ , and let  $p$  be an infinite block of  $P$ . Let  $Q$  be a refinement of  $P$  such that  $Q$  supports each element of  $X$ , and such that  $Q_p = \{q \in Q : q \subseteq p\}$  consists of singletons and one infinite block. Then  $(Q \setminus Q_p) \cup \{p\}$  is a support of every element of  $X$ .*

*Proof.* Make similar adjustments to the proof of Lemma 3.16.  $\square$

*Proof of Proposition 3.17.* Let  $W$  be a well-ordered family of well-orderable sets, and let  $X \in W$ . Since  $W$  is well-ordered,  $W$  has a support  $P$  such that  $\text{fix}(P) \subseteq \text{fix}(W)$ . Similarly,  $X$  has a support  $Q$  such that  $\text{fix}(Q) \subseteq \text{fix}(X)$ . Assume, without loss of generality, that all finite blocks of  $P$  and  $Q$  are singletons and that  $Q$  is a refinement of  $P$ . Using the extensions of Lemmas 3.15 and 3.16, apply the argument in the proof of Proposition 3.11 to show that  $P$  supports each element of  $X$ . Any choice function on  $W$  will be hereditarily symmetric, supported by  $P$ .  $\square$

**Proposition 3.18.** *The statement “The union of any well-ordered family of well-orderable sets is well-orderable” is true in  $\mathcal{V}_p$ .*

*Proof.* Let  $W$  be a well-ordered family of well-orderable sets. Since  $W$  is well-ordered,  $W$  has a support  $P$  such that  $\text{fix}(P) \subseteq \text{fix}(W)$ . By the proof of Proposition 3.17, we know further that  $\text{fix}(P) \subseteq \text{fix}(X)$  for all  $X \in W$ . That is,  $\text{fix}(P) \subseteq \text{fix}(\cup W)$ . Hence,  $\cup W$  is well-orderable.  $\square$

This completes the presentation of our main results.

## Future research

There are at least two natural, related directions for future research involving the finite partition model.

The first is to continue the project of this paper; that is, classifying  $\mathcal{V}_p$ . A standard reference for that objective is Howard and Rubin [3], a comprehensive list of statements that have been proved using the axiom of choice. One might



continue the aim of this paper by determining which of those statements are true, and which ones are false, in the finite partition model.

A second direction for future research could be to explore variants of the finite partition model. For example, assume  $A$  is uncountable, let  $\mathcal{G}$  be the group of all permutations of  $A$ , and let the supports be countable partitions of  $A$ . The corresponding permutation model would be to  $\mathcal{V}_p$  as  $\mathcal{V}_{F^+}$  is to  $\mathcal{V}_F$ .

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