Tropical Constructions and Lifts

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1 The Algebraic Torus and M

Let K denote a field of characteristic zero and K^* denote the associated multiplicative group. A character on $(K^*)^n$ is a monomial in the associated Laurent polynomial ring with coefficient 1. For example, $(K^*)^2$ has coordinate ring $K[x^{\pm 1}, y^{\pm 1}]$ and characters on $(K^*)^2$ are of the form $x^i y^j$ for $i, j \in \mathbb{Z}$. $(K^*)^n$ is called the algebraic n-torus.

First we must describe functions on the torus in a coordinate-free way. We shall initially define M to be a general rank n lattice. As a group M is isomorphic to $\mathbb{Z}^n$ but it is not canonically isomorphic: there are many possible isomorphisms from M to $\mathbb{Z}^n$. Moreover, $\mathbb{Z}^n$ is often given additional structure, such as an inner product, which we do not wish to impart to M. We define the ring

$$K[M] := \left\{ \sum_{v \in M} a_v \chi^v : a_v \in K \right\}$$

where χ is a formal variable and only finitely many a_v are nonzero. Multiplication is defined by

$$\left(\sum_{v \in M} a_v \chi^v \right) \left(\sum_{v \in M} b_v \chi^v \right) = \sum_{v \in M} \left(\sum_{x+y=v} a_x b_y \right) \chi^v$$

This ring is (non-canonically) isomorphic to the usual coordinate ring on $(K^*)^n$, which is

$$K[x_1^{\pm 1}, ..., x_n^{\pm 1}] := \left\{ \sum_{v \in \mathbb{Z}^n} a_v \left(\prod_{1 \leq i \leq n} x_i^{v_i} \right) : a_v \in K \right\}$$

with only finitely many coefficients a_v nonzero. To construct an isomorphism, first choose a $\mathbb{Z}$-basis for M, say $\{e_i : i \leq 1 \leq n\}$. The existence of such a basis and its having n elements is what it means for M to be a rank n lattice. Now consider the map

$$\begin{aligned} p : K[x_1^{\pm 1}, ..., x_n^{\pm 1}] &\rightarrow K[M] \\ \sum_{v \in \mathbb{Z}^n} a_v \left(\prod_{1 \leq i \leq n} x_i^{v_1 e_1 + \cdots + v_n e_n} \right) &\mapsto \sum_{v \in \mathbb{Z}^n} a_v \chi^v \end{aligned}$$

Injectivity and surjectivity follow immediately from $\{e_i\}$ being a basis, and it is straightforward to check that this map preserves addition and multiplication. Thus, it is a ring isomorphism.

We may now define the n-torus in a coordinate-free manner as the space on which $K[M]$ is the coordinate ring. That is,

$$\mathbb{T} := \mathrm{Hom}_{K-alg}(K[M], K)$$

where $\mathbb{T}$ is our new coordinate-free notation for the n-torus. One may read this definition as saying that $\mathbb{T}$ is the set of points on which we evaluate the polynomial functions in $K[M]$.

In light of this new definition, a group structure may be defined on $\mathbb{T}$ in a coordinate-free manner as follows: let $\phi, \psi \in \mathbb{T}$. Then ϕ and ψ are K-algebraic homomorphisms from $K[M]$ to K and as such distribute over addition and multiplication and preserve constants. Thus they are completely determined by their values on characters χ^v. Define multiplication on $\mathbb{T}$ by

$$(\psi\phi)(\chi^v) := \psi(\chi^v)\phi(\chi^v)$$

Inverses may be defined by $\phi^{-1}(\chi^v) = \phi(\chi^{-v})$. Note that we can never have $\phi(\chi^v) = 0$ since $\phi(\chi^v)\phi(\chi^{-v}) = \phi(1) = \mathbb{1}_{\mathbb{T}}$. The isomorphism of $\mathbb{T}$ and $(K^*)^n$ may be seen by choosing a basis for M and using a map analogous to p as defined above. We now assert without proof that

$$M \cong \mathrm{Hom}_{\mathbf{AlgGrp}}(\mathbb{T}, K^*)$$

where $\mathrm{Hom}_{\mathbf{AlgGrp}}(A, B)$ denotes the group of algebraic group homomorphisms from A to B.

2 M, N, and Tropicalization

We now define N as the set of 1-parameter subgroups of $\mathbb{T}$:

$$N := \mathrm{Hom}_{\mathbf{AlgGrp}}(K^*, \mathbb{T})$$

Note that, since the dual of a rank n lattice is a rank n lattice, this gives immediately

$$N \cong \mathbb{Z}^n$$

although the isomorphism is not canonical. It shall be asserted that

$$N \cong \mathrm{Hom}_{\mathbf{Grp}}(M, \mathbb{Z})$$

We also define $N_{\mathbb{R}} := \mathrm{Hom}_{\mathbf{Grp}}(M, \mathbb{R})$ and assert $N \cong \mathbb{R}^n$.

We now define the tropicalization map. First, for $\xi \in \mathbb{T}$, that is $\xi : K[M] \to K$, let $\xi|_M$ denote the restriction of ξ to characters. That is,

$$\begin{aligned} (-)|_M : \{K[M] \to K\} &\to \{\{\chi^v\} \to K^*\} \\ \xi &\mapsto \xi|_M \end{aligned}$$

Since M is isomorphic to the group of characters, we have equivalently

$$\xi| : M \to K^*$$

hence the notation. We will now apply a valuation map. A valuation on a field is a map $val : K \to \mathbb{R} \cup \{\infty\}$ satisfying

$$\begin{aligned} val(k) = \infty \quad &\iff \quad k = 0 &(1)\\ val(k_1 k_2) \quad &= \quad val(k_1) + val(k_2) &(2)\\ val(k_1 + k_2) \quad &\geq \quad \min(val(k_1), val(k_2)) &(3) \end{aligned}$$

A few basic properties of valuations are

$$\begin{aligned} val \text{ is a valuation} \quad &\Rightarrow \quad \forall \alpha \in \mathbb{R}^* : \alpha \cdot val \text{ is a valuation} &(4)\\ val(1) \quad &= \quad 0 &(5)\\ val(k_1) \neq val(k_2) \quad &\Rightarrow \quad val(k_1 + k_2) = \min(val(k_1), val(k_2)) &(6) \end{aligned}$$

Property 1 above implies that, if our valuation is nontrivial (not mapping every nonzero element to zero), then we can always rescale val so that 1 is in the image.

If our field is algebraically closed, then $\forall k \in K : \forall q \in \mathbb{Q} : k^q \in K$. These together with axiom 2 imply that the image of val contains the rationals (and so is dense in $\mathbb{R}$). Thus, composing $\xi|M : M \to K^*$ with a valuation $val : K \to \mathbb{R} \cup \{\infty\}$yields a map from M to $\mathbb{R}$ where the image is dense. Axiom 2 implies that all these maps are group homomorphisms, meaning we have a map from $\mathbb{T}$ to $N_{\mathbb{R}}$. This map is the tropicalization of the torus.

3 The Projective Plane and Extended Tropicalization

We now introduce projective space. Define $\mathbb{P}^n(K)$ to be the set of $n+1$-tuples of elements of K where not all entries are zero, modulo equivalence under scalar multiplication by nonzero elements in K. We say that $\mathbb{P}^n(K)$ is the set of *homogeneous coordinates* over K with $n+1$ entries. Equivalently, $\mathbb{P}^n(K)$ is the space of lines through the origin in K^{n+1}. We wish to tropicalize $\mathbb{P}^n(K)$. To do so, we will cover it in charts we know how to tropicalize and then glue the tropicalizations of the charts. Note that since not all the coordinates of a point may be zero we can cover $\mathbb{P}^n(K)$ as follows:

$$\mathbb{P}^n(K) = \bigcup_{1 \leq i \leq n+1} U_i$$

where

$$U_i := \{(x_1 : ... : x_{n+1}) : x_i \neq 0\}$$

We may dehomogenize U_i in the i^{th} coordinate to obtain

$$U_i = \{(x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_{n+1})\}$$

Thus we see immediately that U_i is in bijection with K^n. The intersection of all the U_i is the algebraic n-torus $(K^*)^n$. To see this, note

$$\bigcap_i U_i = \{(x_1 : ... : x_{n+1}) : \forall i : x_i \neq 0\}$$

We can dehomogenize by any of the coordinates, say x_{n+1} to obtain

$$\bigcap_i U_i \cong \{(x_1, ..., x_n, 1) : \forall i : x_i \neq 0\}$$

We use $\cong$ in place of $=$ here to emphasize that we could have equivalently chosen to dehomogenize by any coordinate.

We will now restrict ourselves to the case $n = 2$ and introduce several sets of coordinates on M. Let $\{x, y\}$ be a $\mathbb{Z}$-basis for M. Then $\{x - y, -y\}$ and $\{y - x, -x\}$ are also $\mathbb{Z}$-bases. Let

$$\begin{aligned} u &= x - y \\ v &= -y \\ s &= y - x \\ t &= -x \end{aligned}$$

Note that these choices are symmetric in the sense that we could start with u and v or s and t and obtain equivalent relationships, defining for example x as $u - v$. Now let

$$\begin{aligned} S_0 &= < x, y >_{\mathbb{N}} \\ S_1 &= < u, v >_{\mathbb{N}} \\ S_2 &= < s, t >_{\mathbb{N}} \end{aligned}$$

where $< p, q >_{\mathbb{N}}$ denotes the nonnegative integer span of vectors p and q: the S_i are cones in M. We may define $K[S_i]$ as the coordinate rings over K with characters $\chi^u, u \in S_i$. Note that these are exactly the coordinate rings on the U_i. We choose to associate U_i with $K[S_i]$. That is,

$$U_i = \mathrm{Hom}_{\mathbf{K-Alg}}(K[S_i], K)$$

which gives us another way to see how the torus is embedded in each U_i: the torus is the subset of U_i whose elements do not send any characters to 0.

Each U_i contains the torus, which we know how to tropicalize, plus some extra points that we don't. Since the U_i are all isomorphic, we need only show how to tropicalize (the extra points of) U_0. For all $\xi \in U_0$, ξ is determined by $\xi(\chi^x)$ and $\xi(\chi^y)$. Tropicalization takes the torus to $N_{\mathbb{R}}$. To find out what we have to add, we need to consider the cases where one or both of $\xi(\chi^x), \xi(\chi^y)$ are zero. If $\xi(\chi^x)$ is zero and $\xi(\chi^y)$ is not, valuation sends ξ to $(\infty, val(\xi(\chi^y)))$. Thus, we must add a copy of the real line at $x^* = \infty$ that is not collinear with y^*, where x^* is a coordinate in $N_{\mathbb{R}}$ defined by the element of $N_{\mathbb{R}}$ that sends x

to 1 and y to 0, and y^* is the analogue for y. Similarly, we must add a copy of the real line at $y^* = \infty$ that is not collinear with x^*. The last case, in which $\xi(\chi^x) = \xi(\chi^y) = 0$ adds a point at $x^* = y^* = \infty$, and this point connects the two lines at infinity previously added in the natural way. Thus, the tropicalization of U_0 is a copy of the real plane with two rays emanating from (∞, ∞). Similarly, the tropicalization of U_1 is a copy of the plane with additional edges at $u^* = \infty$ and $v^* = \infty$. The edges at $u^* = \infty$ and $x^* = \infty$ are actually the same edge: to see this, note

$$x^*(u) = x^*(x - y) = x^*(x) - x^*(y) = 1$$
$$x^*(v) = -x^*(y) = 0$$
$$\Rightarrow x^* = u^*$$

Similarly we have

$$y^* = s^*$$
$$t^* = v^*$$

Assembling the pieces, we find that the tropicalization of $\mathbb{P}^2(K)$ is a triangle with the edges at infinity.

4 Tropical Lines

The set of lines in $\mathbb{P}^2(K)$ is in bijection with the set of planes through the origin in K^3, and every line in $\mathbb{P}^2(K)$ intersects some U_i. Thus all lines in projective space may be described in one of the following ways:

$$\begin{aligned} a\chi^x + b\chi^y + c &= 0 \\ a\chi^u + b + c\chi^v &= 0 \\ a + b\chi^s + c\chi^t &= 0 \end{aligned}$$

Note that if a line intersects the torus, then it may be described in any of the pairs of coordinates.

Define the tropicalization trop(f) of any Laurent polynomial $f = \sum a_u \chi^u$ by

$$\text{trop}(\sum a_u \chi^u) = \min(val(a_u) + u)$$

This expression seems to make no sense, since we appear to be adding a vector to a number. However, it is actually describing a functional on $N_{\mathbb{R}}$: apply u to an element of $N_{\mathbb{R}}$ then add $val(a_u)$.

The Fundamental Theorem of Tropical Geometry sates: let $V(f)$ denote the variety of the algebraic function f. The tropicalization of $V(f)$ as a subset of $\mathbb{P}^n(K)$ produces the same set as that obtained in the following way: tropicalize f as above and take the closure of the set of all points on which the minimum is achieved at least twice, i.e. the *bend loci of the tropical polynomial.* Thus, tropicalizations of lines in projective space may be obtained by tropicalizing the above linear equations and taking their bend loci. In the torus, we obtain

$$\min(val(a) + x, val(b) + y, val(c))\text{obtained at least twice}$$

$$\begin{aligned} \Rightarrow x^* &= y^* + val(b) - val(a) \leq val(c) \\ \vee x^* &= val(c) - val(a) \leq y + val(b) \\ \vee y^* &= val(c) - val(b) \leq x + val(a) \end{aligned}$$

where none of $val(a), val(b), val(c)$ are infinity. Thus the vertex of the tropical line, the point where all three of these rays meet, lies in the interior of the extended tropical plane. Taking the closure, besides filling in the points with irrational coordinates, adds the natural points on the boundary:

$$\begin{aligned} \{x^* = val(c) - val(a), y^* = \infty\} \\ \{x^* = \infty, y^* = val(c) - val(b)\} \\ \{x^* - y^* = val(b) - val(a), v^* = \infty\} \end{aligned}$$

If, in the algebraic equation defining the line, $c = 0$ and $a \neq 0 \neq b$, we have $val(c) = \infty$, giving

$$\min(val(a) + x, val(b) + y, \infty)\text{obtained at least twice}$$

$$\Rightarrow x^* = y^* + val(b) - val(a)$$

The closure is a straight line plus boundary points at one corner and in the relative interior of the opposite edge. Analogous cases in the other coordinate systems are, shockingly, analogous. If $b = c = 0 \neq a$, then we have

$$\min(val(a) + x, \infty)\text{obtained at least twice}$$

$$\Rightarrow x^* = \infty$$

which is one of the boundary edges. Analogous situations in the other coordinate systems give the other two boundary edges.

5 Incidence Configurations, Geometric Constructions, and the Fano Matroid

We now need to introduce some general geometric notions. An *incidence configuration* is a set of geometric objects (points, lines, conics, etc.) together with a set of relationships between these objects. For example, a set of three (non-parallel) lines and three points, where the points are defined to be the pairwise intersections of the lines, is an incidence configuration. A *geometric construction* is a procedure that takes as input some set of geometric objects, defines new objects in terms of those given, defines more new objects in terms of those,

etc. A construction might work as follows:
(1) Take three generic lines as input.
(2) Define 3 points as the pairwise intersections of these lines.
(3) Define a new line as the span of two of these points.
For our purposes, we need only consider constructions in $\mathbb{P}^2(K)$ and $\text{Trop}(\mathbb{P}^2(K))$ involving points and lines. A formal definition is given in [1]. Note that in both of these spaces, any two lines are guaranteed to intersect on at least one point.

A *matroid* is a generalization of the notion of independent sets of vectors in linear algebra. There are many equivalent ways to define a matroid. We will present only one definition:

A finite matroid M is an ordered pair (E, I) where E is a finite set and $I \subset \mathcal{P}(E)$ satisfying
(1) $\emptyset \in I$
(2) $A \in I \wedge B \subset A \Rightarrow B \in I$
(3) $A, B \in I \wedge |A| > |B| \Rightarrow \exists a \in A : B \cup \{a\} \in I$

A matroid M is said to be realizeable over a field K if there is a set of vectors over K that is isomorphic to M as a matroid. The Fano matroid may be defined as the set of all nonzero vectors in $\mathbb{F}_2^3$ with the usual notion of independence. Interestingly, there is no set of vectors over any field of characteristic other than 2 to which it is isomorphic as a matroid. The Fano matroid may also be regarded as an incidence configuration: the minimal dependent sets define planes that intersect each other at specific points in $\mathbb{F}_2^3$. Since the Fano matroid is a configuration of 7 planes and 7 lines in $\mathbb{F}_2^3$, it is a configuration of 7 lines and 7 points in $\mathbb{P}^2(\mathbb{F}_2)$.

For our purposes, we will want to work over a larger characteristic 2 field. First, define Laurent series over a field as formal power series in which finitely many powers are allowed to be negative. Formally,

$$\mathbb{F}((t)) := \left\{ \sum_{i=n}^{\infty} a_i t^i : a_i \in \mathbb{F}, n \in \mathbb{Z} \right\}$$

We may then define Puiseux series in terms of Laurent series:

$$\mathbb{F}\{\{t\}\} := \bigcup_{m=1}^{\infty} \mathbb{F}((t^{1/n}))$$

Addition of Puiseux series is straightforward. Multiplication is defined by

$$\left(\sum_q a_q t^q \right) \left(\sum_p b_p t^p \right) = \sum_k \left(\sum_{p+q=k} a_q b_p \right) t^k$$

The inner sum is guaranteed to be finite by the fact that, in a given Puiseux series, all the powers that appear share a common denominator. Under these operations, Puiseux series form a field. Furthermore, if the base field is algebraically closed then so are Puiseux series over that field. In most of what follows we will let our field $K = \overline{\mathbb{F}}_2\{\{t\}\}$, where $\overline{\mathbb{F}}_2$ denotes the algebraic closure of $\mathbb{F}_2$. The Fano configuration exists over this field, and in fact there are many configurations of planes that satisfy the Fano incidence relationships: any action of $\mathrm{GL}_3(K)$ on the original configuration produces another that is isomorphic as a matroid. Thus, over this larger characteristic 2 field we may speak of 'Fanos' or 'a Fano.' The field of Puiseux series possesses a natural valuation, which is the function that takes a Puiseux series to the value of the lowest power which appears in a term with nonzero coefficient. Thus, we may run tropicalization on our new field K.

6 Tropical Fano and Lifting

It is interesting to note that the Fano incidence configuration is also realizeable in the tropical plane. In fact, it must be since we can tropicalize a Fano configuration in $\mathbb{P}^2(K)$ and tropicalization (being, at the very least, a function) must preserve set-theoretic intersections and containments since, for example, if a point lies in a line, tropicalizing the point and the line cannot separate them. For the same reason, any incidence configuration realizeable in Puiseux series over any field must be realizeable tropically.

The notion of intersection in the tropical plane in not as clear-cut as it is in the projective or euclidean planes: two tropical lines are guaranteed to intersect, but their set-theoretic intersection may consist of a half ray rather than a single point, in which case the two lines are said to be in special position. However, there is a notion of *stable tropical intersection*: there is only one point in the set-theoretic intersection whose position changes continuously with respect to perturbations of the two lines. There is an analogous notion of the stable join of two points. These can be computed using a tropical version of Cramer's Rule, described in [1]. However, there are situations where one must consider nonstable intersection points.

A *lift* of a tropical geometric construction is a formally identical construction performed in projective space which tropicalizes to the tropical construction. There will always be many arrangements of lines and points in projective space which map to a given tropical configuration, but they will not necessarily have the same incidence structure as in the tropical case: a line and a point which do not intersect in projective space may tropicalize to a line and a point which do intersect. The question then is this: given a tropical geometric construction, does there exist a projective realization of this construction which tropicalizes correctly? That is, does there exist a lift of the tropical construction?

First, we must put some constraints on the kinds of constructions we will consider. The most import restriction is that the graph associated with the construction be a tree, which is to say free of loops. This graph is constructed as follows: every object in the construction is assigned a node. Nodes corresponding to objects are linked exactly when one of them was used to define the other: if two points are used to define a line, then there will be edges from each of the nodes representing the points to the node representing the line. There is an important sense in which constructions satisfying this criterion are claim-free, meaning that no step of the construction could fail over any field; there is always a line containing two given points, always a point at which two lines intersect (in projective space), etc.

The construction we will focus on primarily is that of the Fano configuration, which proceeds as follows:

Choose points 1,2,3,7

Define lines 1,2,3,4,5,6 as pairwise spans of these points

Define points 4,5,6 as intersections of line pairs (3,4), (1,5), and (2,6) respectively

(Thesis Node/Claim) Define line 7 as a line containing points 4,5,6

Note that in the Fano configuration, the construction is loop-free except for a single projective line, namely the one that only exists over a characteristic 2 field, which corresponds to the plane specified by the minimal dependent set $\{(1,1,0),(0,1,1),(1,0,1)\}$. This is labeled line 7 above. We shall call the incidence configuration/construction obtained by removing this line from a Fano configuration a *pre-Fano* configuration, or simply a pre-Fano. Thus, on the question of lifting tropical Fano configurations, the strategy will be as follows: given a tropical pre-Fano configuration, where intersection points and spans are not necessarily stable, can we find a lift of this configuration to projective space? Equivalently, what constraints must be placed on such a tropical configuration to ensure that a lift exists? For if we can find such a lift, we know we can complete it to a full Fano in projective space which will of necessity tropicalize to a tropical Fano corresponding with the initial tropical pre-Fano.

This question has been solved not only for Fano configurations but in a much more general setting in [1], under the additional constraint that the tropical construction be stable, meaning that all intersection points and spans are chosen using tropical Cramer's rule. The question we are investigating is under what circumstances a potentially nonstable configuration can be lifted. The motivation for this question is this: as noted above, any action by an element of $\mathrm{GL}_3(K)$ on the original Fano configuration produces a new Fano configuration whose tropicalization will of necessity satisfy the same incidence relations set-theoretically (here we mean the positive incidence relations, e.g. a point lies in the intersection of two lines. After tropicalization there may be many more incidences, for example projectively distinct points may map to the same tropical point). However, it is interesting to note that the tropicalizations of many of the Fanos thus produced do not satisfy the Fano incidence relations if we only consider stable intersections and joins. For example, the matrix

$$\begin{bmatrix} 1 & t^2 & 1 \\ t+t^2 & t & 0 \\ 1+t & t & 1 \end{bmatrix}$$

acting upon K^3 and by extension upon $\mathbb{P}^2(K)$ produces a Fano configuration in whose tropicalization all the lines are identical, incidentally the line with vertex at the origin. The stable intersection of two identical tropical lines is their shared vertex. However, the tropicalizations of the projective intersections are, in (x^*, y^*) coordinates,

$$(0,1),(1,0),(0,\infty),(-\infty,-\infty),(0,2),(\infty,0),(0,1)$$

where $(-\infty,-\infty)$ indicates the point at the intersection of the v^*-axis and the boundary of the extended tropical plane. None of these points are the origin and therefore none of them are the stable intersection of any two of the (indentical) lines in this tropical Fano. Thus, we have a very natural situation in which a nonstable tropical configuration and its lift occur. For the sake of transparency, we will now have a brief digression on how the above computation was performed.

6.1 Action of $\mathbf{GL}_3(K)$ on Projective Fano and Corresponding Tropicalizations

The 'algebraic' Fano consists of 7 points in $\mathbb{P}^2(K)$ (1 dimensional linear subspaces of K^3) and 7 lines in $\mathbb{P}^2(K)$ (2 dimensional linear subspaces of K^3). These points are

$$(1:0:0) \tag{7}$$
$$(0:1:0) \tag{8}$$
$$(0:0:1) \tag{9}$$
$$(1:1:0) \tag{10}$$
$$(0:1:1) \tag{11}$$
$$(1:0:1) \tag{12}$$
$$(1:1:1) \tag{13}$$

The lines are given by the equations

$$x = 0 \tag{1}$$
$$y = 0 \tag{2}$$
$$v = 0 \tag{3}$$
$$x+y = 0 \tag{4}$$
$$s+t = 0 \tag{5}$$
$$u+v = 0 \tag{6}$$
$$x+y+1 = 0 \tag{7}$$

which may be represented by vectors specifying the coefficients in the equations, those vectors having the same entries as the points in the configuration, respective of the numbering (e.g. $u+v=0$ may be represented $(0:1:1)$); lines are vectors in the dual space. Now we need to know how an invertible matrix acing on K^3 moves these vectors in projective space and the dual of projective space. The action on points is obvious: the matrix given as an example above moves the seven points to

$$(1 : t + t^2 : 1 + t) \tag{1}$$
$$(t : 1 : 1) \tag{2}$$
$$(1 : 0 : 1) \tag{3}$$
$$(1 + t^2 : t^2 : 1) \tag{4}$$
$$(1 + t^2 : t : 1 + t) \tag{5}$$
$$(0 : 1 + t : 1) \tag{6}$$
$$(t^2 : t^2 : 0) \tag{7}$$

which tropicalize to

$$(0, 1)_{x^*,y^*} \tag{1}$$
$$(1, 0)_{x^*,y^*} \tag{2}$$
$$(0, \infty)_{x^*,y^*} \tag{3}$$
$$(0, 2)_{x^*,y^*} \tag{4}$$
$$(0, 1)_{x^*,y^*} \tag{5}$$
$$(\infty, 0)_{x^*,y^*} \tag{6}$$
$$(0, \infty)_{x^*,v^*} \tag{7}$$

where the subscript indicates the coordinate system.

Now for the actions on lines. Let f be the function whose variety is the line in question, and $g \in \mathrm{GL}_3(K)$. We wish to find the set

$$\{gx : fx = 0\}$$

This is identical to the set

$$\{x : fg^{-1}x = 0\}$$

Thus we must compute the inverse of our matrix g (up to a constant factor) and multiply on the left by the row vectors representing our lines. We may then use the Fundamental Theorem to find the desired tropical variety. In the example above, the inverse of our matrix is (a multiple of)

$$\begin{bmatrix} 1 & 1+t & 1 \\ 1+t & 1 & 1+t \\ 1+t^2 & 1+t+t^2 & 1+t^2+t^3 \end{bmatrix}$$

giving the new lines (dual space vectors)

$$(1:1:1) \tag{1}$$
$$(1:1:1) \tag{2}$$
$$(1:1:1) \tag{3}$$
$$(1:1:1) \tag{4}$$
$$(1:1:1) \tag{5}$$
$$(1:1:1) \tag{6}$$
$$(1:1:1) \tag{7}$$

which all tropicalize to the line with vertex at the origin as claimed.

6.2 Degrees of Freedom in Lifts

Any valuation induces a metric in the following way:

$$|x-y| = e^{-val(x-y)}$$

It is easy to check that this is in fact a metric. Thus, we have a topology on $\mathbb{P}^2(K)$ with respect to which tropicalization is continuous. Given the valuation introduced above for Puiseux series, we can be explicit about how much freedom we have in choosing the lift of, say, a tropical point. If the point is in the interior with coordinates (a, b) then its lift must be of the form

$$(t^a(1+p_1(t)) : t^b(1+p_2(t)) : 1)$$

where $p_1(t), p_2(t)$ are Puiseux series with lowest power strictly positive. Thus a lift of this tropical point is exactly a projective point lying the product of an open interval of width e^{-a} and one of width e^{-b}. The region of lifts of a line is similarly an open polydisk in the dual space.

Considering a pre-Fano configuration specifically, we are lifting 6 lines and 7 points. Ignoring the requirements of the incidence relationship, our space of lifts is of the form

$$D^{26} \subset (V^*)^{12}(K^*)^{14}$$

where V denotes the dual space of K and D^n is some n-dimensional polydisk. The constraints imposed by the incidence relationship may be regarded as hypersurfaces in this 26-dimensional space and are defined by the following equations:

$$\begin{array}{lll}
\textcircled{1}\cdot\overline{2}=0, & \textcircled{1}\cdot\overline{3}=0, & \textcircled{1}\cdot\overline{5}=0\\
\textcircled{2}\cdot\overline{3}=0, & \textcircled{2}\cdot\overline{1}=0, & \textcircled{2}\cdot\overline{6}=0\\
\textcircled{3}\cdot\overline{1}=0, & \textcircled{3}\cdot\overline{2}=0, & \textcircled{3}\cdot\overline{5}=0\\
\textcircled{4}\cdot\overline{3}=0, & \textcircled{4}\cdot\overline{4}=0, & \textcircled{4}\cdot\overline{7}=0\\
\textcircled{5}\cdot\overline{1}=0, & \textcircled{5}\cdot\overline{5}=0, & \textcircled{5}\cdot\overline{7}=0\\
\textcircled{6}\cdot\overline{2}=0, & \textcircled{6}\cdot\overline{6}=0, & \textcircled{6}\cdot\overline{7}=0
\end{array}$$

where circled numbers are the lines of that index, overlined numbers are the points, and the dot product represents evaluating the function for the line at that point. Now we can see that the condition that a lift exists is equivalent to the condition that these 18 hyperplanes together with the polydisk defined above have a nonempty intersection. If all the hyperplanes intersect transversely and this intersection passes through the polydisk, then we have an 8-dimensional slice of a polydisk of viable lifts, corresponding, say, to choosing each coordinate of points 1,2,3, and 7.

6.3 Principle Coefficients in Lifts

Another way to approach the problem is to consider the principal coefficients of the lifts of the elements in a construction, which do not affect their tropicalization. Much of what follows is taken directly from from [1]. The *principal coefficient* of an element of Puiseux series is simply the coefficient of the lowest order term. They can be defined more generally, but for our purposes this definition is sufficient. The principal coefficients of course are nonzero elements of the base field, in our case $\overline{\mathbb{F}_2}$.

Definition 1. *Let* $(\mathbb{R}, \min, +)$ *be the tropical semiring. For a given* $n \times n$ *matrix over* $(\mathbb{R}, \min, +)$*, the* tropical determinant *is defined*

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_t = \bigoplus_{\sigma \in \Sigma_n} a_{1\sigma(1)} \odot \cdots \odot a_{n\sigma(n)}$$

where Σ_n *is the set of permutations on* n *elements and the circled addition* $\oplus$ *and multiplication* $\odot$ *are those operations in the tropical semiring.*

Definition 2. *Let* $O = (o_{ij})$ *be an* $n \times n$ *matrix over* $(\mathbb{R}, \min, +)$*. Let* $A = (a_{ij})$ *be an* $n \times n$ *matrix in a ring. Denote by* $|O|_t$ *the tropical determinant of* O*. Define the* pseudodeterminant *of* A *with respect to* O *to be*

$$\Delta_O(A) = \sum_{\substack{\sigma \in \Sigma_n \\ o_{1\sigma(1)} \odot \cdots \odot o_{n\sigma(n)} = |O|_t}} (-1)^{i(\sigma)} a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

where $i(\sigma)$ *denotes the sign of the permutation* σ*.*

Lemma 3. *Let* $B = (b_{ij})$ *be a matrix over* K^**,* $A = (a_{ij})$ *be the matrix of principal coefficients of elements of* B*, and* O *be the tropicalization matrix of* B*,* $o_{ij} = T(b_{ij})$*. If* $\Delta_O(A) \neq 0$*, then the principal coefficient of* $|B|$ *is equal to* $\Delta_O(A)$*, and furthermore* $T(B)$ *coincides with the tropical determinant* $|O|_t$*.*

Proof. In the expansion of the determinant $|B|$ of B, those permutations σ corresponding to the summands $b_{1\sigma(1)} \dots b_{n\sigma(n)}$ having the lowest order are by construction exactly the permutations in the expansions of $|O|_t$, i.e. the ones where $|O|_t$ is attained. So, the coefficient of the term $t^{|O|_t}$ in $|B|$ is $\Delta_O(A)$, and so if $\Delta_O(A) \neq 0$ then the order of $|B|$ is $|O|_t$. □

This lemma can be applied immediately in the context of Cramer's Rule:

Definition 4. *Let $O = O_{ij}$ be an $n \times (n+1)$ tropical matrix. Let $A = (a_{ij})$ be a matrix over any ring, also $n \times (n+1)$. Define*

$$Cram_O(A) = (S_1, \dots, S_{n+1})$$

where $S_i = \Delta_{O^i} A^i$ and A^i (O^i) denotes the matrix obtained by removing the i^{th} column from the matrix A (O).

Lemma 5. *Suppose we have a system of n linear equations in n+1 homogeneous variables over $(\mathbb{R}, \min, +)$. Let O be the coefficient matrix of the system and B be any lift of O, i.e. $T(B) = O$. Let A be the principal coefficient matrix of B. If $\forall i : S_i \neq 0$, then the linear system defined by B has exactly one solution in projective space and it tropicalizes to the stable tropical solution $[|O^1|_t : \dots : |O^{n+1}|_t]$.*

Proof. Apply the previous lemma to every component of the projective solution. □

It is straightforward to modify the above results, taken from [1], so that they instead talk about nonstable tropical intersections and spans:

Lemma 6. *Let $B = (b_{ij})$ be a matrix over K^*, $A = (a_{ij})$ be the matrix of principal coefficients of elements of B, and O be the tropicalization matrix of B, $o_{ij} = T(b_{ij})$. If $\Delta_O(A) = 0$, then the order of $|B|$ is strictly greater than $|O|_t$. That is, $T(|B|) > |O|_t$.*

Proof. The result follows immediately from the definition of the valuation on Puiseux series and that of the principle coefficient. □

Lemma 7. *Suppose we have a system of n linear equations in n+1 homogeneous variables over $(\mathbb{R}, \min, +)$. Let O be the coefficient matrix of the system and B be any lift of O, i.e. $T(B) = O$. Let A be the principal coefficient matrix of B. If some but not all of the elements of $Cram_O(A)$, say $\{S_j\} \subset Cram_O(A)$, are zero, then the linear system defined by B has exactly one projective solution that does not tropicalize to the stable tropical solution; the coordinates of the tropicalized projective solution corresponding to the $\{S_j\}$ will be greater than the coordinates of the stable tropical solution.*

Proof. Apply previous lemma to each component of the projective solution. We require that not all components of $\mathrm{Cram}_O(a)$ be zero because we are working in homogeneous coordinates, and so if all components were zero, then all the coordinates of the tropicalized projective solution would be greater than all the coordinates of the stable tropical solution and we would not be able to determine without more information whether they agreed or not. □

Thus, the problem of lifting a nonstable tropical construction becomes the following: in choosing the lifts of the various nodes of the construction, can we choose their principle coefficients in such a way as to make some of the elements of some of those $\mathrm{Cram}_O(A)$ corresponding to those steps of the projective construction analogous to the nonstable steps of the tropical construction zero while leaving the rest nonzero? (As an aside, given that this could be done we would still have to select the higher order terms of the lifts in such a way as to set the orders of the various pseudodeterminants to the desired values).

References

[1] L. F. Tabera, *Tropical Constructive Pappus' Theorem*, February 8, 2008.