INVARIANT DISTRIBUTIONS ON l-SPACES

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Abstract. In a paper published by Ranga Rao [4] in the Annals of Mathematics, it is shown that the invariant measure carried by a nilpotent orbital can be extended to an invariant distribution on the ambient Lie algebra. This paper provides an alternative proof by a method proposed by Gordon Savin.

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1. Preliminary Definitions and Motivation

Definition 1.1. An l-group is a topological group with a neighborhood basis of the identity consisting of compact open subgroups.

Definition 1.2. An l-space is a topological space that is Hausdorff, locally compact, and zero dimensional; that is, each point has a neighborhood basis consisting of compact open neighborhoods.

Definition 1.3. For an l-space X, the support of $f : X \to \mathbb{C}$ is defined as the closure of the set $\{x \in X : f(x) \neq 0\}$.

Definition 1.4. For an l-space X,

$$C_c^\infty(X) = \{f : X \to \mathbb{C} : f \text{ is locally constant and has compact support}\}$$

Recall, f is locally constant if $\forall x \in X \exists U$, a neighborhood of x, such that f is constant on U.

Definition 1.5. Let $\mathcal{D}(X)$ denote the space of invariant measures on X.

Let X be an l-space and let G be an l-group that acts continuously on X. Also assume that $k^\times = GL_1(k)$ where k is a p-adic field acts on X. Suppose that there are 2 orbits in X, $\{0\}$ and $\mathcal{O}$, such that $\bar{\mathcal{O}} = \mathcal{O} \cup \{0\}$. We know from work in the Appendix that 0 and $\mathcal{O}$ carry invariant measures, call them $\mu_{\{0\}}$ and $\mu_{\mathcal{O}}$ respectively. The problem with extending $\mu_{\mathcal{O}}$ to an invariant distribution on all of X is that for $f \in C_c^\infty(X)$, it is not necessarily true that $\text{res}_{\mathcal{O}} f \in C_c^\infty(\mathcal{O})$. Therefore $\mu_{\mathcal{O}}(\text{res}_{\mathcal{O}} f)$ is not well defined.

The idea, as proposed by Gordon Savin, is as follows. Consider $f \in C_c^\infty(X)$. Then $\text{res}_{\bar{\mathcal{O}}} f \in C_c^\infty(\bar{\mathcal{O}})$. Now define $f_{x^2}(y) = f(x^{-2}y)$ for $x \in k^\times$ and $y \in X$. Throughout this paper, assume that for an invariant measure, μ, on a G-orbit in X, $\mathcal{O}'$, and $h \in C_c^\infty(\mathcal{O}')$ then $\mu(h_{\varpi^2}) = q^{-\dim \mathcal{O}'}\mu(h)$ where $\varpi \in k^\times$ is a uniformizer. Thus

$$\text{res}_{\bar{\mathcal{O}}}(f - f_{\varpi^2}) \in C_c^\infty(\bar{\mathcal{O}})$$

and vanishes at 0. This implies that $\text{res}_{\mathcal{O}}(f - f_{\varpi^2}) \in C_c^\infty(\mathcal{O})$, so $\mu_{\mathcal{O}}(f - f_{\varpi^2})$ is well defined.

Define

$$T_{\mathcal{O}}(f) = \frac{\mu_{\mathcal{O}}(f - f_{\varpi^2})}{1 - q^{-\dim(\mathcal{O})}} \tag{1.6}$$

Lemma 1.7. *(a) $T_{\mathcal{O}}(f_{\varpi^2}) = q^{-\dim(\mathcal{O})}T_{\mathcal{O}}(f)$ and (b) $T_{\mathcal{O}}$ agrees with Ranga Rao*

Proof. (a)

$$\begin{aligned} T_{\mathcal{O}}(f_{\varpi^2}) &= \frac{\mu_{\mathcal{O}}(f_{\varpi^2} - f_{\varpi^4})}{1 - q^{-\dim(\mathcal{O})}} \\ &= \frac{q^{-\dim(\mathcal{O})}\mu_{\mathcal{O}}(f - f_{\varpi^2})}{1 - q^{-\dim(\mathcal{O})}} \\ &= q^{-\dim(\mathcal{O})}T_{\mathcal{O}}(f) \end{aligned}$$

(b) By [4], there exists an invariant distribution on X, $\mu_{\mathcal{O}}^{RR}$, such that $\mu_{\mathcal{O}}^{RR}(f) = \mu_{\mathcal{O}}(f)$ for $f \in C_c^\infty(\mathcal{O})$. Then,

$$\begin{aligned} T_{\mathcal{O}}(f) &= \frac{\mu_{\mathcal{O}}(f - f_{\varpi^2})}{1 - q^{-\dim(\mathcal{O})}} \\ &= \frac{\mu_{\mathcal{O}}^{RR}(f - f_{\varpi^2})}{1 - q^{-\dim(\mathcal{O})}} \\ &= \frac{\mu_{\mathcal{O}}^{RR}(f) - \mu_{\mathcal{O}}^{RR}(f_{\varpi^2})}{1 - q^{-\dim(\mathcal{O})}} \\ &= \frac{\mu_{\mathcal{O}}^{RR}(f) - q^{-\dim(\mathcal{O})}\mu_{\mathcal{O}}^{RR}(f)}{1 - q^{-\dim(\mathcal{O})}} \\ &= \mu_{\mathcal{O}}^{RR}(f) \end{aligned}$$

□

Now consider the sequence

$$0 \to \mathcal{D}(\{0\}) \xrightarrow{\varphi} \mathcal{D}(\bar{\mathcal{O}}) \xrightarrow{\gamma} \mathcal{D}(\mathcal{O}) \to 0 \tag{1.8}$$

Where φ is the continuation by zero and γ is the restriction mapping. Clearly φ is injective, and by (1.6) and Lemma 1.7, each element of $\mathcal{D}(\mathcal{O})$ can be extended to an element of $\mathcal{D}(\bar{\mathcal{O}})$. Thus γ is surjective, so the above sequence is exact. Therefore $\mathcal{D}(\bar{\mathcal{O}})$ is 2 dimensional with basis $\mu_{\{0\}}, \mu_{\mathcal{O}}$.

2. The next step

This next lemma will be useful in the following sections.

Lemma 2.1. *Let X be an l-space and Y a closed subset of X. Suppose $f \in C_c^\infty(X)$ such that for each $T \in \mathcal{D}(Y)$, $T(f) = 0$. Then there exists $H_j \in C_c^\infty(X)$ and $g_j \in G$ such that if $H := \sum {}^{g_j}H_j - H_j$,*

$$res_Y H = res_Y f$$

Proof. Consider $C_c^\infty(Y)_G$ and it's dual space $\mathcal{D}(Y)$ where $C_c^\infty(Y)_G$ is the quotient of $C_c^\infty(Y)$ by the span of all functions of the form ${}^gh - h$ with $g \in G$ and $h \in C_c^\infty(Y)$. Let $\{h_1, ..., h_n\}$ be a basis for $C_c^\infty(Y)$ and $\{h_1^*, ..., h_n^*\}$ the dual basis. Then we can write $f = a_1h_1 + \cdots + a_nh_n$. Since $\forall T \in \mathcal{D}(Y), T(f) = 0$ implies that for each i, $h_i^*(f) = a_i = 0$. Thus the image of f vanishes in $C_c^\infty(Y)_G$. Therefore, there exists $g_j \in G$ and $h_j \in C_c^\infty(Y)$ such that

$$\text{res}_Y f = \sum {}^{g_j}h_j - h_j$$

By [1, p19], for each $j, \exists H_j \in C_c^\infty(X)$ such that $\text{res}_Y H_j = h_j$. Let $H = \sum {}^{g_j}H_j - H_j$. Clearly $\text{res}_Y H = \text{res}_Y f$. □

Now suppose that there are three G-orbits in X such that

$$0 \subset \bar{\mathcal{O}}_1 = \{0\} \cup \mathcal{O}_1 \subset \bar{\mathcal{O}}_2 = \{0\} \cup \mathcal{O}_1 \cup \mathcal{O}_2$$

From the appendix, each orbit carries an invariant measure, call them $\mu_0, \mu_{\mathcal{O}_1}, \mu_{\mathcal{O}_2}$.

Let $f_1 = f - q^{\dim(\mathcal{O}_1)}f_{\varpi^2}$ for $f \in C_c^\infty(X)$. Then

$$\begin{aligned}\mu_{\mathcal{O}_1}(f_1) &= \mu_{\mathcal{O}_1}(f) - q^{\dim(\mathcal{O}_1)}\mu_{\mathcal{O}_1}(f_{\varpi^2}) \\ &= \mu_{\mathcal{O}_1}(f) - \mu_{\mathcal{O}_1}(f) = 0\end{aligned} \tag{2.2}$$

Note that $\mu_{\mathcal{O}_1}(f)$ is well defined because it was shown in the previous section that $\mu_{\mathcal{O}_1}$ can be extended to a distribution on X whose restriction to Y agrees with $\mu_{\mathcal{O}_1}$.

If $f_2 = f_1 - (f_1)_{\varpi^2}$ then also note that $\mu_0(f_2) = 0$. Thus, by (1.8) $\forall T \in \mathcal{D}(\bar{\mathcal{O}}_1), T(f_2) = 0$.

By Lemma 2.1 $\exists H_j \in C_c^\infty(X)$ and $g_j \in G$ such that $\mathrm{res}_{\bar{\mathcal{O}}_1}(f_2 - H) = 0$ where $H = \sum {}^{g_j}H_j - H_j$, so $\mu_{\mathcal{O}_2}(f_2 - H)$ makes sense. Thus we set

$$T_{\mathcal{O}_2}(f) = \frac{\mu_{\mathcal{O}_2}(f_2 - H)}{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(1 - q^{-\dim(\mathcal{O}_2)})}$$

We now reprove Lemma 1.4 but substituting $\mathcal{O}_2$ for $\mathcal{O}$.

Proof. (a)

$$\begin{aligned}T_{\mathcal{O}_2}(f_{\varpi^2}) &= \frac{\mu_{\mathcal{O}_2}((f_{\varpi^2})_1 - ((f_{\varpi^2})_1)_{\varpi^2} - H_{\varpi^2})}{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(1 - q^{-\dim(\mathcal{O}_2)})} \\ &= \frac{\mu_{\mathcal{O}_2}(f_{\varpi^2} - q^{\dim(\mathcal{O}_1)}f_{\varpi^4} - (f_{\varpi^4} - q^{\dim(\mathcal{O}_1)}f_{\varpi^6}) - H_{\varpi^2})}{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(1 - q^{-\dim(\mathcal{O}_2)})} \\ &= \frac{q^{-\dim(\mathcal{O}_2)}\mu_{\mathcal{O}_2}(f - q^{\dim(\mathcal{O}_1)}f_{\varpi^2} - (f_{\varpi^2} - q^{\dim(\mathcal{O}_1)}f_{\varpi^4}) - H)}{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(1 - q^{-\dim(\mathcal{O}_2)})} \\ &= q^{-\dim(\mathcal{O}_2)}T_{\mathcal{O}_2}(f)\end{aligned}$$

(b)

$$\begin{aligned}T_{\mathcal{O}_2}(f) &= \frac{\mu_{\mathcal{O}_2}(f_2 - H)}{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(1 - q^{-\dim(\mathcal{O}_2)})} \\ &= \frac{\mu^{RR}_{\mathcal{O}_2}(f_2 - H)}{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(1 - q^{-\dim(\mathcal{O}_2)})} \\ &= \frac{\mu^{RR}_{\mathcal{O}_2}(f_2)}{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(1 - q^{-\dim(\mathcal{O}_2)})} \\ &= \frac{\mu^{RR}_{\mathcal{O}_2}(f_1 - (f_1)_{\varpi^2})}{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(1 - q^{-\dim(\mathcal{O}_2)})} \\ &= \frac{\mu^{RR}_{\mathcal{O}_2}(f_1) - q^{-\dim(\mathcal{O}_2)}\mu^{RR}_{\mathcal{O}_2}(f_1)}{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(1 - q^{-\dim(\mathcal{O}_2)})} \\ &= \frac{\mu^{RR}_{\mathcal{O}_2}(f - q^{\dim(\mathcal{O}_1)}f_{\varpi^2}) - q^{-\dim(\mathcal{O}_2)}\mu^{RR}_{\mathcal{O}_2}(f - q^{\dim(\mathcal{O}_1)}f_{\varpi^2})}{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(1 - q^{-\dim(\mathcal{O}_2)})} \\ &= \frac{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(\mu^{RR}_{\mathcal{O}_2}(f) - q^{-\dim(\mathcal{O}_2)}\mu^{RR}_{\mathcal{O}_2}(f))}{(1 - q^{\dim(\mathcal{O}_1) - \dim(\mathcal{O}_2)})(1 - q^{-\dim(\mathcal{O}_2)})} \\ &= \mu^{RR}_{\mathcal{O}_2}(f)\end{aligned}$$

□

3. General Case

Now suppose that there is a G-orbit in X, $\mathcal{O}_k$ such that $\bar{\mathcal{O}}_k = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_k$ and for each $i \in \{0, ..., k\}$,

$$\overline{\mathcal{O}}_i = \mathcal{O}_i \cup (\bigcup_{j \in J_i} \mathcal{O}_j)$$

where $J_i \subset \{0, ..., k\}$. The orbits in this closure have a partial ordering defined by $\mathcal{O}_m \leq \mathcal{O}_n \iff \mathcal{O}_m \subset \overline{\mathcal{O}}_n$. Let $d_j = \dim(\mathcal{O}_j)$ and for $f \in C_c^\infty(X)$ define $f_1 = f - q^{d_k}f$. For $2 \leq j \leq k$ define

$$f_j = f_{j-1} - q^{d_{k-j}}(f_{j-1})_{\varpi^2}$$

We proceed by induction. The base case was considered in the previous two sections and the inductive hypothesis is as follows: suppose that $\mathcal{O}_{k_1}, \cdots, \mathcal{O}_{k_n}$ are the orbits directly under $\mathcal{O}_k$. Let $\mathcal{O} = \overline{\mathcal{O}}_{k_1} \cup \cdots \cup \overline{\mathcal{O}}_{k_n}$ and further suppose that $\mathcal{D}(\mathcal{O})$ is k dimensional with basis $\{\mu_{\mathcal{O}_0}, ..., \mu_{\mathcal{O}_{k-1}}\}$. Note that $\mu_{\mathcal{O}_0}(f_k) = f_k(0) = 0$ and for $1 \le i \le k-1$,

$$\begin{aligned}\mu_{\mathcal{O}_i}(f_{k-i}) &= \mu_{\mathcal{O}_i}(f_{k-i-1}) - q^{d_i}\mu_{\mathcal{O}_i}((f_{k-i-1})_{\varpi^2}) \\ &= \mu_{\mathcal{O}_i}(f_{k-i-1}) - \mu_{\mathcal{O}_i}(f_{k-i-1}) \\ &= 0\end{aligned}$$

This implies that for $0 \le i \le k-1$ $\mu_{\mathcal{O}_i}(f_k) = 0$ because $\mu_{\mathcal{O}_i}(f_k)$ can be written as a linear combination of $\mu_{\mathcal{O}_i}(f_{k-i})$. Therefore, by Lemma 2.1, choose $H = \sum {}^{g_j}H_j - H_j \in C_c^\infty(X), g_j \in G$ such that $\text{res}_{\mathcal{O}} H = \text{res}_{\mathcal{O}} f_k$. Define

$$T_{\mathcal{O}_k}(f) = \frac{\mu_{\mathcal{O}_k}(f_k - H)}{\prod_{j=0}^{k-1}(1 - q^{d_j - d_k})}$$

Again we reprove Lemma 1.4

Proof.

$$\begin{aligned}T_{\mathcal{O}_k}(f_{\varpi^2}) &= \frac{\mu_{\mathcal{O}_k}((f_k)_{\varpi^2} - H_{\varpi^2})}{\prod_{j=0}^{k-1}(1 - q^{d_j - d_k})} \\ &= \frac{\mu_{\mathcal{O}_k}((f_k - H)_{\varpi^2})}{\prod_{j=0}^{k-1}(1 - q^{d_j - d_k})} \\ &= q^{-d_k}\frac{\mu_{\mathcal{O}_k}(f_k - H)}{\prod_{j=0}^{k-1}(1 - q^{d_j - d_k})} \\ &= q^{-d_k} T_{\mathcal{O}_k}(f)\end{aligned}$$

$$\begin{aligned}
T_{\mathcal{O}_k}(f) &= \frac{\mu_{\mathcal{O}_k}(f_k - H)}{\prod_{j=0}^{k-1}(1-q^{d_j-d_k})} \\
&= \frac{\mu_{\mathcal{O}_k}^{RR}(f_k - H)}{\prod_{j=0}^{k-1}(1-q^{d_j-d_k})} \\
&= \frac{\mu_{\mathcal{O}_k}^{RR}(f_k)}{\prod_{j=0}^{k-1}(1-q^{d_j-d_k})} \\
&= \frac{\mu_{\mathcal{O}_k}^{RR}(f_{k-1} - (f_{k-1})_{\varpi^2})}{\prod_{j=0}^{k-1}(1-q^{d_j-d_k})} \\
&= \frac{\mu_{\mathcal{O}_k}^{RR}(f_{k-1}) - \mu_{\mathcal{O}_k}^{RR}((f_{k-1})_{\varpi^2})}{\prod_{j=0}^{k-1}(1-q^{d_j-d_k})} \\
&= \frac{\mu_{\mathcal{O}_k}^{RR}(f_{k-1})(1-q^{-d_k})}{\prod_{j=0}^{k-1}(1-q^{d_j-d_k})} \\
&= \frac{\mu_{\mathcal{O}_k}^{RR}(f_{k-2} - q^{d_1}(f_{k-2})_{\varpi^2})(1-q^{-d_k})}{\prod_{j=0}^{k-1}(1-q^{d_j-d_k})} \\
&= \frac{\mu_{\mathcal{O}_k}^{RR}(f_{k-3})(1-q^{d_1-d_k})(1-q^{-d_k})}{\prod_{j=0}^{k-1}(1-q^{d_j-d_k})} \\
&\vdots \\
&= \frac{\mu_{\mathcal{O}_k}^{RR}(f_1)\prod_{j=0}^{k-2}(1-q^{d_j-d_k})}{\prod_{j=0}^{k-1}(1-q^{d_j-d_k})} \\
&= \frac{\mu_{\mathcal{O}_k}^{RR}(f - q^{d_{k-1}}f_{\varpi^2})\prod_{j=0}^{k-2}(1-q^{d_j-d_k})}{\prod_{j=0}^{k-1}(1-q^{d_j-d_k})} \\
&= \mu_{\mathcal{O}_k}^{RR}(f)
\end{aligned}$$

□

4. Appendix: Invariant measures on orbits in X

Let G and X be as above. A G-orbit, $\mathcal{O}$, is homeomorphic to the quotient space G/G_x where G_x is the centralizer of $x \in \mathcal{O}$ by the map $gG_x \mapsto g.x$. The goal of this section is to show that G/G_x has an invariant measure.

Lemma. *For $A \subset G/G_x$, A is compact if and only if there exists $B \subset G$ compact such that $A = BG_x$.*

Proof. Let $q : G \to G/G_x$ be the quotient map where $q(g) = gG_x$ for $g \in G$. Suppose that $\exists B \subset G$ compact such that $A = BG_x$. Then, let $\mathcal{C}$ be an open cover for A. Note that if $U \subset G/G_x$ is open then, by the definition of the quotient topology, $q^{-1}(U)$ is open, so q is continuous. Thus $q^{-1}(\mathcal{C})$ is an open cover for B, so $\exists$ finite $\{C_i\}_{i\in I}$ such that $\{q^{-1}(C_i)\}_{i\in I}$ cover B which implies that $\{C_i\}_{i\in I}$ is a finite subcover of $A \implies A$ is compact. Suppose that A is compact. Since X is Hausdorff, $\{x\}$ is closed. Consider the continuous function $\varphi : G \to X$ with $\varphi(g) = g.x$. By the continuity of φ, $\varphi^{-1}(\{x\}) = G_x$ is closed. This implies that G/G_x is Hausdorff, so A being compact implies that A is closed. Therefore it is enough to show that $\exists \bar{D} \subset G/G_x$ compact such that $A \subset \bar{D}$. Let $\mathcal{V} = \{V_i\}$ be an open cover of $q^{-1}(A)$ consisting of compact open sets. Then $\bar{\mathcal{V}} = \{\bar{V}_i\}$ is an open cover of A. Since A is compact there exists a finite subcover $\{V_{n_1}, \cdots, V_{n_m}\}$. Let $D = \bigcup_{1\le i\le m} V_{n_i}$. Then $\bar{D}$ is compact and $A \subset \bar{D}$. □

Lemma 4.1. *For $f \in C_c^\infty(G/G_x)$ $\exists n \in \mathbb{N}$ and a compact open subgroup (c.o.s.) $K_f \le G$ such that*

$$f = \sum_{i=1}^{n} c_i [K_f g_i G_x]$$

with $c_i \in \mathbb{C}$ *and* $g_i \in G$.

Proof. Since f is locally constant, for each $g \in G$ $\exists$ c.o.s. $K_g \leq G$ such that

$$f(kgG_x) = f(gG_x)\, \forall k \in K_g$$

Note that $\{K_g gG_x : g \in G\}$ is an open cover for $supp(f)$, the support of f. Since $supp(f)$ is compact, $\exists \{g_1, ..., g_m\} \subset G$ such that

$$supp(f) \subset \bigcup_{i=1}^{m} K_{g_i} g_i G_x$$

Let $K_f = \cap K_{g_i}$. Then $\{K_f hG_x : h \in G\}$ is an open cover for $supp(f)$ so by compactness there exists $\{h_1, ..., h_n\} \subset G$ such that

$$supp(f) \subset \bigcup_{i=1}^{n} K_f h_i G_x \tag{4.2}$$

and

$$K_f h_i G_x \cap K_f h_j G_x \neq \emptyset \implies i = j \tag{4.3}$$

Let

$$f' = \sum_{i=1}^{n} f(h_i G_x) \cdot [K_f h_i G_x]$$

Clearly if $gG_x \in supp(f)$, then $f'(gG_x) = f(gG_x)$. If $gG_x \notin supp(f)$ then $\forall i$ either $[K_f h_i G_x](gG_x) = 0$ or $f(h_i G_x) = 0$. Thus, $f = f'$. $\square$

Let $L(G/G_x) = \{T \in C_c^\infty(G/G_x)^* : T(L_g f) = T(f)\, \forall g \in G\, \forall f \in C_c^\infty(G/G_x)\}$ where $L_g f(h) = f(g^{-1}h)$

Lemma 4.4. *For* $S \in L(G/G_x)$ *and c.o.s.* $K, K' \leq G$ *then*

$$S([KG_x]) = \frac{[KG_x : KG_x \cap K'G_x]}{[K'G_x : KG_x \cap K'G_x]} \cdot S([K'G_x])$$

Proof. Note that $KG_x \cap K'G_x$ is open, so $\{kKG_x \cap K'G_x : k \in K\}$ and $\{kKG_x \cap K'G_x : k \in K'\}$ are open covers of KG_x and $K'G_x$, respectively. By the compactness of KG_x and $K'G_x$, $\{kKG_x \cap K'G_x : k \in K\}$ and $\{kKG_x \cap K'G_x : k \in K'\}$ admit finite subcovers. So $[KG_x : KG_x \cap K'G_x]$ and $[K'G_x : KG_x \cap K'G_x]$ are finite.

$$\begin{aligned} S([KG_x]) &= \sum_{\overline{gG_x} \in KG_x/(KG_x \cap K'G_x)} S([g(KG_x \cap K'G_x)]) \\ &= [KG_x : KG_x \cap K'G_x] \cdot S([KG_x \cap K'G_x]) \end{aligned}$$

Similarly $S([K'G_x]) = [K'G_x : KG_x \cap K'G_x] \cdot S([KG_x \cap K'G_x])$. Together these give us the desired result. $\square$

Lemma 4.5. *If* $T, S \in L(G/G_x)$ *with* $T \neq 0$, *then* $\exists c \in \mathbb{C}$ *such that* $S = cT$

Proof. Fix a c.o.s. $K \leq G$ and let $c = S([KG_x])/T([KG_x])$. Suppose $f \in C_c^\infty(G/G_x)$, then by Lemma 4.1 there exists c.o.s. $K_f \leq G$ and finite $(c_i, g_i) \in \mathbb{C} \times G$ such that $K_f g_i G_x \cap K_f g_j G_x \neq \emptyset \implies i = j$ and $f = \sum c_i \cdot [K_f g_i G_x]$. Then, by Lemma 4.2,

$$\begin{aligned} S(f) &= S([K_f G_x]) \cdot \sum c_i \\ &= S([KG_x]) \cdot \frac{[K_f G_x : KG_x \cap K_f G_x]}{[KG_x : KG_x \cap K_f G_x]} \cdot \sum c_i \\ &= cT([KG_x]) \cdot \frac{[K_f G_x : KG_x \cap K_f G_x]}{[KG_x : KG_x \cap K_f G_x]} \cdot \sum c_i \\ &= cT([K_f G_x]) \cdot \sum c_i \\ &= cT(f) \end{aligned}$$

$\square$

Note that this implies that $\dim L(G/G_x) \leq 1$

4.1. **Construction of measure.** Fix c.o.s. $K \leq G$ and suppose $f \in C_c^\infty(G/G_x)$. From Lemma 4.1 there exists c.o.s. $K_f \leq G$ and finite $(c_i, g_i) \in \mathbb{C} \times G$ such that $K_f g_i G_x \cap K_f g_j G_x \neq \emptyset \implies i = j$ and $f = \sum c_i [K_f g_i G_x]$. Define

$$\mu_{K,K_f}(f) = \frac{[K_f G_x : K_f G_x \cap K G_x]}{[K G_x : K_f G_x \cap K G_x]} \cdot \sum c_i$$

Lemma 4.6. *$\mu_{K,K_f}(f)$ is independent of the choice of K_f.*

Proof. Suppose that there exists K_1, K_2 and $\{(a_1, g_1), ..., (a_n, g_n)\} \subset \mathbb{C}^n \times G^n, \{(b_1, h_1), ..., (b_m, h_m)\} \subset \mathbb{C}^m \times G^m$ such that

$$f = \sum_{i=1}^n a_i [K_1 g_i G_x] = \sum_{i=1}^m b_i [K_2 h_i G_x]$$

It will be enough to show that $\mu_{K,K_i}(f) = \mu_{K,K_1 \cap K_2}$ for $i \in \{1, 2\}$. Therefore, it is enough to assume that $K_2 \leq K_1$ and to show that $\mu_{K,K_1}(f) = \mu_{K,K_2}(f)$. Note that for $g_1, g_2 \in G$ such that $K_2 g_2 G_x \subset K_1 g_1 G_x$ then $f(g_1 G_x) = f(g_2 G_x)$ and therefore $\sum_{i=1}^m b_i = \sum_{i=1}^n a_i \cdot [K_1 G_x : K_2 G_x]$. So,

$$\begin{aligned}
\mu_{K,K_2}(f) &= \frac{[K_2 G_x : K_2 G_x \cap K G_x]}{[K G_x : K_2 G_x \cap K G_x]} \cdot \sum b_i \\
&= \frac{[K_2 G_x : K_2 G_x \cap K G_x]}{[K G_x : K_1 G_x \cap K G_x]} \cdot \frac{[K_1 G_x : K_2 G_x]}{[K_1 G_x \cap K G_x : K_2 G_x \cap K G_x]} \cdot \sum a_i \\
&= \frac{[K_2 G_x : K_2 G_x \cap K G_x]}{[K G_x : K_1 G_x \cap K G_x]} \cdot \frac{[K_1 G_x : K_2 G_x]}{[K_2 G_x : K_2 G_x \cap K G_x]/[K_1 G_x : K_1 G_x \cap K G_x] \cdot [K_1 G_x : K_2 G_x]} \cdot \sum a_i \\
&= \frac{[K_1 G_x : K_1 G_x \cap K G_x]}{[K G_x : K_1 G_x \cap K G_x]} \cdot \sum a_i \\
&= \mu_{K,K_1}(f)
\end{aligned}$$

□

Define $\mu_K(f) = \mu_{K,K_f}(f)$ where K_f is chosen by the procedure of Lemma 4.1. From Lemma 4.4, μ_K is independent of the choice of K_f. Note that $\mu_K([K G_x]) = 1$. All that is left to show is that μ_K is linear and left invariant.

Lemma 4.7. *μ_K is linear.*

Proof. Suppose that $f_1, f_2 \in C_c^\infty(G/G_x)$ and $c \in \mathbb{C}$. By Lemma 4.1, there exists a c.o.s. $K' \leq G$ and $\{g_1, ..., g_n\} \in G$ such that

$$f_1 + c f_2 = \sum_{i=1}^n (f_1(g_i) + c f_2(g_i)) \cdot [K' g_i G_x]$$

Thus,

$$\begin{aligned}
\mu_K(f_1 + c f_2) &= \frac{[K' G_x : K G_x \cap K' G_x]}{[K G_x : K G_x \cap K' G_x]} \cdot \sum_{i=1}^n (f_1(g_i) + c f_2(g_i)) \\
&= \frac{[K' G_x : K G_x \cap K' G_x]}{[K G_x : K G_x \cap K' G_x]} \cdot \sum_{i=1}^n (f_1(g_i)) + c \frac{[K' G_x : K G_x \cap K' G_x]}{[K G_x : K G_x \cap K' G_x]} \cdot \sum_{i=1}^n (f_2(g_i)) \\
&= \mu_K(f_1) + c \mu_K(f_2)
\end{aligned}$$

□

Lemma 4.8. *μ_K is left invariant.*

Proof. Suppose $f \in C_c^\infty(G/G_x)$. Then there exists a c.o.s. $K_f \leq G$ and finite $g_i \in G$ such that $f = \sum f(g_i G_x) \cdot [K_f g_i G_x]$. Note that for all $g \in G$, $L_g f = \sum f(g_i G_x) \cdot [g K_f g_i G_x]$. Thus it is clear that $\mu_K(f) = \frac{[K_f G_x : K_f G_x \cap K G_x]}{[K G_x : K_f G_x \cap K G_x]} \cdot \sum f(g_i) = \mu_K(L_g f)$. □

Thus, $\mu_K \in L(G/G_x)$ and is non-zero. Together with Lemma 4.3, this implies that $\dim L(G/G_x) = 1$

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