Energetics of Elastic Curves

Jaeyoon Kim and Charles Devlin
Advisor: Ian Tobasco

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Abstract

In 1954, John Nash proved that any smooth embedding of a Riemannian manifold which shrinks distances may be approximated arbitrarily closely by C^1 isometric embeddings. While such isometric embeddings are necessarily rough, featuring large or infinite curvatures in general, quantifying the precise amount of roughness required remains an open problem. Inspired by connections with elasticity theory, we consider the problem of minimizing the "bending energy" or total square curvature of an arbitrary arc-length parameterized curve approximating a given short one. We produce an explicit helical construction and prove that it is nearly optimal in the large curvature limit.

1 Introduction

Suppose we are given a curve $\gamma : [0, L] \to \mathbb{R}^3$ with arc length parameterization. Euler showed that if we approximate a thin elastic rod by γ, the total elastic energy of the rod is proportional to the total squared curvature of γ up to higher order terms [2]. In Section 2, we study the minimum elastic energy required to confine a curve of fixed length to a ball. We prove that the minimum energy is proportional to inverse of the square of the radius of the ball. We further argue that the curves which attain this minimum energy are approximately given by the great arcs of the sphere. Then, in Section 3 we study the energetics of the approximation of short curves by curves of unit speed. We prove that the asymptotic behavior of the energy in the limit of arbitrarily close approximation depends only on the speed of the target curve.

2 Isometric Restriction

Question Given a thin elastic rod of length L, what is the minimum elastic energy necessary to place it into a ball of radius ϵ?

Notation. *We write $||\cdot||$ (without subscripts) for the standard Euclidean norm on $\mathbb{R}^n$.*

Definition 1. *For $\gamma : [0, L] \to \mathbb{R}^3$ in H^2, we define the elastic energy of γ as*

$$E(\gamma) = \int_0^L ||\gamma''(t)||^2 dt$$

Notation. *For $\epsilon > 0$, we denote the minimum elastic energy of the curves of length L confined to the closed ball of radius ϵ by*

$$E_L^*(\epsilon) = \min\{E(\gamma) : \gamma : [0, L] \to \mathbb{R}^3 \text{ is } H^2, ||\gamma'(t)|| = 1 \forall t, ||\gamma||_{C^0} \leq \epsilon\}$$

Where $||f||_{C^0} \equiv \sup_{t\in[0,L]} ||f(t)||$ for a function f.

With these definitions in place, we turn to studying the asymptotic behavior of the minimum energy for a fixed length L as we take ϵ to 0. We also argue that the great arcs of the sphere are asymptotically optimal.

Theorem 1. *For $0 < \epsilon < \frac{L}{2}$,*

$$\frac{L}{\epsilon^2} \geq E_L^*(\epsilon) \geq \frac{L}{\epsilon^2} - \frac{4}{\epsilon} + \frac{4}{L}$$

Proof. Fix $\epsilon > 0$. Choose $\gamma : [0, L] \to B_\epsilon(0)$ in H^2 with unit speed.

$$\begin{aligned}
\left(\int_0^L ||\gamma''||^2\right)^{\frac{1}{2}} \left(\int_0^L ||\gamma||^2\right)^{\frac{1}{2}} &\geq \int_0^L ||\gamma''|| ||\gamma|| \\
&\geq \left|\int_0^L \gamma'' \cdot \gamma\right| \\
&= \left|\int_0^L \sum_i \gamma_i \gamma_i''\right| = \left|\sum_i \int_0^L \gamma_i \gamma_i''\right| \\
&\geq \left|\int_0^L \sum_i (\gamma_i')^2\right| - \sum_i \left|[\gamma_i \gamma_i']_0^L\right| \\
&\geq \left|\int_0^L ||\gamma'||^2\right| - 2\epsilon
\end{aligned}$$

Thus we get that

$$\sqrt{E(\gamma)}\sqrt{\epsilon^2 L} \geq L - 2\epsilon$$

For $\epsilon < \frac{L}{2}$, this means that

$$E(\gamma) \geq \frac{(L-2\epsilon)^2}{\epsilon^2 L}$$

Since this gives us an lower bound, the infimum is just as large. Thus

$$E_L^*(\epsilon) \geq \frac{L}{\epsilon^2} - \frac{4}{\epsilon} + \frac{4}{L}$$

To obtain the upper bound, we construct a curve along a great arc of the sphere. Define $\gamma : [0, L] \to \mathbb{R}^3$ by

$$\gamma(t) = \left(\epsilon \cos\left(\frac{t}{\epsilon}\right), \epsilon \sin\left(\frac{t}{\epsilon}\right), 0\right)$$

We note that γ is arc-length parameterized and $E(\gamma) = \frac{L}{\epsilon^2}$ Combining, we arrive at the desired result

$$\frac{L}{\epsilon^2} - \frac{4}{\epsilon} + \frac{4}{L} \leq E_L^*(\epsilon) \leq \frac{L}{\epsilon^2}$$

It follows that $\lim_{\epsilon \to 0^+} \epsilon^2 E_L^*(\epsilon) = L$. □

By the construction of the upper bound in Theorem 1, we may conclude that the energies of curves γ along great arcs of the sphere are asymptotically optimal.

Now we note that the optimal curve must touch the boundary at least twice. Otherwise, by scaling, we may produce a new curve which touches the boundary at least twice and strictly reduces the energy by increasing the square curvature of the original curve. Now suppose the optimal curve touches the boundary at two distinct values of t. We now provide a construction of a curve which matches the original up to the first of these values of t, lies on the great arc between them, and decreases the total energy.

Claim. *Suppose we have $\gamma : [0, L] \to \overline{B}_\epsilon(0)$ in H^2 with unit speed such that*

1. *γ is confined to a plane in $\mathbb{R}^3$*
2. *There are $t_0 < t_1$ such that $||\gamma(t_0)|| = ||\gamma(t_1)|| = \epsilon$*

Then there exists $\mu : [0, L] \to \overline{B}_\epsilon(0)$ such that $\mu \in H^2$, μ is confined to the same plane as γ, $\mu|_{[0,t_0]} = \gamma$, $||\mu(t)|| = \epsilon$ for all $t \in [t_0, t_1]$, and $\mu|_{[t_1,L]}$ is obtained by a rotation of $\gamma|_{[t_1,L]}$. Furthermore, $E_\epsilon(\mu) \leq E_\epsilon(\gamma)$

Proof. By rotational symmetry, we may assume without loss of generality that γ lies in the plane $z = 0$ and $\gamma(t_0) = 0$. As $\gamma \in H^2$, there is $h \in H^1$ such that h is a weak derivative of γ. Write $\gamma_1 = \gamma|_{[0,t_0]}$, $\gamma_2 = \gamma|_{[t_0,t_1]}$, and $\gamma_3 = \gamma_{[t_1,L]}$. It is easily verified that $h_1 \equiv h|_{[0,t_0]}$ and $h_3 \equiv h|_{[t_1,L]}$ are in H^1 and are weak derivatives for γ_1 and γ_3 respectively. Define $\mu : [0, L] \to \mathbb{R}^3$ as follows: Let l denote the length of $\gamma|_{[t_0,t_1]}$ and $\theta = \frac{l}{\epsilon}$. Let ϕ denote the angle between $\gamma(t_0)$ and $\gamma(t_1)$. Let $R_{\theta-\phi}$ denote the linear map which rotates $\mathbb{R}^3$ about the line $z = 0$ by an angle $\theta - \phi$. Set $\nu(t) = R_{\theta-\phi}\gamma|_{[t_1,L]}(t)$ so that ν is a function $[t_1, L] \to \mathbb{R}^3$. We set

$$\mu(t) = \begin{cases} \gamma(t) & t \leq t_0 \\ (\epsilon\cos(\frac{t}{\epsilon}), \epsilon\sin(\frac{t}{\epsilon}), 0) & t_0 \leq t \leq t_1 \\ \nu(t) & t \geq t_1 \end{cases}$$

By inspection, we see that the image of μ lies in $\overline{B}_\epsilon(0)$ and $|\mu(t)| = \epsilon$ for all $t \in [t_0, t_1]$. We verify now that $\mu \in H^2$. Since $\gamma \in L^2$, both $\mu|_{[0,t_0]}$ and $\mu|_{[t_1,L]}$ do as well. And $\int_{t_0}^{t_1} ||\mu|_{[t_0,t_1]}||^2 = \epsilon^2(t_1 - t_0) < \infty$. We therefore conclude that $\mu \in L^2$.

Define

$$k(t) = \begin{cases} h_1(t) & t \leq t_0 \\ (-\sin(\frac{t}{\epsilon}), \cos(\frac{t}{\epsilon}), 0) & t_0 \leq t \leq t_1 \\ R_{\theta-\phi}h_3(t) & t_1 \leq L \end{cases}$$

Observe that $|k(t)| = 1$ for all t. As $\gamma \in H^2$, it is true that γ is essentially bounded, so we may apply the Chain Rule to $\nu(t)$ to see that a weak derivative for ν is $R_{\theta-\phi}h_3$. By direct computation, we see that for all $\varphi(t) \in C_C^\infty((0, L))$,

$$\begin{aligned} \int_0^L \mu(t)\varphi'(t)dt &= \int_0^{t_0} \gamma(t)\varphi'(t)dt + \int_{t_0}^{t_1} (\epsilon\cos(\tfrac{t}{\epsilon}), \epsilon\sin(\tfrac{t}{\epsilon}), 0)\varphi'(t)dt + \int_{t_1}^L R_{\theta-\phi}\gamma(t)\varphi'(t)dt \\ &= -\int_0^{t_0} h_1\varphi(t)dt - \int_{t_0}^{t_1} (-\sin(\tfrac{t}{\epsilon}), \cos(\tfrac{t}{\epsilon}), 0)\varphi(t)dt - \int_{t_1}^L R_{\theta-\phi}h_3(t)\varphi(t)dt \\ &= -\int_0^L k(t)\varphi(t)dt \end{aligned}$$

So that k is a weak derivative for μ. As $h \in L^2$ and the function $t \mapsto (-\sin(\frac{t}{\epsilon}), \cos(\frac{t}{\epsilon}), 0) \in L^2([t_0, t_1])$ clearly, it follows that $k \in L^2$ and so $\mu \in H^1$.

The argument that $\mu \in H^2$ follows similarly. $h \in H^1$ so has a weak derivative h'. An argument similar to that made above shows that the function

$$\begin{cases} h'(t) & t \leq t_0 \\ (-\frac{1}{\epsilon}\cos(\frac{t}{\epsilon}), -\frac{1}{\epsilon}\sin(\frac{t}{\epsilon}), 0) & t_0 \leq t \leq t_1 \\ R_{\theta-\phi}h'(t) & t_1 \leq t \end{cases}$$

is a weak derivative for $k(t)$ which is in L^2.

We now argue that the energy of μ is, at most, that of γ. Note that because $\mu(t) = \gamma(t)$ for $t \in [0, t_0]$, we know that the elastic energy on that segment is equal. i.e.

$$\int_0^{t_0} ||\mu''||^2 = \int_0^{t_0} ||\gamma''||^2$$

Furthermore, by rotational symmetry we have equal energy on the interval $[t_1, L]$. i.e.

$$\int_{t_1}^{L} ||\mu''||^2 = \int_{t_1}^{L} ||\gamma''||^2$$

Now we refer back to our proof of Theorem 1. We showed that

$$\left(\int_{t_0}^{t_1} ||\gamma''||^2\right)^{\frac{1}{2}} \left(\int_{t_0}^{t_1} ||\gamma||^2\right)^{\frac{1}{2}} \geq t_1 - t_0 - [\gamma \cdot \gamma']|_{t_0}^{t_1}$$

Note that because $\gamma(t_0)$ and $\gamma(t_1)$ lies on the boundary of the sphere, it's derivative must lie tangent to the surface of the sphere. Thus $\gamma(t_0) \cdot \gamma'(t_0) = \gamma(t_1) \cdot \gamma'(t_1) = 0$ This shows that the elastic energy of γ on the interval $[t_0, t_1]$ is at least $\frac{t_1 - t_0}{\epsilon^2}$. On this interval, μ is on the great arc. It follows that the energy of μ along the interval $[t_0, t_1]$ is $\frac{t_1 - t_0}{\epsilon^2}$. This completes our proof that energy of μ is, at most, that of γ. □

Even though we know that the midsection of the curve is a great circle, we do not know what happens at the ends of the curve. This becomes more apparent from our proof of Theorem 1. The boundary conditions suggest that if the ends of the curve moves in a radial direction, the elastic energy may be lower than that of the great circle.

3 Isometric Approximation

In this section, we explore the idea of approximating curves while controlling potential energy. We borrow the idea of elastic energy from Section 2. In addition, we add an "approximation energy" term, measuring how well our target curve is approximated. By shrinking a parameter ϵ, we will emphasize the importance of the "approximation energy" term in the total energy to force the approximation to become arbitrarily close.

Definition 2. *Given two curves $\gamma, \alpha : [0, L] \to \mathbb{R}^3$, $\gamma \in H^2$ and arc-length parameterized, we define*

$$E_{\alpha,\epsilon}(\gamma) = \epsilon \int_0^L ||\gamma''(t)||^2 dt + \frac{1}{\epsilon}\int_0^L ||\gamma(t) - \alpha(t)||^2 dt$$

Here α is our target curve and γ is the unit speed approximation. The first term is the elastic energy and the second term denotes the approximation energy. Note that we may shrink the parameter ϵ to place greater emphasis on the closeness of the approximation and less emphasis on maintaining low curvature.

Notation. *Given a target curve $\alpha : [0, L] \to \mathbb{R}^3$, we denote the minimum energy of all approximations of α by*

$$E_\alpha^*(\epsilon) = \min\{E_{\alpha,\epsilon}(\gamma) : \gamma \in H^2, ||\gamma'(t)|| = 1 \forall t\}$$

Furthermore, a unit speed curve γ_ϵ is said to be an optimal curve provided that

$$E_{\alpha,\epsilon}(\gamma_\epsilon) = E_\alpha^*(\epsilon)$$

In this section, we study the behavior of $E_\alpha^*(\epsilon)$ and optimal curves as $\epsilon \to 0$.

We are particularly interested in optimal curves that approximate our target curve α "well" in some sense. To formalize this idea, we give the following definitions.

Definition 3. *We will say that the sequence of optimal curves γ_ϵ approximates the target curve α well provided that* $\lim_{\epsilon \to 0} ||\gamma_\epsilon - \alpha||_{L^2} = 0$.

Definition 4. *Let $\alpha : [0, L] \to \mathbb{R}^3$ be a target curve. We say that the target curve α is short provided that $||\alpha'(t)|| \leq 1$ for all $t \in [0, L]$.*

Throughout the rest of the paper, we will focus on short target curves. As we will see in Theorem 2, energy optimal approximations of a target curve α well approximate α if and only if α is short.

Theorem 2. *Fix $\alpha \in C^2([0, L] \to \mathbb{R}^n)$. For $\epsilon > 0$ let γ_ϵ minimize $E_{\epsilon,\alpha}(\gamma)$. Then $\gamma_\epsilon \to \alpha$ in the L^2 sense $\iff ||\alpha'(t)|| \leq 1$ $\forall t \in [0, L]$.*

Proof. $\Longrightarrow$: Suppose $\alpha \in C^1$ and that there is t_0 such that $||\alpha'(t_0)|| > 1$. Choose $r_1 > 0$ such that $||\alpha'|| > 1$ on $B_{r_1}(t_0) \subset [0, L]$. By applying the Mean Value Theorem to each component, we can choose $r_2 \in (0, r_1)$ sufficiently small so that $||\alpha(t_0 + r_2) - \alpha(t_0 - r_2)|| > 2r_2 + 2\delta$ for some fixed $\delta > 0$. Note that $||\gamma(t_0 + r_2) - \gamma(t_0 - r_2)|| \leq 2r_2$. We see that

$$\begin{aligned} 2r_2 + 2\delta &\leq ||\alpha(t_0 + r_2) - \alpha(t_0 - r_2)|| \\ &\leq |\alpha(t_0 + r_2) - \gamma(t_0 + r_2)|| + ||\alpha(t_0 - r_2) - \gamma(t_0 - r)2)|| + ||\gamma(t_0 + r_2) - \gamma(t_0 - r_2)|| \\ &\implies 2\delta \leq ||\alpha(t_0 + r_2) - \gamma(t_0 + r_2)|| + ||\alpha(t_0 - r_2) - \gamma(t_0 - r_2)|| \end{aligned}$$

Assume without loss of generality that $||\alpha(t_0 + r_2) - \gamma(t_0 + r_2)|| \geq \delta$. Define $C = max||\alpha'|| + 1$. Observe that

$$\left|\frac{d}{dt}||\alpha - \gamma||\right| = \left|\frac{-1}{|\alpha - \gamma|}(\alpha - \gamma) \cdot (\alpha' - \gamma')\right| \leq ||\alpha' - \gamma'|| \leq C$$

We will argue that $||\alpha(t_0 + r_2 + s) - \gamma(t_0 + r_2 + s)|| \geq \delta - Cs$ for $t_0 + s \in B_{r_1}(t_0)$. Suppose not. Then by the Mean Value Theorem, there is some z between $t_0 + r_2$ and $t_0 + r_2 + s$ such that $s\frac{d}{dt}|_{t=z}||\alpha - \gamma|| = ||\alpha(t_0 + r_2 + s) - \gamma(t_0 + r_2 + s)|| - ||\alpha(t_0 + r_2) - \gamma(t_0 + r_2)||$ Then $\frac{d}{dt}|_{t=z}||\alpha - \gamma|| > C$. This is a contradiction. Thus

$$||\alpha - \gamma||_{L^2} \geq \frac{1}{L}\int_{t_0 - r_1}^{t_0 + r_1} max(\delta - C(s - t_0 - r_2), 0) > 0$$

.

Note that C, δ, r_1, r_2, t_0 only depended on α.

$\Longleftarrow$: For the other direction, suppose that $||\alpha'(t)|| < 1$. Here, we construct γ_n using the convex integration method [1]. Let $h : [0, L] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a function satisfying

1. $||h(x, y)|| = 1$
2. $\alpha'(t) = \int_0^L h(t, u)du$ for all $t \in [0, L]$

Define

$$\gamma_n(t) \equiv \alpha(0) + \int_0^t h(u, nu)du$$

By [1], there is a constant $H > 0$ such that $||\gamma_n - \alpha||_{C^0} \leq \frac{1}{n}H$. Note that $\gamma_n'(t) = h(t, nt)$. Thus $\gamma_n''(t) = \frac{\partial h}{\partial x}|_{(t,nt)} + n\frac{\partial h}{\partial y}|_{(t,nt)}$. So for sufficiently large n, there is a constant $A > 0$ such that

$$||\gamma_n''(t)|| \leq nA$$

Thus for sufficiently large n,

$$E_{\alpha,\epsilon}(\gamma_n) \leq \epsilon L n^2 A^2 + \frac{LH^2}{\epsilon n^2}$$

Plugging in $n = \sqrt{\frac{H}{\epsilon A}}$, we get that

$$E_{\alpha,\epsilon}(\gamma) \leq 2\frac{LH}{A}$$

Now suppose that $||\alpha'(t)|| \leq 1$ for all t and that there is a point t_0 such that $||\alpha'(t_0)|| = 1$. Fix $l > 0$. We will show that $||\gamma - \alpha||_{L^2} < l$ for sufficiently small ϵ. Define $\delta = \min\{\frac{l}{2||\alpha||_{L^2}}, \frac{1}{2}\}$. We now define $\beta : [0, L] \to \mathbb{R}^n$ by $\beta(t) = (1 - \delta)\alpha(t)$. Note that $|\beta'(t)| \leq (1 - \delta) < 1$ for all t. Thus we know that there is some $C > 0$ such that for all ϵ sufficiently small, there is a unit speed curve γ_ϵ such that $E_{L,\beta}(\gamma_\epsilon) \leq C$. This then gives

$$||\gamma_\epsilon - \alpha||_{L^2} \leq ||\gamma_\epsilon - \beta||_{L^2} + ||\alpha - \beta||_{L^2}$$

Thus $E_{L,\alpha}(\gamma_\epsilon) \leq C + \frac{1}{\epsilon}||\alpha - \beta||_{L^2}$. But by our construction of β, we have that $||\alpha - \beta||_{L^2} = \frac{l}{2}$. So for ϵ sufficiently small, $||\gamma - \alpha||_{L_2} \leq C\epsilon + \frac{l}{2} < l$. □

Lemma 1. *Suppose γ_ϵ is a sequence of unit speed minimizers, where α is a short curve. Then $\gamma_\epsilon(0) \to \alpha(0)$ as $\epsilon \to 0$.*

Proof. Note that

$$\frac{d}{dt}||\gamma_\epsilon(t) - \alpha(t)||^2 = 2(\gamma_\epsilon - \alpha)'(t) \cdot (\gamma - \alpha)(t) \leq C||\gamma(t) - \alpha(t)||$$

where $C = 2 + 2\sup_{t\in[0,L]} ||\alpha'(t)||$. If the claim does not hold, then there is $\frac{CL}{6} > \mu > 0$ and $\epsilon_n < \frac{1}{n}$ for each $n \in \mathbb{N}$ such that $||\gamma_{\epsilon_n}(0) - \alpha(0)|| > \mu$. Consider the equation $f(t) = \left(\frac{2\mu - Ct}{2}\right)^2$. Note that $f(0) = \mu^2$ and $f'(t) = -C\frac{2\mu - Ct}{2} = -C\sqrt{f(t)}$. Thus, for all n and all t, $||\gamma_{\epsilon_n}(t) - \alpha(t)||^2 \geq f(t)$ so that

$$\int_0^L ||\gamma_{\epsilon_n}(t) - \alpha(t)||^2 dt \geq \int_0^L f(t)dt = \int_0^L \left(\frac{2\mu - Ct}{2}\right)^2 dt = \mu^2 L - \frac{1}{2}\mu C L^2 + \frac{C^2}{12}L^3$$

We see that this quantity is positive because if it is 0, then

$$\frac{C^2}{12}L^3 < \mu^2 L + \frac{C^2}{12}L^3 = \frac{1}{2}\mu C L^2 \leq \frac{C^2}{12}L^3$$

which is absurd. It follows then that $||\gamma_\epsilon - \alpha||_{L^2} \not\to 0$ as $\epsilon \to 0$, a contradiction.

□

Lemma 2. *Suppose (γ_ϵ) is a sequence of minimizers. Then*

$$\int_0^L \gamma_\epsilon' \cdot \alpha' \to \int_0^L |\alpha'|^2$$

Proof. Suppose $\beta \in C_C^\infty((0,L))$. Then

$$\begin{aligned}
\left|\int_0^L \beta \cdot (\gamma_\epsilon - \alpha)'\right| &= \left|-\int_0^L \beta' \cdot (\gamma_\epsilon - \alpha)\right| \\
&\leq \int_0^L |\beta' \cdot (\gamma_\epsilon - \alpha)| \\
&\leq \int_0^L ||\beta'||||\gamma_\epsilon - \alpha|| \\
&\leq \left(\int_0^L ||\beta'||^2\right)^{\frac{1}{2}} \left(\int_0^L ||\gamma_\epsilon - \alpha||^2\right)^{\frac{1}{2}}
\end{aligned}$$

β is smooth and compactly supported, so the left integral is bounded by some positive constant. The right integral tends to 0 as $\epsilon \to 0$ by the proof that $\limsup_{\epsilon\to 0} E_\alpha^*(\epsilon) \leq 2\int_0^L 1 - ||\alpha'||^2$. It follows that

$$\int_0^L \beta \cdot \gamma_\epsilon' \to \int_0^L \beta \cdot \alpha'$$

Now, let $\delta > 0$ and let $\beta \in C_C^\infty((0,L))$ be a function such that $\beta(t) = \alpha'(t)$ on $[\delta^2, L - \delta^2]$ and $\beta(t) = 0$ for $t = 0$ and $t = L$. Choose ϵ sufficiently small that

$$\left|\int_0^L (\gamma_\epsilon' - \alpha') \cdot \beta\right| < \delta$$

β and α are both smooth, so set

$$M = \max_{t\in[0,L]} |\alpha' - \beta| < \infty$$

Observe then that

$$\begin{aligned}
\left|\int_0^L (\gamma_\epsilon' - \alpha') \cdot \alpha'\right| &\leq \left|\int_0^L (\gamma_\epsilon' - \alpha') \cdot \beta\right| + \int_0^L |(\gamma_\epsilon' - \alpha') \cdot (\alpha' - \beta)| \\
&\leq \delta + \left(\int_0^L |\gamma_\epsilon' - \alpha'|^2\right)^{\frac{1}{2}} \left(\int_0^L |\alpha' - \beta|^2\right)^{\frac{1}{2}} \\
&\leq \delta + \left(2\sqrt{L}\right)\left(\sqrt{2}\delta M\right)
\end{aligned}$$

Take $\delta \to 0$. □

Theorem 3. *Suppose $\alpha : [0,L] \to \mathbb{R}^3$ is a smooth map and $||\alpha'(t)|| \in (0,1)$ for all $t \in [0,L]$. Then*

$$\lim_{\epsilon\to 0} E_\alpha * (\epsilon) = 2\int_0^L 1 - ||\alpha'(t)||^2$$

.

Proof. Choose $R : [0,L] \to O(3)$ to be smooth such that $\alpha'(t) = ||\alpha'(t)||R(t)\vec{e}_1$. We define

$$D = \max_{i,j} \max_{t\in[0,L]} \frac{d^2}{dt^2}\sqrt{1 - ||\alpha'(t)||^2} R_{ij}(t)$$

For $\epsilon > 0$, we define $w_\epsilon : [0,L] \to \mathbb{R}^3$ by

$$w_\epsilon(t) = (||\alpha'(t)||, \sqrt{1 - ||\alpha'(t)||^2}\cos(\tfrac{t}{\sqrt{\epsilon}}), \sqrt{1 - ||\alpha'(t)||^2}\sin(\tfrac{t}{\sqrt{\epsilon}}))$$

Now define $\gamma_\epsilon : [0,L] \to \mathbb{R}^3$ by

$$\gamma_\epsilon(t) = -\sqrt{1 - ||\alpha'(t)||^2} R(0)\vec{e}_3 + \int_0^t R(s)w_\epsilon(s)$$

We will argue that for any $\epsilon \in (0, \epsilon_0)$, $E(\gamma_\epsilon) < \delta + 2\int_0^L 1 - ||\alpha'(t)||^2$
Note that

$$\gamma_\epsilon'(t) = R(t)w_\epsilon(t)$$
$$\gamma_\epsilon''(t) = R'(t)w_\epsilon(t) + R(t)w_\epsilon'(t)$$

We have

$$\begin{aligned}
w_\epsilon'(t) = \Bigg(&\frac{1}{2}\frac{\alpha'(t)\cdot\alpha''(t)}{||\alpha'(t)||}, \frac{\alpha'\cdot\alpha''}{\sqrt{1 - ||\alpha'(t)||^2}}\cos(\frac{t}{\sqrt{\epsilon}}) \\
&- \sqrt{\frac{1 - ||\alpha'(t)||^2}{\epsilon}}\sin(\frac{t}{\sqrt{\epsilon}}), \frac{\alpha'\cdot\alpha''}{\sqrt{1 - ||\alpha'(t)||^2}}\sin(\frac{t}{\sqrt{\epsilon}}) + -\sqrt{\frac{1 + ||\alpha'(t)||^2}{\epsilon}}\cos(\frac{t}{\sqrt{\epsilon}})\Bigg)
\end{aligned}$$

$$||w_\epsilon(t)||^2 = \frac{1}{4}\frac{(\alpha'(t)*\alpha''(t))^2}{||\alpha'(t)||^2} + \frac{(\alpha'\cdot\alpha'')^2}{1-||\alpha'||^2} + \frac{1}{\epsilon}(1-||\alpha'||^2)$$

Note the following bound on the curvature energy:

$$\epsilon\int_0^L ||\gamma''||^2 \le \epsilon\int_0^L (||R'(t)|| + ||w_\epsilon'(t)||)^2 \qquad = \epsilon\int_0^L ||R'(t)||^2 + \epsilon\int_0^L ||R'(t)||*||w_\epsilon|| + \epsilon\int_0^L ||w_\epsilon||^2$$

Using the inequality $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$, we obtain

$$\begin{aligned}\epsilon\int_0^L ||\gamma''||^2 \le \epsilon\int_0^L ||R'(t)||^2 + \epsilon\int_0^L ||R'(t)|| * \sqrt{\frac{1}{4}\frac{(\alpha'(t)*\alpha''(t))^2}{||\alpha'(t)||^2} + \frac{(\alpha'*\alpha'')^2}{1-||\alpha'||^2}} \\ + \sqrt{\epsilon}\int_0^L ||R'(t)||\sqrt{1-||\alpha'||^2} \\ + \epsilon\int_0^L \frac{1}{4}\frac{(\alpha'(t)*\alpha''(t))^2}{||\alpha'(t)||^2} + \frac{(\alpha'*\alpha'')^2}{1-||\alpha'||^2} + \int_0^L (1-||\alpha'||^2)\end{aligned}$$

Since $\epsilon<\epsilon_0$, we see that $\epsilon\int_0^L ||\gamma''||^2 < \frac{4}{41}\delta + \int_0^L (1-||\alpha'(t)||^2)$. Now we argue that the approximation term is close to $\int_0^L (1-||\alpha'(t)||^2)$. Choose $n\in\mathbb{N}\cup\{0\}$ such that $2\pi\sqrt{\epsilon}n\in[0,L]$. Then note that by applying the Mean Value Theorem twice, we obtain

$$||R_{ij}(s+h)-R_{ij}(s)|| \le D|h|^2$$

For ease of notation, we denote $f_i:[0,L]\to\mathbb{R}^3$ as the i^{th} column of $\sqrt{1-||\alpha'||^2}R(t)$.

$$\begin{aligned}\left|\left|\int_0^{2\pi\sqrt{\epsilon}n} f_2(s)\cos\left(\frac{s}{\sqrt{\epsilon}}\right)\right|\right| &= \left|\left|\sum_{i=0}^{n-1}\int_0^{2\pi\sqrt{\epsilon}} f_2(h+2\pi\sqrt{\epsilon}i)\cos\left(\frac{h}{\sqrt{\epsilon}}\right)\right|\right| \\ &\le \sum_0^{n-1}\left|\left|\int_0^{2\pi\sqrt{\epsilon}} f_2(h+2\pi\sqrt{\epsilon}i)\cos\left(\frac{h}{\sqrt{\epsilon}}\right)\right|\right|\end{aligned}$$

By linear approximation, we see that

$$\left|\left|\int_0^{2\pi\sqrt{\epsilon}} f_2(h+2\pi\sqrt{\epsilon}i)\cos\left(\frac{h}{\sqrt{\epsilon}}\right) - \int_0^{2\pi\sqrt{\epsilon}} (f_2(2\pi\sqrt{\epsilon}i)+f_2'(2\pi\sqrt{\epsilon}i)h)\cos\left(\frac{h}{\sqrt{\epsilon}}\right)\right|\right| \le D(2\pi\sqrt{\epsilon})^3$$

Note that

$$\int_0^{2\pi\sqrt{\epsilon}} (f_2(2\pi\sqrt{\epsilon}i)+f_2'(2\pi\sqrt{\epsilon}i)h)\cos\left(\frac{h}{\sqrt{\epsilon}}\right) = 0$$

since cos is even around π.

Thus

$$\left|\left|\int_0^{2\pi\sqrt{\epsilon}n} f_2(s)\cos\left(\frac{s}{\sqrt{\epsilon}}\right)\right|\right| \le 4\pi^2 LD\epsilon$$

Once again applying Mean Value Theorem, we see that

$$\left|\left|\int_{2\pi\sqrt{\epsilon}n}^t f_2(s)cos\left(\frac{s}{\sqrt{\epsilon}}\right) - f_2(t)\int_{2\pi\sqrt{\epsilon}n}^t \cos\left(\frac{s}{\sqrt{\epsilon}}\right)\right|\right| \le 4\epsilon\pi^2 D$$

$$\implies \left|\left|\int_{2\pi\sqrt{\epsilon}n}^t f_2(s)cos\left(\frac{s}{\sqrt{\epsilon}}\right) - \sqrt{\epsilon}\sin\left(\frac{t}{\sqrt{\epsilon}}\right)\right|\right| \le 4\epsilon\pi^2 D$$

Similarly, since sin is even around $\frac{\pi}{2}$, we have that

$$\left|\left|\int_{\frac{1}{2}\pi\sqrt{\epsilon}}^{(2n+\frac{1}{2})\pi\sqrt{\epsilon}} f_3(s)\sin\left(\frac{s}{\sqrt{\epsilon}}\right)\right|\right| \le 4\pi^2 LD\epsilon$$

We also observe

$$\left|\left|\int_0^{\frac{1}{2}\pi\sqrt{\epsilon}} f_3(s)\sin(\frac{s}{\sqrt{\epsilon}}) - f_3(0)\int_0^{\frac{1}{2}\pi\sqrt{\epsilon}} \sin(\frac{s}{\sqrt{\epsilon}})\right|\right| \le \frac{D}{4}\pi^2\epsilon$$

We have

$$f_3(0)\int_0^{\frac{1}{2}\pi\sqrt{\epsilon}} sin\left(\frac{s}{\sqrt{\epsilon}}\right) = f_3(0) = \sqrt{1-||\alpha'(0)||^2}R(0)\vec{e}_3$$

Choose n such that $|(2n+\frac{1}{2})\pi\sqrt{\epsilon}-t|\le 2\pi\sqrt{\epsilon}$. By Mean Value Theorem,

$$\left|\left|\int_{(2n+\frac{1}{2})\pi\sqrt{\epsilon}}^t f_3(s)\sin\left(\frac{s}{\sqrt{\epsilon}}\right) - f_3(t)\int_{(2n+\frac{1}{2})\pi\sqrt{\epsilon}}^t \sin\left(\frac{s}{\sqrt{\epsilon}}\right)\right|\right| \le 4\epsilon\pi^2 D$$

$$\implies \left|\left|\int_{(2n+\frac{1}{2})\pi\sqrt{\epsilon}}^{t} f_2(s)\sin\left(\frac{s}{\sqrt{\epsilon}}\right) - \sqrt{\epsilon}\cos\left(\frac{t}{\sqrt{\epsilon}}\right)\right|\right| \leq 4\epsilon\pi^2 D$$

Note that

$$\begin{aligned}\gamma_\epsilon(t) - \alpha(t) &= -\sqrt{1-||\alpha'(0)||^2}R(0)\vec{e}_3 + \int_0^t R(s)(w_\epsilon(s) - \alpha(s))ds \\ &= \int_0^t f_2(s)\cos\left(\frac{s}{\sqrt{\epsilon}}\right) + \int_0^t f_3(s)\sin\left(\frac{s}{\sqrt{\epsilon}}\right) - \sqrt{1-||\alpha'(0)||^2}R(0)\vec{e}_3\end{aligned}$$

Thus

$$\left|\left|\gamma_\epsilon(t) - \alpha(t) - R(t)\left(0, \sqrt{\epsilon}\sin\left(\frac{t}{\sqrt{\epsilon}}\right), \sqrt{\epsilon}\cos\left(\frac{t}{\sqrt{\epsilon}}\right)\right)\right|\right| \leq (12+8L)\pi^2 D\epsilon$$

It follows that the approximation energy is bounded above:

$$\frac{1}{\epsilon}\int_0^L ||\gamma_\epsilon - \alpha|| \leq \int_0^L 1 - ||\alpha'||^2 + O(\sqrt{\epsilon})$$

Combining these bounds on the curvature energy and approximation energy, we see that

$$\limsup_{\epsilon\to 0} E_\alpha^*(\epsilon) \leq 2\int_0^L 1 - ||\alpha'||^2$$

We now argue that $\liminf_{\epsilon\to 0} E_\alpha^*(\epsilon) \geq 2\int_0^L 1 - ||\alpha'||^2$. Observe that for a sequence of optimizing curves γ_ϵ, we have

$$\begin{aligned}E_\alpha^*(\epsilon) &= \epsilon\int_0^L ||\gamma_\epsilon''||^2 + \frac{1}{\epsilon}\int_0^L ||\gamma_\epsilon - \alpha||^2 \\ &\geq 2\int_0^L ||\gamma_\epsilon''|| ||\gamma_\epsilon - \alpha|| \\ &\geq \left|2\int_0^L \gamma_\epsilon'' \cdot (\gamma_\epsilon - \alpha)\right| \\ &= \left|2\left(\gamma_\epsilon' \cdot (\gamma_\epsilon - \alpha)|_0^L - \int_0^L 1 - \gamma_\epsilon' \cdot \alpha'\right)\right| \\ &\geq \left|2\int_0^L 1 - \gamma_\epsilon' \cdot \alpha'\right| - 2(||\gamma_\epsilon(L) - \alpha(L)|| + ||\gamma_\epsilon(0) - \alpha(0)||)\end{aligned}$$

By Lemma 1,

$$||\gamma_\epsilon(0) - \alpha(0)|| \to 0$$

as $\epsilon \to 0$, and likewise for the other endpoint. By an application of Lemma 2, we see that

$$\liminf_{\epsilon\to 0} E_\alpha^*(\epsilon) \geq 2\int_0^L 1 - ||\alpha'||^2$$

□

4 Future Works

From our construction, optimal curves seem to be helical in nature and can be constructed by convex integration. This is further supported by the fact that

$$\lim_{\epsilon\to 0} ||\sqrt{\epsilon}(\gamma_\epsilon - \alpha)'' - \frac{1}{\sqrt{\epsilon}}(\gamma_\epsilon - \alpha)||_{L^2} = 0$$

for energy optimizing curves γ_ϵ, which follows from the asymptotic behavior of $E_\alpha^*(\epsilon)$. By applying Lagrange Multipliers for Banach spaces, one can conclude that for a given target curve α and energy minimizer γ_ϵ, there exists $\lambda \in \mathbb{R}$ such that

$$\epsilon^2\gamma_\epsilon^{(4)} + \lambda\gamma_\epsilon'' + (\gamma_\epsilon - \alpha) = 0$$

We believe that by applying this differential equation to the Fourier Series expansion of γ_ϵ will demonstrate that optimizing curves are approximately helical in the limit.

References

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