GALOIS REPRESENTATIONS VALUED IN REDUCTIVE GROUPS AND THEIR CENTRALIZERS

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Abstract. For $\widehat{G}$ a classical reductive group over $\mathbf{C}$, we describe representations of $\mathrm{Gal}(\bar{\mathbf{Q}}_p, \mathbf{Q}_p)$ modulo its wild inertia such that the image of this quotient in $\widehat{G}$ has finite centralizer. For each such centralizer, we also describe its representations.

1. Introduction

1.1. **History.** This project is motivated by a refined version of the local Langlands conjecture. Roughly speaking, the local Langlands conjecture provides a correspondence between representations of a p-adic group and certain maps from the Weil group into the dual group. These maps are called "Langlands parameters" and are conjectured to partition the p-adic group representations into finite sets called "L-packets". We explain this conjecture in its basic form below and show how it relates to our problem. For a more complete exposition of the Langlands program and the further technical properties stated in the conjecture, we refer the reader for example to [Gel84],[Bor79], and [DR09].

To state the basic form of the conjecture, we need to set up some terminology. Let $W_{\mathbf{Q}_p}$ denote the *Weil group* for primes $p < \infty$. It is a dense subgroup of $\mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ generated by inertia subgroup I and Frobenius element $\mathrm{Frob} \in \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$. Let $W_{\mathbf{Q}/\mathbf{R}}$ denote the *Weil group* for $p = \infty$. It is defined as the unique nonsplit extension of short exact sequence

$$1 \to \mathbf{C}^\times \to W_{\mathbf{Q}/\mathbf{R}} \to \mathrm{Gal}(\mathbf{C}/\mathbf{R}) \to 1.$$

Now, define the Weil-Deligne group

$$L_p := \begin{cases} W_{\mathbf{C}/\mathbf{R}} & \text{if } p = \infty \\ W_{\mathbf{Q}_p} \times SL_2(\mathbf{C}) & \text{if } p < \infty. \end{cases}$$

Then the simplified conjecture states that there is a finite-to-1 surjection

$$\left\{ L_p \xrightarrow{\Phi} \hat{G} \right\} \longleftarrow \mathrm{Irr}(G(\mathbf{Q}_p))$$

where on the left we require ϕ to be a "tempered, admissible Langlands parameter" [Bor79]. The finite fibers of this map are called *L-packets*. The centralizer $S_\Phi = \mathrm{Cent}(\mathrm{Im}(\phi), \hat{G})$ and its representations roughly parameterize these L-packets, thus giving rise to a "refined" local Langlands correspondence [Kal15]. Furthermore, we are interested in the trace and dimension of these representations due to the conjectural Langlands-Shelstad transfer factors [LS87]. Numerous efforts have been made in studying and constructing these L-packets, see for example [Kal16],[DR09], and [Yu01]. In particular, Kaletha reduced a refined version of the local conjecture to the case of "depth-zero Langlands parameters", i.e when Φ is trivial on $SL_2(\mathbf{C})$ and wild inertia. Our project is to carefully study the depth-zero supercuspidal Langlands parameters.

Date: September 12, 2018.

1.2. **Set-up of the problem.** In what follows, let $\hat{G}$ denote a connected, reductive group with coefficients in $\mathbf{C}$. Let $\hat{G}_{\text{der}}$ denote its derived subgroup, and let $\hat{G}_{\text{sc}}, \hat{G}_{\text{ad}}$ denote the simply connected cover and adjoint quotient of $\hat{G}_{\text{der}}$, respectively. It is a fact that $\hat{G}_{\text{ad}}$ is in 1-1 correspondence with irreducible root systems, there are the classical types A_n, B_n, C_n, D_n, and the exceptional types E_6, E_7, E_8, F_4, G_2. In this paper, we focus on the **classical types**.

Let $\mathbf{F}_p$ be the finite field of p elements and let $\Gamma_p := \text{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$. Let ϕ denote the Frobenius element $\phi : x \to x^p$, which is a (topological) generator of $\text{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p) = \hat{Z}$, the profinite completion of the integers. Define the inertia subgroup $I = \ker\left(\Gamma_p \to \text{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)\right)$. Then the *wild inertia* $I^+ \lhd I$ is the maximal pro-p subgroup, and the *tame inertia group* is I/I^+. The tame inertia is (noncanonically) isomorphic to the direct product over all primes $q \neq p$ of the rings of q-adic integers [Ser71, Section 4], hence is denoted by $\mathbf{Z}^{(p)}$. We obtain a split short exact sequence

$$1 \to \mathbf{Z}^{(p)} \to I^+\backslash\Gamma_p \to \hat{Z} \to 1.$$

Hence $I^+\backslash\Gamma_p \cong \langle s\rangle \rtimes \langle\phi\rangle$ for some generator $s \in \mathbf{Z}^{(p)}$ with relation $\phi s\phi^{-1} = s^q$, where q is some power of p. We will pursue two main goals in this paper.

(1) List all representations $\Phi : I^+\backslash\Gamma_p \to \hat{G}$ such that $S_\Phi := \text{Cent}(Im(\Phi), \hat{G})$ is a finite group,
(2) For each such Φ, list all irreducible representations $\rho : S_\Phi \to GL_n(V)$ and their character values.

Observe, the representation Φ defined here is related to the depth-zero Langlands parameter $\Phi : L_p \to \hat{G}$ defined in Subsection 1.1 above.

1.3. **Main Results.** To achieve goal (1), it is enough to specify the image of (s, ϕ) so that

- $s \in \hat{G}$ is "regular", "semi-simple", and of finite order coprime to p
- $\phi \in \hat{G}$ normalizes the torus $\widehat{T}$, $\widehat{T}^\phi$ is finite, and
- $\phi s\phi^{-1} = s^q$.

Let w_0 denote the projection of $\phi \in N(\widehat{T})$ to $w_0 \in W$, the Weyl group. We now describe some elements of the Weyl group acting on the cocharacter lattice $X_*(\widehat{T})$. This will help present a "normal form" of w_0, i.e a nice presentation of w_0 in which w_0 is in general conjugate to. For a classical root system of rank n, define elements of W with action on $X_*(\widehat{T})$ by $e_i : \varepsilon_i \to -\varepsilon_i$; and $e'_i : \varepsilon_i \to -\varepsilon_{n+1-i}, \varepsilon_{n+1-i} \to \varepsilon_i$; and "little mirror" $m : \varepsilon^*_i \to \varepsilon^*_{n+1-i}$.

For type B_n and C_n, define $w'_0 \in W$ by $w'_0 = \overline{(1, \ldots, i_1)} \ldots \overline{(i_{m-1}+1, \ldots, i_m)}, i_0 := 0 < i_1 < i_2 \cdots < i_m$. This notation means $w'_0 : \varepsilon^*_j \to \varepsilon^*_{j+1}$for $j \notin \{i_1, \ldots, i_m\}$ and $w'_0 : \varepsilon^*_{i_j} \to -\varepsilon^*_{i_{j-1}+1}$. For D_n, due to simplifications of computation later on, define a slight variant of w'_0. Namely, let $v'_0 = \overline{(i_1, \ldots i_2 - 1)} \ldots \overline{(i_m, \ldots, i_{m+1} - 1)}$.

Let $s = (s_1, \ldots, s_n)$ be a representative of an element in $\widehat{T} \cong (\mathbf{C} \otimes_{\mathbf{Z}} X_*(\widehat{T}))/X_*(\widehat{T})$ in the ε^*_i coordinates. Table (1.3.1) presents the pairs of (s, ϕ) which satisfy the first two conditions, where ϕ is some lift of a w_0 in normal form. Note that in the table we only specified the action of ϕ on the first 'half' of coordinates of s in types B_n, C_n, D_n; the action on the second half is just a "big mirror" of the action. Also, if we further require $\phi s\phi^{-1} = s^q$, we direct the reader to item (3) of Section 4, to see what additional restrictions are placed.

To achieve goal (2), we use the short exact sequence

$$1 \to \widehat{T}^\phi_{\text{ad}} \to \text{Cent}(s, \phi; \widehat{G}_{\text{ad}}) \to W^\phi_s \to 1 \qquad (1.3.1)$$

and Clifford theory to classify representations of $\text{Cent}(s, \phi, \widehat{G}_{\text{ad}})$. We say the extension $\text{Cent}(s, \phi, \widehat{G}_{\text{ad}})$ in (1.3.1) is split if the short exact sequence has a splitting. We say it is virtually split if its irreducible representations arise in the same way as if it were split. Section 3.4 provides a precise characterization of an extension being virtually split.

TABLE 1.3.1. Results for Part I

Type	$s \in \widehat{T}_{\text{ad}}$	$w_0 \in W$
A_n	$(e^{2\pi i t_{n+1}}, \dots, e^{2\pi i t_1})$	$w_0 = (1, 2, \dots, n+1)^i$
	$d/(n+1) > t_d > \cdots > t_1 = 0,\ t_{d+i} = d/(n+1) + t_i$	for some i.
B_n	$(e^{2\pi i t_1}, \cdots, e^{2\pi i t_n}, 1, e^{-2\pi i t_n}, \cdots, e^{-2\pi i t_1})$	$w_0 = w_0'$
	$t_1 = 1/2 > t_2 > \cdots t_n > 0$	$i_1 = 1, i_m = 2n$
C_n	$(e^{2\pi i t_1}, \dots, e^{2\pi i t_n}, e^{-2\pi i t_n}, \dots, e^{-2\pi i t_1})$	$w_0 = e w_0' m w_0' m^{-1}$
	$t_i + t_{n+1-i} = 1/2$ and $t_i > t_{i+1}$	$i_m = \lfloor n/2 \rfloor$. $e = \text{id}$ if n even
		$e : \varepsilon_{(n+1)/2} \to -\varepsilon_{(n+1)/2}$ if n odd
D_{2n}	$(e^{2\pi i t_1}, \cdots, e^{2\pi i t_{2n}}, e^{-2\pi i t_{2n}}, \cdots, e^{-2\pi i t_1})$	$w_0 = v_0' m v_0' m^{-1}$
	$t_i + t_{2n-i+1} = 1/2,\ t_1 = 1/2 > t_2 > \cdots > t_{2n-1} > 0$	$i_1 = 1, i_2 = 2, i_m < i_{m+1} := n+1$
D_{2n+1}	$(e^{2\pi i t_1}, \cdots, e^{2\pi i t_{2n}}, 1, e^{-2\pi i t_{2n}}, \cdots, e^{-2\pi i t_1})$	$w_0 = e_1' e_{n+1} v_0' m v_0'^{-1} m^{-1}$
	$t_i + t_{2n-i+2} = 1/2,\ t_1 = 1/2 > t_2 > \cdots > t_{2n} > 0$	$i_1 = 2, i_m < i_{m+1} := n+1$

We will also be interested in the representation theory for T_+, C_+, which are the preimages of $\widehat{T}_{\text{ad}}, \text{Cent}(s, \phi, \widehat{G}_{\text{ad}})$, respectively, into the simply-connected cover. Table (1.3.2) fully records the representation theory for both $\text{Cent}(s, \phi, \widehat{G}_{\text{ad}})$ and C_+.

TABLE 1.3.2. Results for Part II

Type	T_{ad}^{ϕ}	T_+	W_s	$\text{Cent}(s, \phi; \widehat{G})$	C_+
A_{2n}	$\mathbf{Z}/(2n+1)\mathbf{Z}$	$(\mathbf{Z}/(2n+1)\mathbf{Z})^2$	$\mathbf{Z}/(2n+1)\mathbf{Z}$	split	split
A_{2n+1}	$\mathbf{Z}/(2n+2)\mathbf{Z}$	$(\mathbf{Z}/(n+1)\mathbf{Z}) \times \mathbf{Z}/(4n+4)\mathbf{Z}$	$\mathbf{Z}/(2n+2)\mathbf{Z}$	split	virtually-split
B_n	$(\mathbf{Z}/2\mathbf{Z})^m$	$(\mathbf{Z}/4\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})^{m-1}$	$\mathbf{Z}/2\mathbf{Z}$	split	virtually-split
C_n	$(\mathbf{Z}/2\mathbf{Z})^m$ or	$(\mathbf{Z}/2\mathbf{Z})^{m+1}$ or	$\mathbf{Z}/2\mathbf{Z}$	split	virtually-split
	$(\mathbf{Z}/2\mathbf{Z})^{m-2} \times \mathbf{Z}/4\mathbf{Z}$	$(\mathbf{Z}/2\mathbf{Z})^{m-1} \times \mathbf{Z}/4\mathbf{Z}$			
D_{2n}	$(\mathbf{Z}/2\mathbf{Z})^m$ or	$(\mathbf{Z}/2\mathbf{Z})^{m-2} \times \mathbf{Z}/4\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	split	virtually-split
	$(\mathbf{Z}/2\mathbf{Z})^{m-2} \times \mathbf{Z}/4\mathbf{Z}$				
D_{2n+1}	$(\mathbf{Z}/2\mathbf{Z})^{m-2} \times \mathbf{Z}/4\mathbf{Z}$	$(\mathbf{Z}/2\mathbf{Z})^{m-1} \times \mathbf{Z}/8\mathbf{Z}$	$\mathbf{Z}/4\mathbf{Z}$	virtually split	virtually split

1.4. **Acknowledgements.** We would like to graciously thank our advisor Dr. Tasho Kaletha. He generously devoted his time, throughout the extent of the program, in explaining the appropriate concepts and background needed for our project in a very clear and concise way. We would also like to thank the organizers of the University of Michigan REU for making this project possible.

2. PRELIMINARIES

2.1. **Notation.** Let $\widehat{T} \subset \widehat{B}$ be a maximal torus and Borel subgroup of the reductive group $\widehat{G}$ and let Φ be the corresponding root system of $\widehat{T}$ in $\widehat{G}$, with positive system Φ^+ corresponding to $\widehat{B}$. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the corresponding simple roots, where n is the rank of $\widehat{G}$.

Let $X^*(\widehat{T})$ be the character lattice of algebraic homomorphisms from $\widehat{T}$ to $\mathbf{C}^\times$, and $X_*(\widehat{T})$ be the cocharacter lattice of algebraic homomorphisms from $\mathbf{C}^\times$ to $\widehat{T}$. Composition defines a $\mathbf{Z}$-bilinear pairing

$$\langle\ ,\ \rangle : X^*(\widehat{T}) \times X_*(\widehat{T}) \to \text{Hom}(\mathbf{C}^\times, \mathbf{C}^\times) = \mathbf{Z}.$$

Since $\widehat{T} \cong (\mathbf{C}^\times)^n$, we know that both $X^*(\widehat{T})$ and $X_*(\widehat{T})$ are free abelian groups of rank n and that the pairing is nondegenerate.

Now define two vector spaces

$$V = \mathbf{C} \otimes_{\mathbf{Z}} X_*(\widehat{T})$$

and

$$V^* = \mathbf{C} \otimes_{\mathbf{Z}} X^*(\widehat{T}).$$

These are both n-dimensional $\mathbf{C}$-vector spaces, and we can extend $\langle \ , \ \rangle$ to a nondegenerate $\mathbf{C}$-bilinear pairing $V^* \times V \to \mathbf{C}$. Having done this, define the coweights $\{\check{\omega}_1, \ldots, \check{\omega}_n\}$ in V such that $\check{\omega}_i$ is dual to α_i with respect to the pairing. Furthermore, we can naturally identify $X^*(\widehat{T})$ and $X_*(\widehat{T})$ with the subgroups $\mathbf{Z} \otimes_{\mathbf{Z}} X^*(\widehat{T})$ of V^* and $\mathbf{Z} \otimes_{\mathbf{Z}} X_*(\widehat{T})$ of V, respectively. We conclude that $X_*(\widehat{T})$ and $X^*(\widehat{T})$ are dual to each other with respect to the pairing.

Given a root $\alpha \in \Phi$, there is a unique element $\check{\alpha} \in V$ such that $\langle \alpha, \check{\alpha} \rangle = 2$ and the map

$$\sigma_\alpha : V^* \to V^* \quad x \mapsto x - \langle x, \check{\alpha} \rangle \alpha$$

preserves Φ. Collectively, these elements are called *coroots*, and are denoted $\check{\Phi}$.

For a general reductive group $\widehat{G}$, we have the relation

$$\check{Q} \subset X_*(\widehat{T}) \subset \check{P}$$

where $\check{P}$ is the coweight lattice, i.e the $\mathbf{Z}$-span of coweights $\check{\omega}$, and $\check{Q}$ is the coroot lattice $= \mathbf{Z}(\check{\Phi})$. Moreover it is known $\check{P} = X_*(\widehat{T})$ if and only if $\widehat{G}$ is adjoint and $X_*(T) = \check{Q}$ if and only if $\widehat{G}$ is simply-connected.

Let $N(\widehat{T})$ denote the normalizer of $\widehat{T}$ in $\widehat{G}$ and $W := N(\widehat{T})/\widehat{T}$ denote the Weyl group. Note that W acts on $\widehat{T}$ by conjugation, making $\widehat{T}$ into a W-module. This defines a W-module structure on $X_*(\widehat{T})$ which can be extended to one on $\mathbf{C} \otimes_{\mathbf{Z}} X_*(\widehat{T})$ by letting W act trivially on $\mathbf{C}$. With this action, V becomes a $\mathbf{C}$-linear W representation.

With these W-module structures, the $\mathbf{Z}$-bilinear map

$$\mathbf{C} \times X_*(\widehat{T}) \to \widehat{T} \quad (t, \check{\omega}) \mapsto \check{\omega}\left(e^{2\pi i t}\right)$$

gives rise to a W-module homomorphism

$$\exp : V \to \widehat{T}.$$

This map is surjective, and its kernel is $\mathbf{Z} \otimes_{\mathbf{Z}} X_*(\widehat{T}) \cong X_*(\widehat{T})$, so we obtain the following isomorphism of W-modules

$$\widehat{T} \cong V/X_*(\widehat{T}).$$

Finally, note that for any $x \in V$ and any $\alpha \in X_*(\widehat{T})$, we have

$$\alpha(\exp(x)) = e^{2\pi i \langle \alpha, x \rangle}. \tag{2.1.1}$$

2.2. **Inner regular elements.** In this section, we will be concerned with pairs (s, φ) of automorphisms of $\widehat{G}_{ad}$ which are in particular *inner* - i.e the s, φ action on $\widehat{G}_{ad}$ is given by conjugation by s, φ, respectively. So $\widehat{G}^s_{ad}$ is understood as the Ad-s invariants of $\widehat{G}$. We will see that on the Lie algebra level, an inner element is conjugate to an element of the standard alcove. Our following exposition closely follows that of [R09].

An element $s \in \widehat{G}_{ad}$ is *semisimple* if s acts diagonally on $\mathfrak{g} := \mathrm{Lie}(\widehat{G}_{ad})$. Any torsion element $s \in \widehat{G}_{ad}$ is semisimple and $\widehat{G}_{ad}$-conjugate to an element of $\widehat{T}_{ad}$, and there is $x \in V_{\mathbf{Q}} := \mathbf{Q} \otimes \check{P}$ such that $s = \exp(x)$. We have $x, x' \in V_{\mathbf{Q}}$ give rise to $\hat{G}$-conjugate elements $\exp(x)$, $\exp(x')$ if and only if x, x' are conjugate under the extended affine Weyl group

$$\widetilde{W} := W \ltimes \check{P}.$$

Here, $\check{P}$ acts on V by translations. The (unextended) affine Weyl group is the normal subgroup

$$\widetilde{W}^\circ := W \ltimes \check{Q} \lhd \widetilde{W}.$$

It can also be thought of as the group of reflections in V about the affine root hyperplanes $\alpha = n$ for $\alpha \in \Phi$ and $n \in \mathbf{Z}$. We call C an *alcove* if it is a connected component of the set of $x \in V$ not lying in any root hyperplane. A *wall* of C is the intersection of a root hyperplane $H_{\alpha,n}$ with the alcove closure $\overline{C}$. Thus each alcove has $n+1$ walls. From [Bou02, V.3.2], $\widetilde{W}^\circ$ is a Coxeter group generated by the $n+1$ hyperplanes about a fixed alcove and $\widetilde{W}^\circ$ permutes the alcoves in V freely and transitively. We now describe the standard alcove that will be used for our computations.

To the base $\Delta = \{\alpha_1, .., \alpha_n\}$, let $\tilde{\alpha} = \sum_1^n a_i\alpha_i$ be the highest root. Let $\alpha_0 := 1 - \tilde{\alpha}$ be an affine linear function, so

$$\sum_{i=0}^{n} a_i\alpha_i = 1,$$

where $a_0 := 1$. Then

$$C = \{x \in V : \langle \alpha_i, x\rangle > 0 \quad \text{for} \quad 0 \le i \le n\}$$

is the *standard alcove* associated to base Δ. In terms of barycentric coordinates,

$$C = \Big\{ \sum_{i=0}^{n} x_i\check{\omega}_i/a_i : x_i > 0 \quad \text{and} \quad \sum_{i=0}^{n} x_i = 1 \Big\},$$

where $\check{\omega}_0 := 0$. The following information can be also found in [Bou02, VI]. Since $\widetilde{W}^\circ$ acts transitively on the set of alcoves, $\widetilde{W}$ does too. Hence $\overline{C}$ intersects all $\widetilde{W}$-orbits in V. This means each torsion element $s \in \widehat{G}_{ad}$ is conjugate to $\exp(x)$ for some $x \in \overline{C} \cap V_{\mathbf{Q}}$. However, the extended affine Weyl group $\widetilde{W}$ does not act freely on the alcoves in V like $\widetilde{W}^\circ$, so we consider the alcove stabilizer

$$\Omega := \{\rho \in \widetilde{W} : \rho \cdot C = C\}.$$

We have the decomposition

$$\widetilde{W} = \Omega \ltimes \widetilde{W}^\circ.$$

It then follows for $x, x' \in \overline{C}$, $\exp(x)$ and $\exp(x')$ are $\widehat{G}_{ad}$-conjugate if and only if x and x' are conjugate under Ω.

It is useful to construct an explicit isomorphism

$$\Omega \cong X_*(\widehat{T})/\mathbf{Z}\check{\Phi}.$$

For each coset in $X_*(\widehat{T})/\mathbf{Z}\check{\Phi}$, there exists a unique coweight $\check{\omega}$ which is a vertex of alcove C. We call such a coweight *miniscule*. For each miniscule coweight, there is exists a unique $\rho_i \in \Omega$ such that $\rho_i \cdot \check{\omega}_0 = \check{\omega}_i$. This bijection is a group homomorphism.

We call an element $s \in \widehat{G}$ *regular* if the identity component of its centralizer $\mathrm{Cent}(s; \widehat{G})^\circ$ is a maximal torus. We call an element $s \in \widehat{T}$ *strongly regular* if $\mathrm{Cent}(s; \widehat{G})$ is a maximal torus. By [R09, Prop 2.1],

$$\mathrm{Cent}(s; \widehat{G})/\mathrm{Cent}(s; \widehat{G})^\circ \cong \Omega_x := \{\rho \in \Omega : \rho \cdot x = x\},$$

so $s = \exp(x)$ is strongly regular precisely when x is not fixed by an element of the affine Weyl group that stabilizes the alcove. Furthermore, $\Omega_x \cong W_s/W_s^\circ$, where W_s is the stabilizer of s in W and W_s° is generated by reflections for the roots in $\{\alpha \in \Phi^+ : \alpha(s) = 1\}$. To visualize the action of Ω on $x = \sum_{i=0}^n s_i\check{\omega}_i$, label the nodes of the extended Dynkin diagram $\widetilde{\mathcal{D}}(\mathfrak{g})$ by s_i; Ω acts on x via the symmetries of $\widetilde{\mathcal{D}}(\mathfrak{g})$. This gives an efficient way of explicitly describing the regular, but not strongly regular elements.

Consider an element $s \in \widehat{G}$ which is semisimple, regular, but not strongly regular. Then W_s° is trivial, $\Omega_x = W_s$, $\mathrm{Cent}(s;\widehat{G})^\circ = \widehat{T}$, and we have the short exact sequence

$$0 \to \widehat{T} \to \mathrm{Cent}(s;\widehat{G}) \to W_s \to 0$$

Let $\varphi \in N(\widehat{T})$ be such that $\widehat{T}^\varphi$ is finite and $\varphi s \varphi^{-1} = s^q$ for q a power of a prime. Denote by w_0 the projection of $\varphi \in N(\widehat{T})$ onto the Weyl group W. Then $\mathrm{Cent}(s;\widehat{G})^\varphi = \mathrm{Cent}(s,\varphi;\widehat{G})$ and we have the exact sequence

$$0 \to \widehat{T}^\varphi \to \mathrm{Cent}(s,\varphi;\widehat{G}) \to W_s^\varphi \to H^1(\langle\varphi\rangle,\widehat{T}) \to \cdots$$

Observe

$$0 \to \widehat{T}^\varphi \to \widehat{T} \xrightarrow{\varphi-1} \widehat{T}$$

is an exact sequence of algebras, and since $\widehat{T}^\varphi = \ker(\varphi-1)$ is finite, $\varphi - 1$ is surjective. Hence the first cohomology group $H^1(\langle\varphi\rangle,\widehat{T}) = \ker(N)/\mathrm{im}(\varphi-1) = \widehat{T}/\widehat{T} = \{1\}$, where $N = \sum_0^{\mathrm{ord}(\varphi)-1}\varphi^i$ is the norm map, and we have proven

$$0 \to \widehat{T}_{\mathrm{ad}}^\varphi \to \mathrm{Cent}(s,\varphi;\widehat{G}_{\mathrm{ad}}) \to W_s^\varphi \to 1$$

is an exact sequence. Furthermore, if one takes $\widehat{T}_+$ and $\mathrm{Cent}(s,\varphi,\widehat{G}_{\mathrm{ad}})_+$ to be the preimages of $\widehat{T}_{\mathrm{ad}}^\varphi$ and $\mathrm{Cent}(s,\varphi;\widehat{G}_{\mathrm{ad}})$ in $\widehat{G}_{sc}$, we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{T}_+ & \longrightarrow & \mathrm{Cent}(s,\varphi,\widehat{G}_{\mathrm{ad}})_+ & \longrightarrow & W_s^\varphi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow\| & & \\ 1 & \longrightarrow & \widehat{T}_{ad}^\varphi & \longrightarrow & \mathrm{Cent}(s,\varphi,\widehat{G}_{\mathrm{ad}}) & \longrightarrow & W_s^\varphi & \longrightarrow & 1 \end{array} \tag{2.2.1}$$

We conclude this subsection with a useful criterion for the group of toral φ-fixed points to be finite.

Lemma 2.2.1. *For any $\varphi \in N(\widehat{T})$, $\widehat{T}^\varphi$ is finite if and only if the linear operator $\varphi - \mathbf{1} : V \to V$ is invertible. If this is the case, then $|\widehat{T}^\varphi| = \det(\varphi - \mathbf{1}) = p_\varphi(1)$, where p_φ is the characteristic polynomial of φ.*

Proof. Explicitly, $\widehat{T}^\varphi = \{t \in \widehat{T} \mid (\varphi \cdot t)t^{-1} = 1\}$. Since $\exp : V \to \widehat{T}$ is a surjective group homomorphism, $\widehat{T}^\varphi \cong F_\varphi/X_*(\widehat{T})$, where F_φ is the preimage of $\widehat{T}^\varphi$ under exp. Because exp is φ-equivariant, we can write F_φ explicitly as $\left\{x \in V \mid \varphi\cdot x - x \in X_*(\widehat{T})\right\}$. Now consider the linear operator $\varphi - \mathbf{1}$ acting on V. If for some $x \in V$ we know $x \in \ker(\varphi - \mathbf{1})$, then x is surely in F_φ. If $F_\varphi/X_*(\widehat{T})$ is finite, F_φ must be covered by a finite number of translates of $X_*(\widehat{T})$. Since $\ker(\varphi - \mathbf{1}) \subseteq F_\varphi$, we know $\ker(\varphi - \mathbf{1})$ is covered by those same translates. We also know that $\ker(\varphi - \mathbf{1}) \cong \mathbf{C}^n$ for some n, so $n = 0$. Since $\varphi - \mathbf{1}$ is injective and sends $V \to V$, it must be an isomorphism.

On the other hand, suppose that $\varphi - \mathbf{1}$ is invertible. Then we can describe F_φ explicitly as $(\varphi - \mathbf{1})^{-1}X_*(\widehat{T})$, so F_φ is also a free abelian group of rank n. Since φ maps $X_*(\widehat{T})$ into itself, we see that $(\varphi - \mathbf{1})$ is an injective endomorphism of F_φ with image $X_*(\widehat{T})$. As a result, $F_\varphi/X_*(\widehat{T})$ is finite. Furthermore, using the Smith Normal Form for $\varphi - \mathbf{1}$, we see that $|\widehat{T}^\varphi| = |F_\varphi/X_*(\widehat{T})| = \det(\varphi - \mathbf{1}) = p_\varphi(1)$. □

Corollary 2.2.2. *Whether $\widehat{T}^\varphi$ is finite depends only on the conjugacy class of φ in W.*

Corollary 2.2.3. *Whether $\widehat{T}^\varphi$ is finite doesn't depend on the choice of group in a given isogeny class.*

2.3. **Basic properties of $\widehat{G}$.** The tables below provide fundamental properties of $\widehat{G}$. The root system provided is for Lie($\widehat{G}$). Here, $\Delta = \{\alpha_1, .., \alpha_n\}$, $\check{\Delta} = \{\check{\alpha}_1, .., \check{\alpha}_n\}$, and $\widetilde{\alpha}$ is the highest root with respect to Δ.

TABLE 2.3.1. Root Systems

Type	Φ	$\Delta, \check{\Delta}$	$\widetilde{\alpha}$
A_n	$\{\varepsilon_i - \varepsilon_j : i \neq j\}$	$\{\varepsilon_1 - \varepsilon_2, .., \varepsilon_n - \varepsilon_{n+1}\}$	$\alpha_1 + .. + \alpha_n$
		$\{\varepsilon_1^* - \varepsilon_2^*, .., \varepsilon_n^* - \varepsilon_{n+1}^*\}$	
B_n	$\{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) : i \neq j\}$	$\{\varepsilon_1 - \varepsilon_2, .., \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$	$\alpha_1 + 2\alpha_2 .. + 2\alpha_n$
		$\{\varepsilon_1^* - \varepsilon_2^*, .., \varepsilon_{n-1}^* - \varepsilon_n^*, 2\varepsilon_n^*\}$	
C_n	$\{2\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) : i \neq j\}$	$\{\varepsilon_1 - \varepsilon_2, .., \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$	$2\alpha_1 + 2\alpha_2 .. + 2\alpha_{n-1} + \alpha_n$
		$\{\varepsilon_1^* - \varepsilon_2^*, .., \varepsilon_{n-1}^* - \varepsilon_n^*, \varepsilon_n^*\}$	
D_n	$\{\pm(\varepsilon_i \pm \varepsilon_j) : i \neq j\}$	$\{\varepsilon_1 - \varepsilon_2, .., \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$	$\alpha_1 + 2\alpha_2 + .. + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$
		$\{\varepsilon_1^* - \varepsilon_2^*, .., \varepsilon_{n-1}^* - \varepsilon_n^*, \varepsilon_{n-1}^* + \varepsilon_n^*\}$	

TABLE 2.3.2. Weyl Groups

$\widehat{G}_{\mathrm{ad}}$	$\widehat{G}_{\mathrm{sc}}$	W	Ω
$\mathrm{PSL}_{n+1} = \mathrm{SL}_{n+1}/\mu_{n+1}$	SL_{n+1}	Σ_{n+1}	$\mathbf{Z}/(n+1)\mathbf{Z}$
SO_{2n+1}	Spin_{2n+1}	$(\mathbf{Z}/2\mathbf{Z})^n \rtimes \Sigma_n$	$\mathbf{Z}/2\mathbf{Z}$
$\mathrm{PSp}_{2n} = \mathrm{Sp}_{2n}/\mu_2$	Sp_{2n}	$(\mathbf{Z}/2\mathbf{Z})^n \rtimes \Sigma_n$	$\mathbf{Z}/2\mathbf{Z}$
$\mathrm{PSO}_{2n} = \mathrm{SO}_{2n}/\mu_2$	Spin_{2n}	$(\mathbf{Z}/2\mathbf{Z})^{n-1} \rtimes \Sigma_n$	n even: $(\mathbf{Z}/2\mathbf{Z})^2$
			n odd: $\mathbf{Z}/4\mathbf{Z}$

3. REPRESENTATION THEORY FOR GROUP EXTENSIONS

3.1. **Group Extensions.** In this section we state the theory needed to describe irreducible representations of $\mathrm{Cent}(s, \varphi, \widehat{G}_{\mathrm{ad}})$ and $\mathrm{Cent}(s, \varphi, \widehat{G}_{\mathrm{ad}})_+$, both of which are extensions of a finite abelian group by a finite abelian group.

First let us recall some basic results of group extensions and Clifford theory. Given a group G, and an abelian group K, we say E is an *extension* of K by G if it fits in the short exact sequence

$$1 \to K \to E \to G \to 1$$

Choose a set-theoretic section $s : G \to E$. This defines a G action on K by

$$g \cdot k := s(g)ks(g)^{-1}.$$

Also, s determines a 2-cocyle

$$\xi_s(g_1, g_2) := s(g_1)s(g_2)s(g_1g_2)^{-1} \in Z^2(G, K),$$

which "measures" how far s is from being a group homomorphism. Changing the section s replaces ξ by a cohomologous 2-cocycle, so the class $[\xi] \in H^2(G, E)$ is independent of choice of section.

Given a group G, a G-module K, and a normalized 2-cocycle $\xi \in Z^2(G, K)$, we can construct the group $K \boxtimes_\xi G$ whose underlying set is the Cartesian product $K \times G$, and has the group law

$$(k_1, g_1)(k_2, g_2) := \big(k_1{}^{g_1}k_2\xi(g_1, g_2), g_1g_2\big)$$

(here ${}^{g_1}k_2 = g_1 \cdot k_2$). Now if we have an extension E of K by G, with a chosen section s, we see there is an isomorphism $E \to K \boxtimes_{\xi_s} G$ given by sending $e = ks(g) \in E$ to (k, g).

A short lemma now shows the section determines the group structure on E completely.

Lemma 3.1.1. [Bro82, Section 4.3] *There is a* $1-1$ *correspondence*

$$\left\{ \begin{array}{c} \textit{Equivalence classes of extensions of } K \textit{ by } G \\ \textit{with prescribed } G\textit{-action on } K \end{array} \right\} \longleftrightarrow H^2(G, K)$$

given by

$$[E] \longrightarrow [\xi_s]$$
$$K \boxtimes_\xi G \longleftarrow [\xi]$$

where the "prescribed action" is the one determined by a choice of section. E *and* E' *are said to be equivalent extensions if there exists a commutative diagram of short exact sequences.*

3.2. **Clifford theory.** Clifford theory is the study of representations of a finite group extension G in terms of that of its normal subgroup N and quotient H. We will now quote two lemmas from [Kal18] on the subject. In what follows, $\mathrm{Irr}(G)$ will denote the isomorphism classes of irreducible G-representations.

Lemma 3.2.1. *Suppose we have an exact sequence of finite, not necessarily abelian groups*

$$1 \to N \to G \to H \to 1$$

Let $\pi \in Irr(G)$ *and* $S_\pi \subset Irr(N)$ *denote the irreducible* N*-representation occurring in* $\pi|_N$*. Then*

1. S_π *is a single* H *orbit and each member of* S_π *occurs with the same multiplicity, denoted* m_π.
2. *The map*
$$Irr(G) \to Irr(N)/H$$
defined by $\pi \to S_\pi$ *is surjective.*

Lemma 3.2.2. *Suppose* H *is abelian and* $\sigma \in Irr(N)$*. Let* $G_\sigma := Stab(\sigma, G)$ *and* $G'_\sigma \subset G_\sigma$ *be the largest subgroup to which* σ *extends, i.e* $\widetilde{\sigma}$ *is a (linear) representation of* G'_σ *such that* $\widetilde{\sigma}|_N = \sigma$*. Then*

$$Ind_{G'_\sigma}^{G} \widetilde{\sigma}$$

is an irreducible G*-representation and every irreducible* G*-representation arises this way.*

It is easy to show that when $H_\sigma = \mathrm{Stab}(\sigma, H)$ is cyclic, then $G'_\sigma = G_\sigma$. Using the previous lemma, we can recover the classical classification of irreducible representations for a semi-direct product of abelian normal group by quotient:

Corollary 3.2.3. [Ser71, Section 8.2] *Suppose* $G = N \rtimes H$ *for an abelian normal subgroup* N *and subgroup* H *of* G*. Let* $\chi : N \to \mathbf{C}^\times$ *be a character and let*

$$H_\chi = \{h \in H : \chi(hnh^{-1}) = \chi(n), \forall n \in N\}.$$

Then extend χ *to the character* $\widetilde{\chi} : N \rtimes H_\chi \to \mathbf{C}^\times$ *by setting* $\widetilde{\chi}(nh) = \chi(n)$*. For any irreducible representation* ρ *of* H_χ*, composing with the standard projection* $N \rtimes H_\chi \to H_\chi$ *gives an irreducible representation* ρ' *of* $G_\chi := N \rtimes H_\chi$*. Then*

$$Ind_{G_\chi}^{G} (\widetilde{\chi} \otimes \rho')$$

is an irreducible G*-representation and every irreducible* G*-representation is uniquely determined this way.*

We now want to generalize this result to the case when the extension is not split.

Lemma 3.2.4. *Suppose we have the exact sequence of groups $1 \to A \to B \to C \to 1$ where A is abelian, and a given representation χ of A. Choose a set-theoretic section $s : C \to B$ and identify $B \cong A \boxtimes_s C$. Then $\widetilde{\chi}$ is an extension of χ to B if and only if $\widetilde{\chi}(a, c) = \chi(a)\sigma(c)$, where σ is a set-theoretic function $C \to \mathbf{C}^\times$ that satisfies $\chi \circ \xi_s = \xi_\sigma$.*

Proof. Given an extension $\widetilde{\chi}$ of χ to B, define $\sigma : C \to \mathbf{C}^\times$ to be the function $c \mapsto \widetilde{\chi}(1, c)$. Then because $\widetilde{\chi}$ extends χ, we have that

$$\widetilde{\chi}(a, c) = \widetilde{\chi}(a, 1)\widetilde{\chi}(1, c) = \chi(a)\sigma(c).$$

Also, since $\widetilde{\chi}$ is a group homomorphism $B \to \mathbf{C}^\times$, we know that

$$\begin{aligned} 1 &= \widetilde{\chi}(a_1, c_1)\widetilde{\chi}(a_2, c_2)\widetilde{\chi}(a_1{}^{c_1}a_2\xi_s(c_1, c_2), c_1c_2)^{-1} \\ &= \chi(a_1)\sigma(c_1)\chi(a_2)\sigma(c_2)\sigma(c_1c_2)^{-1}\chi(a_1{}^{c_1}a_2\xi_s(c_1, c_2))^{-1}. \end{aligned}$$

Since $\mathbf{C}^\times$ is abelian and χ is a homomorphism, this implies that

$$1 = \sigma(c_1)\sigma(c_2)\sigma(c_1c_2)^{-1}\chi(a_2)\chi({}^{c_1}a_2)^{-1}\chi(\xi_s(c_1, c_2)))^{-1}, \tag{3.2.1}$$

but because $({}^{c_1}a_2, 1) = (1, c_1)(a_2, 1)(1, c_1)^{-1}$, we see that

$$\chi({}^{c_1}a_2) = \widetilde{\chi}\left((1, c_1)(a_2, 1)(1, c_1)^{-1}\right) = \widetilde{\chi}(a_2, 1) = \chi(a_2),$$

so equation 3.2.1 becomes

$$\chi \circ \xi_s(c_1, c_2) = \xi_\sigma(c_1, c_2). \tag{3.2.2}$$

On the other hand, if we choose a $\sigma : C \to \mathbf{C}^\times$ that satisfies equation 3.2.2 and define the function $\widetilde{\chi} : B \to \mathbf{C}^\times$ by $\widetilde{\chi}(a, c) = \chi(a)\sigma(c)$, then the calculation we just did shows that $\widetilde{\chi}$ is a group homomorphism. Also,

$$\sigma(1) = \sigma(1)\sigma(1)\sigma(1)^{-1} = \xi_\sigma(1, 1) = \chi \circ \xi_s(1, 1) = 1,$$

so $\widetilde{\chi}$ extends χ. □

With the notation being the same as in lemma 3.2.4, we make the following definition:

Definition 3.2.5. Suppose $1 \to A \to B \to C \to 1$ is an exact sequence where A and C are abelian. Then, with notation as in 3.2.3, for a given character χ of A, we obtain an exact sequence $1 \to A \to B_\chi \to C_\chi \to 1$. If there exists a set-theoretic section $s : C_\chi \to B_\chi$ such that $\chi \circ \xi_s = 1$, then we call χ a *virtually split character*. If all of the characters of A are virtually split in this way, then we say that B is a *virtually split extension* of A by C.

All split extensions are virtually split. When B is a virtually split extension, by lemma 3.2.4 we can extend any character χ of A trivially to $\widetilde{\chi}$ by setting $\widetilde{\chi}(as(c)) = \chi(a)$. Then by lemma 3.2.2 all irreducible representations of B are given by

$$\mathrm{Ind}_{B'_\chi}^{B} \widetilde{\chi} \otimes \sigma$$

for characters χ of A and σ of C (here we regard σ as a character of B by pulling it back along the map $B \to C$. In particular, the irreducible representations of virtually split extensions arise in the exact same way as irreducible representations of split extensions (by lemma 3.2.2).

3.3. **Langlands-Shelstad section.** It is a result of Langlands and Shelstad [LS87] that for a connected reductive group $\widehat{G}$, a pinning $(\widehat{T}, \widehat{B}, \{X_\alpha\})$ of $\widehat{G}$ gives a section $n : W \to N(\widehat{T})$ with a particularly nice 2-cocycle formula. The construction is as follows:

For a simple root α, write H_α for the image of $\check{\alpha}$ under the identification of V with $\mathrm{Lie}(\widehat{T})$. Next, choose $X_{-\alpha} \in \mathrm{Lie}(\widehat{G})$ to be the unique element such that $[X_\alpha, X_{-\alpha}] = H_\alpha$. The subspace of $\mathrm{Lie}(\widehat{G})$ spanned by $\{X_\alpha, H_\alpha, X_{-\alpha}\}$ is a Lie subalgebra of $\mathrm{Lie}(\widehat{G})$ which is isomorphic to $\mathfrak{sl}_2(\mathbf{C})$. Because $\widehat{G}$

is connected, there is a unique homomorphism of algebraic groups $\varphi : SL_2(\mathbf{C}) \to \widehat{G}$ whose tangent map at the identity is this inclusion $\mathfrak{sl}_2(\mathbf{C}) \hookrightarrow \mathrm{Lie}(G)$. Furthermore,

$$\varphi \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

acts on $\widehat{T}$ in the same way as $\sigma_\alpha \in W(\widehat{T})$, so define $n(\sigma_\alpha)$ to be equal to this element of $N(\widehat{T})$. Then for a typical element $\sigma \in W$, express σ as a reduced composition of reflections about simple roots $\sigma = \sigma_{\alpha_1} \dots \sigma_{\alpha_n}$ and define $n(\sigma) = n(\sigma_{\alpha_1}) \dots n(\sigma_{\alpha_n})$. This is well defined, because $n(\sigma)$ is independent of the choice of reduced expression for σ. Finally, set $n(1) = 1$.

Lemma 3.3.1. [LS87, Lemma 2.1A] *The 2-cocycle $\xi(\sigma_1, \sigma_2) = n(\sigma_1)n(\sigma_2)n(\sigma_1\sigma_2)^{-1} \in Z^2(W, \widehat{T})$ is given by the formula*

$$\xi(\sigma_1, \sigma_2) = \prod_{\substack{\alpha > 0 \\ \sigma_1^{-1}\alpha < 0 \\ \sigma_2^{-1}\sigma_1^{-1}\alpha > 0}} \check{\alpha}(-1),$$

where $\alpha > 0$ means α is a root in $\widehat{B}$ and $\check{\alpha} \in X_*(\widehat{T})$ is the coroot associated to α.

3.4. **Concrete Conditions for Virtual Splitness.** Given some $\varphi \in N(\widehat{T}_{ad})$ with $|\widehat{T}^\varphi| < \infty$, we can project φ to some $w_0 \in W$ and lift w_0 to $N(\widehat{T}_{sc})$. Since $\widehat{T}^\varphi$ is finite, any lift of w_0 to $N(\widehat{T}_{sc})$ can be conjugated to any other by an element of $\widehat{T}_{sc}$. Conjugating $N(\widehat{T}_{sc})$ by an element of $\widehat{T}_{\mathrm{sc}}$ replaces the extension to an isomorphic one. As a result, if we have some section $n : W \to N_{sc}$ and we write p for the projection $\widehat{G}_{sc} \to \widehat{G}_{ad}$ we may without loss of generality lift w_0 to $n(w_0)$ and replace φ with $p \circ n(w_0)$.

We now describe a procedure for lifting an element from W^φ to $\mathrm{Cent}(\varphi; \widehat{G}_{ad})_+$ and provide conditions for this lift to give a splitting or "virtual splitting". For commuting $w, v \in W$, let

$$c(w, v) := n(w)n(v)n(w)^{-1}n(v)^{-1} = \xi(w, v)\xi(v, w)^{-1}$$

denote the commutator. Now choose a lift $\dot{c}(w, w_0)$ of c to V such that $\exp_{sc} \circ \dot{c} = c$. Since $\varphi - 1$ acts invertibly on V by (2.2.1), we can form

$$\dot{t}_w = (\varphi - 1)^{-1}\dot{c}(w, w_0) \in V. \tag{3.4.1}$$

Define $t_w := \exp_{ad}(\dot{t}_w)$. If we apply $\exp_{ad}$ to the equation $(\varphi - 1)\dot{t}_w = \dot{c}(w, w_0)$ and keep in mind $\exp_{ad} = p \circ \exp_{sc}$, we see that ${}^\varphi t_w = p \circ c(w, w_0)t_w$.

Now define the section

$$s : W^\varphi \to N(\widehat{T}_{ad}) \quad w \mapsto t_w(p \circ n(w)),$$

and note that

$$\begin{aligned} {}^\varphi s(w) &= {}^\varphi t_w \, {}^\varphi(p \circ n(w)) \\ &= t_w(p \circ c(w, w_0))(p \circ c(w_0, w))(p \circ n(w)) \\ &= t_w(p \circ n(w)) \\ &= s(w), \end{aligned}$$

so s actually maps W^φ into $\mathrm{Cent}(\varphi, \widehat{G}_{ad})$.

In what follows, we will use the convention s for a section from Weyl group to centralizer, and n for section from Weyl group into normalizer. If we now define $t'_w = \exp_{sc}(\dot{t}_w)$ and define

$$s' : W^\varphi \to \widehat{G}_{sc} \quad w \mapsto t'_w n(w),$$

we see that $p \circ s'(w) = s(w)$, so that s' actually maps into $\mathrm{Cent}(\varphi; G_{ad})_+$.

If we now choose a lift $\dot{\xi}$ of ξ_n to V along $\exp_{sc}$, for any two $w, v \in W^{\varphi}$, we can define

$$\dot{A}(w,v) := \dot{c}(w,w_0) + w \cdot \dot{c}(v,w_0) - \dot{c}(wv,w_0) + (\varphi - 1)\dot{\xi}(w,v)$$

and define $\dot{B}(w,v) := (\varphi-1)^{-1}\dot{A}(w,v)$. Note that

$$\begin{aligned}
\xi_s(w,v) &= s(w)s(v)s(wv)^{-1} \\
&= t_w(p \circ n(w))t_v(p \circ n(v))(p \circ n(wv))^{-1}t_{wv}^{-1} \\
&= t_w^w t_v(p \circ \xi_n(w,v))t_{wv}^{-1} \\
&= \exp_{ad}(\dot{B}(w,v)),
\end{aligned}$$

and similarly, $\xi_{s'} = \exp_{sc} \circ \dot{B}$. As a result, if $\dot{B}(w,v) \in \check{P}$ for all $w, v \in W_s^{\phi}$, we see that ξ_s is trivial on W_s^{ϕ}, so s gives a splitting of the exact sequence $1 \to T^{\varphi} \to \mathrm{Cent}(s, \varphi; \widehat{G}_{ad}) \to W_s^{\varphi} \to 1$.

Now suppose that

$$\dot{A}(w,v) = (\varphi-1)\cdot q + (w-1)\cdot p' \tag{3.4.2}$$

for some $p' \in \check{P}$ and $q \in \check{Q}$. Define $\dot{f} := (\varphi-1)^{-1}p'$ and note that since $(\varphi-1)\dot{f} = p'$, we have $\exp_{ad}(\dot{f}) \in \widehat{T}_{ad}^{\varphi}$ and therefore $\exp_{sc}(\dot{f}) \in \widehat{T}_{+}$. If we act on (3.4.2) by $(\varphi-1)^{-1}$ and note that $(\varphi-1)^{-1}$ and $(w-1)$ commute, we see that $(w-1)\dot{f} = \dot{B}(w,v) - q$.

Now let χ be a character of $\widehat{T}_+$ and let w and v be elements of $(W_s^{\varphi})_{\chi}$, the subgroup of W_s^{ϕ} which stabilizes χ. If we define $f' = \exp_{sc}(\dot{f})$, we find

$$\begin{aligned}
\chi(\xi_{s'}(w,v)) &= \chi(\exp_{sc}(\dot{B}(w,v))) \\
&= \chi(\exp_{sc}((w-1)\dot{f})) \\
&= \chi({}^{w}ff^{-1}) \\
&= \chi({}^{w}f)\chi(f^{-1}) \\
&= \chi(f)\chi(f)^{-1} \\
&= 1,
\end{aligned}$$

so that χ is virtually split. Since if (3.4.2) holds, this is true for all characters of $\widehat{T}_+$, we see that in this case $\mathrm{Cent}(s, \phi; G_{ad})_+$ is virtually split. A similar argument using ξ_s and $f = \exp_{ad}(\dot{f})$ shows that if (3.4.2) holds, then $\mathrm{Cent}(s, \phi; G_{ad})$ is virtually split.

4. Case Study

We will systematically perform the following computations using the theory developed from sections 2 and 3:

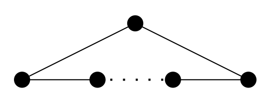
(1) Describe the regular, but not strongly regular elements $s \in \widehat{G}_{\mathrm{ad}}$.
(2) Describe the $\varphi \in N(\widehat{G}_{\mathrm{ad}})$ such that $\widehat{T}_{\mathrm{ad}}^{\varphi}$ is finite and W_s^{φ} is non-trivial.
(3) Describe pairs (s, φ) satisfying (1), (2) and $\varphi s \varphi^{-1} = s^q$ for q a power of a prime.
(4) Describe $\mathrm{Cent}(s, \varphi; \widehat{G}_{\mathrm{ad}})$ for such pairs (s, φ) and its irreducible representations.
(5) Describe $\mathrm{Cent}(s, \varphi; \widehat{G}_{\mathrm{ad}})_+$ for such pairs (s, φ) and its irreducible representations.

For root systems of the classical type, we will begin by introducing a convenient coordinate system to simplify the computation.

4.1. **Type A_n.** Since there is an isomorphism $PGL_n(\mathbf{C}) \cong PSL_n(\mathbf{C})$, we will work with $PGL_n(\mathbf{C})$ for notational convenience.

(1) Choose the standard maximal torus of diagonal matrices and the standard Borel subgroup of upper-triangular matrices. In this case, $\alpha_i \in \Delta$ is the character $\alpha_i(t) = t_i/t_{i+1}$.

In type A_n, we know the highest root is $\alpha_1 + \ldots + \alpha_n$, so $v_i = \check{\omega}_i$ for all i. The extended Dynkin diagram for type A_n is



The alcove stabilizer Ω acts on $\widetilde{\mathcal{D}}(\mathfrak{g})$ via rotations, so $\Omega \cong \mathbf{Z}/(n+1)\mathbf{Z}$. Write w for the generator of Ω which sends $v_i \mapsto v_{i+1}$. The semisimple elements of V fixed by a nontrivial rotation $w^d \in \Omega$ are given by

$$x = \sum_{i=0}^{n} s_i v_i$$

such that $s_i = s_{i+d}$ for all $i = 0, 1, .., n$, and $\sum_0^n s_i = 1$ and $s_i > 0$ for all i. Then by Equation 2.1.1 and Table 2.3.1,

$$(e^{2\pi i t_{n+1}}, \ldots, e^{2\pi i t_1}) \in PGL_{n+1}(\mathbf{C}), \tag{4.1.1}$$

where and the t_i satisfy $d/(n+1) > t_d > \cdots > t_1 = 0$ and $t_{d+i} = d/(n+1) + t_i$ for all i.

(2) Since φ acts on the space W spanned by $\varepsilon_1^* \ldots \varepsilon_n^*$ as a permutation matrix, its characteristic polynomial is $p_{\varphi,W}(\lambda) = \prod_i \lambda^{\ell_i} - 1$, where ℓ_i is the length of the ith cycle of φ. Since this action decomposes into an action on V and an action on $\mathbf{C}(\varepsilon_1^* + \ldots + \varepsilon_n^*)$ (the second of which φ acts on trivially) we see that $p_{\varphi,W}(\lambda) = p_{\varphi,V}(\lambda)(\lambda - 1)$. If $\widehat{T}^{\varphi}$ is finite, then $p_{\varphi,V}(1) \neq 0$, so $p_{\varphi,W}(\lambda)$ can contain only a single factor of $\lambda^{\ell_i} - 1$, so w_0 must be a single cycle in Σ_{n+1}. In this case, $p_{\varphi,V}(\lambda) = \lambda^n + \lambda^{n-1} + \ldots + 1$, so $|\widehat{T}^{\varphi}| = p_{\varphi,V}(1) = n + 1$. If necessary, conjugate w_0 so that it is the cycle $(1\ 2 \ldots n+1) \in \Sigma_{n+1}$. Having done this, we can write the elements of $\widehat{T}^{\varphi}$ explicitly: write $\zeta = e^{2\pi i/(n+1)}$. Then the ith element of $\widehat{T}^{\varphi}$ is $t_i = [\mathrm{diag}(1, \zeta^i, \ldots, \zeta^{i(n-1)})]$. From this, we see that

$$\widehat{T}_{\mathrm{ad}}^{\varphi} \cong \mathbf{Z}/(n+1)\mathbf{Z}$$

via the isomorphism $t_i \mapsto i$.

If $W_s^{\varphi} \neq \{1\}$, there exists $w' \neq 1$ such that w_0 commutes with it and

$$W_s^{\varphi} = \langle w' \rangle \cong \mathbf{Z}/\tfrac{n+1}{dd'}\mathbf{Z},$$

where $w' = w^{dd'}, d' | (n+1)/d$. Lastly, since w_0 is a cycle which commutes with w', there exists i such that $w_0^i = w'$ and $\gcd(i, n+1) = dd'$ For simplicity, we assume $d' = 1, w' = w^d$ in the following steps.

(3) We wish to solve the equation

$$w_0.(s_{n+1},\dots,s_1)=(s_{n+1}^p,\dots,s_1^p)\in PGL_{n+1}(\mathbf{C})$$

for $s=(s_{n+1},\dots s_1)$ as given in (4.1.1). This amounts to solving $s_1=s^p_{w_0(1)}z,\dots,s_{w_0^{i-1}(1)}z=s^p_{d+1}z$ where $z\in\mathbf{C}^\times$. Since our solution is in $PGL_{n+1}(\mathbf{C})$, we may as well take $s_1=1$. Since w_0 is a cycle, all the s_i are now determined once we specify z. After composing the equations i times, we have $s_j=s_{j+d}^{p^i}z^{\frac{p^i-1}{p-1}}$, and using the fact that $s_{j+d}=\zeta^d s_j$, we see that

$$s_j=s_j^{p^i}z^{\frac{p^i-1}{p-1}}\zeta^{p^id}$$

for all $j=1,\dots n+1$. In particular for $j=1$, we have $s_j=1$ which implies (†) $z^{\frac{p^i-1}{p-1}}=\zeta^{-p^id}$. Observe

$$\begin{aligned}\zeta^d=s_{d+1}&=s_{w_0^i(1)}\\&=s^p_{w_0^{i+1}(1)}z\\&=s^p_{d+w_0(1)}z\\&=\zeta^{dp}s^p_{w_0(1)}z=\zeta^{dp},\end{aligned}$$

which implies $d(1-p)\equiv 0 \mod n+1$, or equivalently, $p\equiv 1 \mod (n+1)/d$. So we can simplify (†) to get $z^{\frac{p^i-1}{p-1}}=\zeta^{-d}$, so z is some $\frac{p^i-1}{p-1}$ root of ζ^{-d}.

Now, it just remains to check for a given z whether we have the string of inequalities $\arg(s_{n+1})>\arg(s_n)>\cdots>\arg(s_1)=0$. We see our work fully classify triples (s,ϕ,p) satisfying steps (1), (2), (3). Furthermore once we specify p, we see there are only finitely many pairs (s,ϕ), which can all be found systematically using a computer program.

(4) The map $n:W\to N(\widehat{T})$ which maps every element of W to the equivalence class of its permutation matrix, $w\mapsto[P(w)]$, is a splitting of W into $N(\widehat{T})$. Since $\widehat{T}_{ad}^\varphi$ is finite, by conjugating φ by an element of $\widehat{T}_{ad}$ we can fix $\varphi=n(w_0)$. Then the restriction of n to W_s^φ gives a splitting of W_s^φ into $\mathrm{Cent}(s,\varphi;\widehat{G}_{ad})$. Since each element $w\in W_s^\varphi$ is a power of w_0, it acts trivially on $\widehat{T}_{ad}^\varphi$. As a result,

$$\mathrm{Cent}(s,\varphi;\widehat{G}_{\mathrm{ad}})\cong\mathbf{Z}/(n+1)\mathbf{Z}\times\mathbf{Z}/\tfrac{n+1}{k}\mathbf{Z}.$$

(5) We now want to understand the pullback C_+ of $\mathrm{Cent}(s,\varphi;\widehat{G}_{ad})$ to $\widehat{G}_{sc}=SL_{n+1}(\mathbf{C})$. First, note that

$$\det(\mathrm{diag}(1,\zeta^i,\dots,\zeta^{i(n-1)})=\prod_{j=0}^{n-1}\zeta^{ij}=\prod_{j=1}^{n}\zeta^{ij}=\zeta^{in(n+1)/2}.$$

Also note that

$$\det(P(\varphi))=\mathrm{sign}(1\ 2\dots n+1)=(-1)^n.$$

Case 1: $n+1$ odd.
In this case, $\zeta^{in(n+1)/2}=1$, so the map $r:\widehat{T}_{ad}^\varphi\to\widehat{T}_+$ sending $t_i\mapsto\mathrm{diag}(1,\zeta^i,\dots,\zeta^{i(n-1)})$ gives a splitting of $\widehat{T}^\varphi$ into $\widehat{T}_+$. Since $\widehat{T}_+$ is abelian, we conclude

$$\widehat{T}_+\cong(\mathbf{Z}/(n+1)\mathbf{Z})^2.$$

Also, since $\det(P(w_0^i))=\det(P(w_0))^i=1$, the map $s:W_s^\varphi\to C^+$ sending $w\mapsto P(w)$ gives a splitting of W_s^φ into C_+, so that

$$C_+\cong(\mathbf{Z}/(n+1)\mathbf{Z})^2\rtimes\mathbf{Z}/\tfrac{n+1}{k}\mathbf{Z}.$$

Explicitly, since $w_0^{-1} \cdot r(t_i) = \zeta^i r(t_i)$, we see that an element $w = w_0^{-\ell} \in W_s^\varphi$ acts on $\widehat{T}_+$ by sending $\zeta^j r(t_i)$ to $w \cdot \zeta^j r(t_i) = \zeta^{j+i} r(t_i)$.

Case 2: $n+1$ even.
In this case, $\zeta^{in(n+1)/2} = (-1)^i$. As a result, the map $r : \widehat{T}_{ad}^\varphi \to \widehat{T}_+$ which sends

$$t_i \mapsto \begin{cases} \mathrm{diag}(1, \zeta^i, \ldots, \zeta^{i(n-1)}) & 2|i \\ \zeta^{1/2}\mathrm{diag}(1, \zeta^i, \ldots, \zeta^{i(n-1)}) & 2 \nmid i \end{cases}$$

gives a set-theoretic section $\widehat{T}_{ad}^\varphi \to \widehat{T}_+$. Its cocycle is

$$\xi_r(t_i, t_j) = \begin{cases} \zeta & 2 \nmid i \text{ and } 2 \nmid j \\ 1 & \text{otherwise.} \end{cases}$$

Because of this, we see $r(t_1)^i = \zeta^{\lfloor i/2 \rfloor} r(t_i)$, so $r(t_1)$ has order $2(n+1)$. On the other hand $r(t_2)^i = r(t_{2i})$, so $r(t_2)$ has order $(n+1)/2$. Since $r(t_2)^i$ always has a 1 in its first coordinate, and since the first coordinate of $r(t_1)^i$ is $\zeta^{\lfloor i/2 \rfloor}$, we see that $\langle r(t_1) \rangle \cap \langle r(t_2) \rangle = 1$. Since $\widehat{T}_+$ is abelian, we then have

$$\widehat{T}_+ \cong \langle r(t_1) \rangle \times \langle r(t_2) \rangle \cong \mathbf{Z}/(2n+2)\mathbf{Z} \times \mathbf{Z}/\tfrac{n+1}{2}\mathbf{Z},$$

where the isomorphism sends $(i,j) \in \mathbf{Z}/(2n+2)\mathbf{Z} \times \mathbf{Z}/\frac{n+1}{2}\mathbf{Z}$ to $\zeta^{\lfloor i/2 \rfloor} r(t_{i+2j}) \in \widehat{T}_+$. From here on, we will use this isomorphism freely.

Since $w_0^{-1} \cdot r(t_i) = \zeta^i r(t_i)$, we see that

$$\begin{aligned} w_0^{-1} \cdot (i,j) &= w_0^{-1} \cdot \zeta^{\lfloor i/2 \rfloor} r(t_{i+2j}) \\ &= \zeta^{\lfloor i/2 \rfloor + i + 2j} r(t_{i+2j}) \\ &= \zeta^{\lfloor (3i+4j)/2 \rfloor} r(t_{(3i+4j)+2(-i-j)}) \\ &= (3i+4j, -i-j). \end{aligned}$$

Also note that

$$\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2n+1 & 4n \\ -n & -2n+1 \end{pmatrix} = \begin{pmatrix} 2(n+1)+1 & 4(n+1) \\ -(n+1) & -2(n+1)+1 \end{pmatrix},$$

so explicitly, $w_0^{-\ell} \cdot (i,j) = ((2\ell+1)i + 4\ell j,\ -\ell i + (-2\ell+1)j)$.

Now for a given character χ of A, write B for the maximal subgroup of C_+ to which χ extends, and write E for the image of this subgroup in W_s^φ. As a subgroup of W_s^φ, we know that

$$E \cong \left\langle w_0^{k'} \right\rangle \cong \mathbf{Z}/\tfrac{n+1}{k'}\mathbf{Z}$$

for some $k' | n+1$. Define $v := w_0^{k'}$.

First, suppose that $2|k'$. In this case, $\det(P(v)) = (-1)^{k'} = 1$, so the map $s : E \to B$ sending $v^\ell \mapsto P(v^\ell)$ gives a splitting of E into B. As a result, χ is virtually split.

On the other hand, suppose that $2 \nmid k'$. Since $\chi \in X(\mathbf{Z}/(2n+2)\mathbf{Z} \times \mathbf{Z}/\frac{n+1}{2}\mathbf{Z})$, we know that $\chi(i,j) = x_1^i x_2^j$, where $x_1 \in \mu_{2(n+1)}$ and $x_2 \in \mu_{(n+1)/2}$. Since χ is fixed by E, we know in particular that $\chi(1,0) = v^{-1} \cdot \chi(1,0)$. As a result,

$$x_1 = x_1^{2k'+1} x_2^{-k'},$$

so

$$x_2^{k'} = x_1^{2k'}. \tag{4.1.2}$$

Since $\det(\zeta^{1/2}P(w_0)) = 1$, the map $s : E \to B$ sending $v^\ell \mapsto (\zeta^{1/2}P(w_0))^{k'\ell}$ gives a set-theoretic section of E into B. The cocycle of this section is

$$\xi_s(v^i, v^j) = \begin{cases} 1 & i+j < \frac{n+1}{k'} \\ -1 & i+j \geq \frac{n+1}{k'}. \end{cases}$$

Note that $-1 \in \widehat{T}_+$ corresponds to $(n+1, 0) \in \mathbf{Z}/(2n+2)\mathbf{Z} \times \mathbf{Z}/\frac{n+1}{2}\mathbf{Z}$. Since $2|n+1$ but $2 \nmid k'$, we know that $2|\frac{n+1}{k'}$. Raising Equation 4.1.2 to the power of $\frac{n+1}{2k'}$ and remembering that $x_2 \in \mu_{(n+1)/2}$, we see that

$$x_1^{n+1} = x_2^{(n+1)/2} = 1.$$

As a result, $\chi \circ \xi_s = 1$, so in this case, χ is also virtually split. As a result, the sequence $1 \to \widehat{T}_+ \to C_+ \to W_s^\varphi \to 1$ is virtually split.

4.2. **Type B_n. (1)** Let

$$J = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & & 1 & 0 \\ \vdots & \iddots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{Mat}_{2n+1}.$$

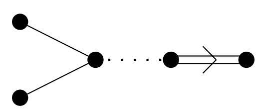
Then in this coordinate system,

$$\widehat{G}_{\mathrm{ad}} = \mathrm{SO}_{2n+1} = \{X \in \mathrm{Mat}_{2n+1} : \det(X) = 1 \text{ and } X^T J X = J\}$$

and

$$\widehat{T}_{\mathrm{ad}} = \{t = \mathrm{diag}(t_1, .., t_n, 1, t_n^{-1}, .., t_1^{-1}) \in \mathrm{SO}_{2n+1} : t_i \neq 0\}.$$

Then $\alpha_n(t) = t_n, \alpha_i(t) = t_i/t_{i+1}$ if $i \neq n$ gives the basis for Δ. The extended Dynkin diagram for type B_n is



The alcove stabilizer Ω acts on $\widetilde{\mathcal{D}}(\mathfrak{g})$ via reflecting α_0 and α_1, so the elements fixed by a nontrivial reflection $\overline{\omega} \in W_s^\varphi$ are given by

$$x = \sum_{i=0}^{n} s_i v_i$$

such that $s_0 = s_1, \sum_0^n s_i = 1$. In type B_n, $v_1 = \check{\omega}_1$ and $v_i = \check{\omega}_i/2$ if $i \neq 1$. Then a nontrivial $\overline{\omega} \in W_s^\varphi$ which fixes x is

$$\overline{\omega} = (1, 0, \cdots, 0) \rtimes \mathrm{id} \in (\mathbf{Z}/2\mathbf{Z})^n \rtimes \Sigma_n.$$

Furthermore, we see

$$W_s^\varphi = \langle \overline{\omega} \rangle = \mathbf{Z}/2\mathbf{Z},$$

and the regular, but not strongly regular elements are of the form

$$s = \mathrm{diag}(s_1, \cdots, s_n, 1, s_n^{-1}, \cdots s_1^{-1}), \tag{4.2.1}$$

where

$$s_i = e^{2\pi i t_j}, \text{ and } t_1 = 1/2 > t_2 > \cdots t_n > 0.$$

(2) Let $w_0 = \tau \rtimes \sigma \in (\mathbf{Z}/2\mathbf{Z})^n \rtimes \Sigma_n$ denote image of φ in W. Observe, T^φ is finite if and only if w_0 is a product of negative cycles. Furthermore, $W_s^\varphi \neq \{1\}$ if and only if $ww_0 = w_0 w$, which implies the first cycle of φ is length 1. Assume, after some conjugation, that $\varphi = \varphi_{i_1} \sqcup \cdots \sqcup \varphi_{i_m}$ where $i_1 = 1$, $\varphi_{i_{k+1}} = \overline{(i_{k1}+1, .., i_{k+1})}$ is a negative cycle whose action on $\widehat{T}$ is given by

$$\varepsilon^*_{i_k+1} \to \varepsilon^*_{i_k+2}, \cdots, \varepsilon^*_{i_{k+1}-1} \to \varepsilon^*_{i_{k+1}}, \varepsilon^*_{i_{k+1}} \to -\varepsilon^*_{i_k+1}.$$

With these restrictions on φ, we see

$$\widehat{T}_{\text{ad}}^{\varphi} \cong (\mathbf{Z}/2\mathbf{Z})^m,$$

where m is the number of (negative) cycles of φ.

(3) Taking the general form of $\varphi = \tau \rtimes \sigma \in (\mathbf{Z}/2\mathbf{Z})^n \rtimes \Sigma_n$ and $s = \text{diag}(s_1, \cdots, s_n, 1, s_n^{-1}, \cdots s_1^{-1})$ where $s_i = e^{2\pi i t_j}$, and $t_1 = 1/2 > t_2 > \cdots t_n > 0$ found in **(1)**, **(2)**, we see

$$w_0.s = \text{diag}\big(-1, s_{\sigma(1)}^{\tau(1)}, s_{\sigma(2)}^{\tau(2)}, \cdots, s_{\sigma(n)}^{\tau(n)}, 1, \cdots, -1\big) = \varphi.s.\varphi^{-1},$$

and we want this to equal

$$s^p = \big((-1)^p, s_1^p, .., s_n^p, 1, .., (-1)^p\big).$$

This places the restrictions

$$t_j = \frac{l_j}{p^{k_j}+1}, \text{ where } k_j = \text{ order of cycle in } \varphi \text{ containing } j \tag{4.2.2}$$

for l_j integers which must make $\{t_j\}$ satisfy $1/2 = t_1 > t_2 \cdots > t_n > 0$, meaning

$$\frac{\tau(j) l_{\sigma(j)} - p l_j}{p^{k_j}+1} \in \mathbf{Z}.$$

The key takeaway is given a prime p and the rank($\widehat{G}$)=n, there are only finitely many pairs (s, φ) which satisfy **(1)**,**(2)**,**(3)**, and the restriction on s described in 4.2.2 allows for one to write a computer program which outputs all possible pairs.

(4) We will show the Langlands-Shelstad section directly gives a splitting of $W_s^{\varphi} \to \text{Cent}(s, \varphi; \widehat{G}_{\text{ad}})$. The set of roots $\alpha \in \Phi^+$ such that $w^{-1}\alpha < 0$ is given in the following table:

TABLE 4.2.1. B_n Roots

$\alpha > 0$	ε_1	$\varepsilon_1 - \varepsilon_j, j > 1$	$\varepsilon_1 + \varepsilon_j, j > 1$
$w^{-1}\alpha < 0$	$-\varepsilon_1$	$-\varepsilon_1 - \varepsilon_j, j > 1$	$-\varepsilon_1 + \varepsilon_j, j > 1$

Using Table 2.3.1 and 4.2.1, we easily compute

$$n(w)^2 = \xi(w, w) = 1/2[2\varepsilon_1^* + \sum_{j>1}((\varepsilon_1^* + \varepsilon_j^*) + (\varepsilon_1^* - \varepsilon_j^*))] = n\varepsilon_1^* \in \check{P}.$$

Recall $\widehat{T}_{\text{ad}} = V/\check{P}$, so $w^2 = 1 \in \widehat{T}_{\text{ad}}$ which shows the Langlands-Shelstad section is a splitting. Using that $\varphi(1) = -1$ and φ is product of negative cycles, we see

$$\{\alpha \in \Phi^+ : w^{-1}\alpha < 0 \text{ and } w_0^{-1}w^{-1}\alpha > 0\} = \{\varepsilon_1, \varepsilon_1 \pm \varepsilon_j, 1 < j\},$$

and

$$\{\alpha \in \Phi^+ : w_0^{-1}\alpha < 0 \text{ and } w^{-1}w_0^{-1}\alpha > 0\} = \{\varepsilon_1, \varepsilon_1 \pm \varepsilon_j, 1 < j\}.$$

Since

$$[n(w), n(w_0)] = \xi_s(w, w_0)\xi(w_0, w)^{-1} = n\varepsilon_1^* + n\varepsilon_1^* = 2ne_1^* \in \check{Q},$$

the lift of w lives in both $\widehat{T}_{\text{ad}}^{\varphi}$ and $\widehat{T}_+$. Finally, observe that w acts trivially on $\widehat{T}_{\text{ad}}^{\varphi}$, hence we have shown

$$\text{Cent}(s, \varphi, \widehat{G}_{\text{ad}}) = \widehat{T}_{\text{ad}} \times W_s^{\varphi} = (\mathbf{Z}/2\mathbf{Z})^m \times \mathbf{Z}/2\mathbf{Z},$$

where m is the number of cycles in φ.

(5) We first prove

$$\widehat{T}_+ \cong (\mathbf{Z}/4\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})^{m-1}.$$

We see a basis of $\widehat{T}_{\rm ad}^{\varphi}$ is given by $\langle 1/2\varepsilon_1^*, 1/2\varepsilon_2^* + \cdots + 1/2\varepsilon_{i_2}^*, \cdots, 1/2\varepsilon_{i_{m-1}+1} + \cdots 1/2\varepsilon_{i_m}\rangle$. Then the pre-image in the simply-connected cover has basis given by $\langle f, e_2, .., e_m\rangle$ where

$$\begin{aligned} f &= 1/2\varepsilon_1^*, \\ e_k &= 1/2\varepsilon_{i_k+1}^* + \cdots + 1/2\varepsilon_{i_{k+1}}^* \text{ if } i_k \text{ even }, \\ e_k &= 1/2\varepsilon_1^* + 1/2\varepsilon_{i_k+1}^* + \cdots + 1/2\varepsilon_{i_{k+1}}^* \text{ if } i_k \text{ odd }. \end{aligned}$$

We can see this indeed gives a basis and implies $\widehat{T}_+ \cong \mathbf{Z}/4\mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z})^{m-1}$. Now it is straigtforward to verify $w.f = f^3$ and $w.e_k = e_k$ if i_k even and $w.e_k = f^3 e_k$ if i_k odd.

By Table 4.2.1, $\xi(w,w) \in \check{Q}$ if n is even, so we again have a splitting: $\mathrm{Cent}(s,\varphi;\widehat{G}_{\rm ad})_+ \cong \widehat{T}_+ \rtimes \mathbf{Z}/2\mathbf{Z}$ and we know its representations. If n is odd, then $n(w)$ has order 4. Let $\chi : T_+^{\varphi} \to \mathbf{C}^\times$ be a character. Then $w.\chi(f) = x_1 = \chi(w.f) = x_1^3$ implies $x_1 = \pm 1$. Thus, $\chi\xi_s(w,w) = \chi(-1) = \chi(f^2) = 1$ and thus from 3.2.4, we see $\sigma : (W_s^{\varphi})_\chi = W_s^{\varphi}$ must be a character. This means all irreducible representations of $\mathrm{Cent}(s,\varphi;\widehat{G}_{\rm ad})_+$ are either of the form

$$\mathrm{Ind}_{\widehat{T}_+^{\varphi}}^{\mathrm{Cent}(s,\varphi;\widehat{G}_{\rm ad})_+}\chi$$

when $w.\chi \neq \chi$ or of the form

$$\widetilde{\chi} : \mathrm{Cent}(s,\varphi;\widehat{G}_{\rm ad})_+ \to \mathbf{C}^\times,$$

defined by

$$\widetilde{\chi}(tn(w)) = \chi(t)\sigma(w)$$

for a character σ of W_s^{φ}.

4.3. **Type C_n.** Define the group homomorphism $P : \Sigma_k \to GL_k(\mathbf{C})$ which sends an element $\sigma \in \Sigma_k$ to its corresponding permutation matrix $P(\sigma)$, i.e. the matrix such that for all $M \in \mathrm{Mat}_k(\mathbf{C})$ we have $(P(\sigma)MP(\sigma)^{-1})_{ij} = M_{\sigma^{-1}(i)\sigma^{-1}(j)}$. Also, given a function $f : \{1,\ldots,k\} \to \mathbf{C}$, define the matrix $\Lambda(f)$ such that

$$\Lambda(f)_{ij} = \begin{cases} f(i) & i = j \\ 0 & i \neq j. \end{cases}$$

With these definitions, note that

$$P(\sigma)\Lambda(f)P(\sigma)^{-1} = \Lambda(f \circ \sigma^{-1}).$$

Now consider the alternating bilinear form

$$J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix} = \Lambda((-1)^i)P(\rho) \in \mathrm{Mat}_{2n}(\mathbf{C}),$$

where $\rho \in \Sigma_{2n}$ is the permutation sending $i \mapsto 2n+1-i$. Having chosen this J, we can present $Sp_{2n}(\mathbf{C})$ as

$$Sp_{2n}(\mathbf{C}) = \{X \in GL_n(\mathbf{C}) \mid X^T JX = J\}.$$

In this presentation, we can choose the maximal torus

$$\widehat{T}_{sc} = \{\mathrm{diag}(t_1,\ldots,t_n,t_n^{-1},\ldots,t_1^{-1})\}.$$

Then in the adjoint case we get

$$\widehat{G}_{ad} = PSp_{2n}(\mathbf{C}) = Sp_{2n}(\mathbf{C})/\mu_2$$

and

$$\widehat{T}_{ad} = \widehat{T}_{sc}/\mu_2.$$

Now write $\{b_1, \ldots, b_n\}$ for the standard basis of $(\mathbf{Z}/2\mathbf{Z})^n$ and define a homomorphism $q_1 : (\mathbf{Z}/2\mathbf{Z})^n \to \Sigma_{2n}$ which sends $b_i \mapsto (i, 2n+1-i)$. Note that $q_1((\mathbf{Z}/2\mathbf{Z})^n)$ commutes with ρ. There is an embedding $\Sigma_n \hookrightarrow \Sigma_{2n}$ given by having an element $\sigma \in \Sigma_n$ act on only the first n coordinates of $\{1, \ldots, 2n\}$. With this identification in mind, define the function $q_2 : \Sigma_n \to \Sigma_{2n}$ which sends $\sigma \mapsto \rho\sigma\rho\sigma$. Note that for any elements $\sigma, \tau \in \Sigma_n$, we know that σ and $\rho\tau\rho$ commute, so that ρ commutes with $q_2(\sigma)$ and q_2 is a homomorphism. Finally, note that for any $x \in (\mathbf{Z}/2\mathbf{Z})^n$ and any $\sigma \in \Sigma_n$, we have that ${}^{q_2(\sigma)}q_1(x) = q_1({}^{\sigma}x)$, where Σ_n acts on $(\mathbf{Z}/2\mathbf{Z})^n$ by permuting coordinates. We therefore obtain an injective group homomorphism $q : W \hookrightarrow \Sigma_{2n}$ which sends $x \rtimes \sigma \mapsto q_1(x)q_2(\sigma)$. Note that $q(W)$ commutes with ρ.

For $\sigma \in \Sigma_{2n}$, define the function $s_\sigma : \{1, \ldots, 2n\} \to \mathbf{C}$ by

$$s_\sigma(i) = \begin{cases} (-1)^{i+\sigma^{-1}(i)} & i \le n \\ 1 & i > n. \end{cases}$$

Note that if σ and ρ commute, we have

$$\begin{aligned} s_\sigma \circ \rho &= \begin{cases} (-1)^{\rho(i)+\sigma^{-1}\rho(i)} & \rho(i) \le n \\ 1 & \rho(i) > n \end{cases} \\ &= \begin{cases} (-1)^{i+\sigma^{-1}(i)} & i > n \\ 1 & i \le n \end{cases} \\ &= (-1)^{i+\sigma^{-1}(i)} s_\sigma. \end{aligned}$$

Now define the map $R : \Sigma_{2n} \to GL_{2n}(\mathbf{C})$ which sends $\sigma \mapsto \Lambda(s_\sigma)P(\sigma)$. Note that for $\sigma \in \Sigma_{2n}^\rho$ we have that

$$\begin{aligned} R(\sigma)^T J R(\sigma) &= P(\sigma^{-1})\Lambda(s_\sigma)\Lambda((-1)^i)P(\rho)\Lambda(s_\sigma)P(\sigma) \\ &= P(\sigma^{-1})\Lambda(s_\sigma)\Lambda((-1)^i)\Lambda(s_\sigma \circ \rho)P(\rho\sigma) \\ &= P(\sigma^{-1})\Lambda((-1)^{\sigma^{-1}(i)})P(\rho\sigma) \\ &= \Lambda((-1)^i)P(\rho) \\ &= J, \end{aligned}$$

so R maps Σ_{2n}^ρ into $Sp_{2n}(\mathbf{C})$. As a result, $n = R \circ q$ is an injective map from $W \to Sp_{2n}(\mathbf{C})$. Since $n(w)$ acts on our chosen $\widehat{T}_{sc}$ in the same way as w, we see that n is a set-theoretic section of the exact sequence $1 \to \widehat{T}_{sc} \to N(\widehat{T}_{sc}) \to W \to 1$. Since $n = R \circ q$, where q is a group homomorphism, we see that $\xi_n = \xi_R \circ q$. Now we compute ξ_R:

$$\begin{aligned} \xi_R(\sigma, \tau) &= \Lambda(s_\sigma)P(\sigma)\Lambda(s_\tau)P(\tau)(\Lambda(s_{\sigma\tau})P(\sigma\tau))^{-1} \\ &= \Lambda(s_\sigma)\Lambda(s_\tau \circ \sigma^{-1})P(\sigma)P(\tau)P(\sigma\tau)^{-1}\Lambda(s_{\sigma\tau}) \\ &= \Lambda(s_\sigma \cdot (s_\tau \circ \sigma^{-1}) \cdot s_{\sigma\tau}). \end{aligned}$$

Observe

$$\begin{aligned} s_\sigma \cdot s_{\sigma\tau} &= \begin{cases} (-1)^{i+\sigma^{-1}(i)+i+\tau^{-1}\sigma^{-1}(i)} & i \le n \\ 1 & i > n \end{cases} \\ &= \begin{cases} (-1)^{\sigma^{-1}(i)+\tau^{-1}\sigma^{-1}(i)} & i \le n \\ 1 & i > n \end{cases} \end{aligned}$$

and

$$s_\tau \circ \sigma^{-1} = \begin{cases} (-1)^{\sigma^{-1}(i)+\tau^{-1}\sigma^{-1}(i)} & \sigma^{-1}(i) \leq n \\ 1 & \sigma^{-1}(i) > n, \end{cases}$$

so that

$$s_\sigma \cdot (s_\tau \circ \sigma^{-1}) \cdot s_{\sigma\tau} = \begin{cases} (-1)^{\sigma^{-1}(i)+\tau^{-1}\sigma^{-1}(i)} & (\sigma^{-1}(i) \leq n) \oplus (i \leq n) \\ 1 & \text{otherwise,} \end{cases}$$

where here $\oplus$ denotes logical exclusive or. Now let $m \in \Sigma_n$ denote the permutation sending $i \mapsto n+1-i$. Note that $q_1(m)$ is the permutation sending $i \mapsto n+1-i$ if $i \leq n$ and $i \mapsto 3n+1-i$ if $i > n$. If we suppose that σ and τ commute with $q_1(m)$, then we see that

$$(s_\sigma \cdot (s_\tau \circ \sigma^{-1}) \cdot s_{\sigma\tau}) \circ q_1(m) = \begin{cases} (-1)^{q_1(m)\sigma^{-1}(i)+q_1(m)\tau^{-1}\sigma^{-1}(i)} & (\sigma^{-1}(i) \leq n) \oplus (i \leq n) \\ 1 & \text{otherwise.} \end{cases}$$

Note that $q_1(m)\sigma^{-1}(i) + q_1(m)\tau^{-1}\sigma^{-1}(i)$ always equals $\sigma^{-1}(i) + \tau^{-1}\sigma^{-1}(i)$ mod 2, so

$$(s_\sigma \cdot (s_\tau \circ \sigma^{-1}) \cdot s_{\sigma\tau}) \circ q_1(m) = s_\sigma \cdot (s_\tau \circ \sigma^{-1}) \cdot s_{\sigma\tau}.$$

We conclude that if we compose n with the projection $p : Sp_{2n}(\mathbf{C}) \to PSp_{2n}(\mathbf{C})$, we obtain a section of the exact sequence $1 \to \widehat{T}_{ad} \to N(\widehat{T}_{ad}) \to W \to 1$ whose cocycle is $p \circ s$.

(1) The highest root of a root system of type C_n is $2\alpha_1 + \ldots + 2\alpha_{n-1} + \alpha_n$ so the vertices of the fundamental alcove are $v_i = \check{\omega}_i/2$ for $i < n$ and $v_n = \check{\omega}_n$. The extended Dynkin diagram for C_n is

Hence $\Omega \cong \mathbf{Z}/2\mathbf{Z}$, where the nontrivial element acts by reflecting the diagram, i.e sending $v_i \mapsto v_{n-i}$. As a result, when written in barycentric coordinates, the nontrivial elements of the fundamental alcove which are fixed by this element are of the form

$$x = \sum_{i=0}^{n} s_i v_i$$

such that $s_i = s_{n-i}$, $\sum_{i=0}^{n} s_i = 1$, and $s_i > 0$ for all i. Since $\check{\omega}_i = \sum_{j=1}^{i} \varepsilon_j^*$ for $i < n$ and $\check{\omega}_n = \sum_{j=1}^{n} \varepsilon_j^*/2$, we see that

$$\exp(x) = (e^{\pi i t_1}, \ldots, e^{\pi i t_n}, e^{-\pi i t_n}, \ldots, e^{-\pi i t_1}),$$

where $t_i = \sum_{j=i}^{n} s_j$, so in particular $t_i + t_{n+1-i} = 1$ and $t_i > t_{i+1}$.

(2) Since any element $\varphi \in W$ can be conjugated into an element which is a product of signed cycles, and since in $Sp_{2n}(\mathbf{C})$ it is obvious that a product of signed cycles has a finite set of fixed points in $\widehat{T}_{sc}$ if and only if it is a product of negative cycles, by 2.2.2 and 2.2.3 we see that $\widehat{T}^{\varphi}$ is finite if and only if φ is conjugate to a product of negative cycles.

Write w for the projection of the nontrivial element of Ω to W. Then

$$w = (1, 1, \ldots, 1) \rtimes m.$$

Since $\mathbf{Z}/2\mathbf{Z}^n$ is abelian and $(1, 1, \ldots, 1)$ is fixed by every element of Σ_n under its action on $\mathbf{Z}/2\mathbf{Z}^n$, we see that

$$w_0 w w_0^{-1} = w_0(1, 1, \ldots, 1)w_0^{-1} \rtimes w_0 m w_0^{-1} = (1, 1, \ldots, 1) \rtimes w_0 m w_0^{-1},$$

so w_0 commutes with w if and only if w_0 commutes with m. If it is the case that both w_0 commutes with m and $\widehat{T}^{w_0}$ is finite, then we can conjugate w_0 by elements of W which commute with m so that it is of the form $em\psi m\psi$, where e sends $\varepsilon^*_{\lceil n/2 \rceil}$ to $-\varepsilon^*_{\lceil n/2 \rceil}$ when n is odd and ψ

is a product of negative cycles acting only on the coordinates $\{1,\ldots,\lfloor\frac{n}{2}\rfloor\}$. Explicitly, there exist $1\le i_1<i_2<\ldots<i_m\le\lfloor\frac{n}{2}\rfloor$ such that

$$\psi(\varepsilon_j^*)=\varepsilon_{j+1}^* \text{ for } j\notin\{i_1,i_2,\ldots,i_m\} \text{ and } \psi(\varepsilon_{i_j}^*)=-\varepsilon_{i_{j-1}+1}^*.$$

On the other hand, if we can conjugate w_0 into such a form by elements of W which commute with m, then $\widehat{T}^{\varphi}$ is finite and φ commutes with m. As a result, we will now assume that w_0 is of the form $em\psi m\psi$.

Write l for the number of cycles of ϕ. We will now show that

$$T_{ad}^{\phi}\cong\begin{cases}(\mathbf{Z}/2)^l & \text{all cycles have even length or all have odd length}\\ (\mathbf{Z}/2)^{l-2}\times\mathbf{Z}/4 & \text{otherwise.}\end{cases}$$

Since $T_{sc}^{\phi}\cong(\mathbf{Z}/2)^l$, we know that $|T_{ad}^{\phi}|=2^l$. We will write out a basis for T_{ad}^{ϕ} to describe it explicitly. For notational convenience, we will present basis elements via elements of V which exponentiate to desired basis element. If w_0 has at least one cycle of even length and one of odd length, then we can conjugate w_0 so that the first cycle has odd length and the second cycle has even length. In this case, if n is even, a basis for T_{ad}^{ϕ} is given by $\{e_3,\ldots,e_m,e_{m'},\ldots e_{1'},f\}$, where

$$\begin{aligned}
e_k&=\frac{1}{2}\varepsilon_{i_k}^*+\ldots+\frac{1}{2}\varepsilon_{i_{k+1}-1}^*,\\
e_{k'}&=\frac{1}{2}\varepsilon_{i'_{k+1}+1}^*+\ldots+\frac{1}{2}\varepsilon_{i'_k}^*,\\
f&=\sum_{i_{k+1}-i_k\text{ even}}\left[\sum_{\substack{j=0\\ j\text{ even}}}^{i_{k+1}-i_k-1}\frac{1}{2}\varepsilon_{i_k+j}^*+\frac{1}{2}\varepsilon_{i'_k-j}^*\right]+\sum_{i_{k+1}-i_k\text{ odd}}\left[\sum_{j=0}^{i_{k+1}-i_k-1}\frac{(-1)^j}{4}\varepsilon_{i_k+j}^*+\frac{(-1)^j}{4}\varepsilon_{i'_k-j}^*\right].
\end{aligned}$$

Here, i'_k denotes $m(i_k)=n+1-i_k$. Note that $\phi e_k\phi^{-1}=e_k$ and $\phi f\phi^{-1}=-f=f$, so these elements all lie in T_{ad}^{ϕ}. We know e_k has order 2, and since there is at least one even cycle and at least one odd cycle, f has order 4. Because all of the basis elements we have written down are linearly independent, we see by a counting argument that they span T_{ad}^{ϕ}. As a result, in this case $T_{ad}^{\phi}\cong(\mathbf{Z}/2)^{l-2}\times\mathbf{Z}/4$. The case when n is odd is only slightly different: just take the basis $\{e_3,\ldots,e_m,e_{m+1},e_{m'},\ldots e_{1'},f\}$, where $e_{m+1}=\frac{1}{2}\varepsilon_{\lceil n/2\rceil}^*$.

On the other hand, if either all of the cycles of w_0 have odd length, or if they all have even length, then f has order 2. In this case, if n is even, take the basis $\{e_2,e_3,\ldots,e_m,e_{m'},\ldots e_{1'},f\}$ and if n is odd, take the basis $\{e_2,e_3,\ldots,e_m,e_{m+1},e_{m'},\ldots e_{1'},f\}$. Since all basis elements have order 2, we see $T_{ad}^{\phi}\cong(\mathbf{Z}/2)^l$.

(3) Assume s is as given in (insert equation) and w_0 is in the normal form discussed in step (2). Let $w_0^{-1}.(s_1,\ldots,s_n)=(s_2,s_3,\ldots,s_{i_1},s_1^{-1},\ldots)$, so we will actually compute pairs $(s,\phi):\phi^{-1}s\phi=s^p$ because it makes the notation easier. We follow the same procedure as in step (3) for type A_n,B_n to find $s=(s_1,\ldots,s_n)$ must satisfy $s_1^{p^{i_1}+1}=\varepsilon^{i_1+1}$, where $\varepsilon^2=1$ and we get a string of inequalities:

$$\begin{aligned}
1/2 &> \frac{k_1(i_1+1)}{2p^{i_1+1}} > \left\{\frac{pk_1(i_1+1)}{2(p^{i_1}+1)}\right\} > \cdots > \left\{\frac{p^{i_1-1}k_1(i_1+1)}{2(p^{i_1}+1)}\right\} > \\
&\vdots \\
&> \frac{k_j(i_j-i_{j-1}+1)}{2(p^{i_j-i_{j-1}}+1)} > \left\{\frac{pk_j(i_j-i_{j-1}+1)}{2(p^{i_j-i_{j-1}}+1)}\right\} > \cdots > \left\{\frac{p^{i_j-i_{j-1}-1}k_j(i_j-i_{j-1}+1)}{2(p^{i_j-i_{j-1}}+1)}\right\} > \\
&\vdots \\
&> 1/4,
\end{aligned}$$

where $k_j \in \mathbf{Z}, j \in \{1,\ldots,m\}, m =$ number of cycles of w_0, are the free parameters chosen to make this inequality hold, and $\{x\}$ denotes the fractional part of x. The key takeaway again is for a given prime p, there exists only finitely many pairs (s,ϕ) which can be determined systematically on a case-by-case basis using a computer program.

(4) In order to apply the machinery of section (3.4) to n, we need to pick lifts of $c(id,w_0)$, $c(w,w_0)$, $\xi(1,1)$, $\xi(1,w)$, $\xi(w,1)$, and $\xi(w,w)$. Lift $c(id,w_0)$, $\xi(1,1)$, $\xi(1,w)$, and $\xi(w,1)$ to 0. Note that

$$\begin{aligned}
\xi_s(w,w) &= \begin{cases} (-1)^{w(i)+i} & (w(i)\le n)\oplus(i\le n) \\ 1 & \text{otherwise} \end{cases} \\
&= \Lambda((-1)^{w(i)+i}) \\
&= (-1)^n,
\end{aligned}$$

so we can lift $\xi(w,w)$ to $n\breve{\omega}_n$. Finally, note that $c(w,w_0) = \Lambda((-1)^{f(i)})$, where $f : \{1,\ldots,2n\} \to \{0,1\}$. We can then lift $c(w,w_0)$ to $\sum_{i=1}^n f(i)x_i^*/2$. Since both w_0 and w commute with m, we know that $m \cdot c(w,w_0) = c(w,w_0)$, so $f \circ m = f$. From this, we can see that $w \cdot \dot{c}(w,w_0) = -\dot{c}(w,w_0)$. As a result, for any $w_1, w_2 \in W_s^\varphi$, we see that

$$\dot{c}(w_1,w_0) + w_1 \cdot \dot{c}(w_2,w_0) - \dot{c}(w_1w_2,w_0) = 0.$$

We can then conclude that

$$\dot{c}(w_1,w_0) + w_1 \cdot \dot{c}(w_2,w_0) - \dot{c}(w_1w_2,w_0) + (\varphi-1)\dot{\xi}(w_1,w_2) = (\varphi-1)\dot{\xi}(w_1,w_2)$$

is equal to $(\varphi-1)p$ for some $p \in \check{P}$ (either 0 or $n\breve{\omega}_n$). As a result, the sequence $1 \to \widehat{T}^\varphi \to \mathrm{Cent}(s,\varphi;\widehat{G}_{ad}) \to W_s^\varphi \to 1$ splits.

(5) First we will show that

$$\widehat{T}_+ \cong \begin{cases} (\mathbf{Z}/2)^{l+1} & \text{all cycles of } w_0 \text{ have even length} \\ (\mathbf{Z}/2)^{l-1}\times\mathbf{Z}/4 & \text{otherwise,} \end{cases}$$

where l is the number of cycles of w_0. First note that $\widehat{T}_+$ is the pullback of $\widehat{T}_{ad}^\phi$ along a map with kernel of size 2, so $|\widehat{T}_+| = 2^{l+1}$. Using the same notation as in part 2, if all of the cycles of w_0 have even length then we can pick a basis for $\widehat{T}_+$ given by $\{e_1,\ldots,e_m,e_{m'},\ldots,e_{1'},f\}$ (Whereas before, e_k was the image of $\frac{1}{2}\varepsilon_{i_k}^* + \ldots + \frac{1}{2}\varepsilon_{i_{k+1}-1}^*$ under the exponential map from V to PSp_{2n}, it is now the image of the same element of V under the exponential map from V to Sp_{2n}). All of these elements lie in $\widehat{T}_+$ and they all have order 2. Since they are independent, we see by a counting argument that they generate $\widehat{T}_+$. As a result, $\widehat{T}_+ \cong (\mathbf{Z}/2)^{l+1}$.

On the other hand, if w_0 has at least one cycle with odd length, then we can conjugate w_0 so that this is the first cycle of w_0, and take the basis $\{e_2,\ldots,e_m,e_{m'},\ldots,e_{1'},f\}$ when n is even and

$\{e_2,\ldots,e_m,e_{m+1},e_{m'},\ldots,e_{1'},f\}$ when n is odd. In this case, f has order 4, while all other basis elements have order 2. As before, all of these basis elements lie in $\widehat{T}_+$, and they are all independent, so they generate $\widehat{T}_+$. As a result, we see that $\widehat{T}_+\cong(\mathbf{Z}/2)^{l-1}\times\mathbf{Z}/4$.

Now note that writing $\varphi=y\rtimes\sigma$ for $y\in\mathbf{Z}/2\mathbf{Z}^n$ and $\sigma\in\Sigma_n$, we see that $\varphi\cdot\check{\omega}_n=y\cdot\check{\omega}_n$, and if we (again employing the standard basis for $\mathbf{Z}/2\mathbf{Z}^n$) write $y=\sum_{i\in I}b_i$ for some $I\subset\{1,\ldots,n\}$, then $y\cdot\check{\omega}_n=\check{\omega}_n-\sum_{i\in I}x_i^*$. As a result, $(\varphi-1)\check{\omega}_n=-\sum_{i\in I}x_i^*$. Now consider $p\in V$ given by

$$p=\sum_{i\in I}\frac{x_i^*}{2}+\sum_{\substack{i\notin I\\ i\leq\lfloor n/2\rfloor}}\frac{x_i^*-x_{m(i)}^*}{2}.$$

Since φ commutes with m, we know that I is preserved by m, so

$$w\cdot p=-\sum_{i\in I}\frac{x_i^*}{2}+\sum_{\substack{i\notin I\\ i\leq\lfloor n/2\rfloor}}\frac{x_i^*-x_{m(i)}^*}{2},$$

so that $(w-1)p=(\varphi-1)\check{\omega}_n$. Since $(p+\check{\omega}_n)\in\mathbf{Z}\{x_1^*,\ldots,x_n^*\}$, we see that $p\in\check{P}$. Because $(\varphi-1)\dot{\xi}(w_1,w_2)$ is always some integer multiple of $\check{\omega}_n$, we can then conclude that

$$\dot{c}(w_1,w_0)+w_1\cdot\dot{c}(w_2,w_0)-\dot{c}(w_1w_2,w_0)+(\varphi-1)\dot{\xi}(w_1,w_2)=(w-1)p'$$

for any w_1,w_2 and some corresponding $p'\in\check{P}$. As a result, the sequence $1\to\widehat{T}_+\to\mathrm{Cent}(s,\varphi;\widehat{G}_{ad})_+\to W_s^{\varphi}\to 1$ is virtually split.

4.4. **Type** D_n**.** Since $\Omega=\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$ if n is even and $\mathbf{Z}/4\mathbf{Z}$ if n is odd, we split into cases of parity of n.

Case D_{2n}

(1) Let

$$J=\begin{pmatrix}0&\cdots&0&1\\0&&1&0\\\vdots&\iddots&&\vdots\\1&0&\cdots&0\end{pmatrix}\in\mathrm{Mat}_{4n}.$$

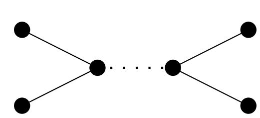
Then in this coordinate system,

$$\widehat{G}_{\mathrm{ad}}=\mathrm{PSO}_{4n}=(\{X\in\mathrm{Mat}_{4n}:\det(X)=1\text{ and }X^TJX=J\})/\mu_2$$

and

$$\widehat{T}_{\mathrm{ad}}=\{t=\mathrm{diag}(t_1,..,t_{2n},t_{2n}^{-1},..,t_1^{-1})\in\mathrm{PSO}_{4n}:t_i\neq 0\}.$$

Then $\alpha_{2n}(t)=t_{2n-1}+t_{2n},\alpha_i(t)=t_i/t_{i+1}$ if $i\neq n$ gives the basis for Δ. The extended Dynkin diagram for type D_{2n} is



The non-trivial symmetries of this Dynkin diagram are

$$\Omega=\langle w_1\rangle\times\langle w_2\rangle=\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z},$$

where

$$w_1:\alpha_j\leftrightarrow\alpha_{2n-j}\text{ for all }0\leq j\leq 2n$$

and

$$w_2:\alpha_0\leftrightarrow\alpha_{2n-1}\quad\alpha_1\leftrightarrow\alpha_{2n}\quad\alpha_j\leftrightarrow\alpha_{2n-j}\text{ for }2\leq j\leq 2n-2.$$

In type D_{2n}, $v_1 = \check{\omega}_1, v_i = \check{\omega}_i/2$ for $i = 2, .., 2n-2$, $v_{2n-1} = \check{\omega}_{2n-1}, v_{2n} = \check{\omega}_{2n}$. So the regular but not strongly regular elements are ones fixed by a subgroup of $\langle w_1 \rangle \times \langle w_2 \rangle$. We treat the nontrivial case when it is fixed by both w_1, w_2 so that W_s^{φ} is not cyclic. Then

$$x = \sum_{i=0}^{2n} s_i v_i,$$

where $s_0 = s_1 = s_{2n-1} = s_{2n}$ and $s_i = s_{2n-i}$ for $2 \leq i \leq 2n-2$. Thus

$$\exp(x) = \mathrm{diag}(e^{2\pi i t_1}, \cdots e^{2\pi i t_{2n}}, e^{-2\pi i t_{2n}}, \cdots, e^{-2\pi i t_1}),$$

where $t_1 = 1/2 > t_2 > \cdots > t_{2n-1} > t_{2n} = 0$ and $t_i + t_{2n-i+1} = 1/2$ for all i. We can now describe the w_1, w_2 action on $\widehat{T}$: $w_1 = e_1 e_{2n}$ and $w_2 = (-1).m$, where $e_i : \varepsilon_i^* \to -\varepsilon_i^*$ and fixes ε_j^* for $j \neq i$ and $m : \varepsilon_j^* \to \varepsilon_{2n+1-j}^*$ for all j.

(2) In order for w_0 to commute with w_1, w_2 and to make $\widehat{T}_{\mathrm{ad}}^{\varphi}$ finite, it is shown, in the same way as for type B_n, C_n, that w_0 must be conjugate to a "normal form" $w_0' \cdot m w_0' m^{-1}$, where $w_0' = \varphi_{i_1} \sqcup \cdots \varphi_{i_m}, 1 = i_1 < 2 = i_2 < i_3 < \cdots < i_m < i_{m+1} := n+1$, is a signed permutation in n variables with action on $\{\varepsilon_1^*, \cdots \varepsilon_n^*\}$ given by

$$(w_0')^{-1} : \varepsilon_j^* \to \varepsilon_{j-1}^* \text{for } j \notin \{i_1, i_2, \cdots i_m\} \text{ and } (w_0')^{-1} : \varepsilon_{i_a}^* \to -\varepsilon_{i_{a+1}-1}^*.$$

We further say w_0' acts as identity on $\{e_{n+1}^*, \cdots, \varepsilon_{2n}^*\}$ to make this an action on $\widehat{T}_{\mathrm{ad}}^{\varphi}$. Notice this φ has the same normal form as that for type C_n, so $\widehat{T}_{\mathrm{ad}}^{\varphi}$ is the same:

$$\widehat{T}_{\mathrm{ad}}^{\varphi} = \begin{cases} (\mathbf{Z}/2\mathbf{Z})^m & \text{if all cycle lengths } i_{k+1} - i_k \text{ are odd} \\ (\mathbf{Z}/2\mathbf{Z})^{m-2} \times \mathbf{Z}/4\mathbf{Z} & \text{else.} \end{cases} \tag{4.4.1}$$

(3) The restrictions that $\phi s \phi^{-1} = s^p$ makes on (s, ϕ, p) for type D_n are very similar to that of type B_n and C_n. We omit it for that reason and invite the reader to attempt it. Let us know if you find a (nicer) characterization!

(4) Let us record the tables arising while computing the Langlands-Shelstad section:

Table 4.4.1. W_s action on D_{2n} Roots

α	$\varepsilon_1 \pm \varepsilon_{2n}$	$\varepsilon_1 \pm \varepsilon_j, j < 2n$	$\varepsilon_i \pm \varepsilon_{2n}, i > 1$	$\varepsilon_i \pm \varepsilon_j, 1 < i < j < 2n$
$w_1^{-1}\alpha$	$-(\varepsilon_1 \pm \varepsilon_{2n})$	$-(\varepsilon_1 \mp \varepsilon_j)$	$\varepsilon_i \mp \varepsilon_{2n}$	$\varepsilon_i \pm \varepsilon_j$

α	$\varepsilon_i - \varepsilon_j, i < j$	$\varepsilon_i + \varepsilon_j, i < j$
$w_2^{-1}\alpha$	$\varepsilon_{2n+1-j} - \varepsilon_{2n+1-i}$	$-(\varepsilon_{2n+1-j} + \varepsilon_{2n+1-i})$

For the action of w_0^{-1}, we only need to record which positive roots get sent to negative. Let

$$B = \{i_a : a = 1, .., k\} \cup \{i_a' : a = 1, .., k\},$$

where $i_a' = 2n + 1 - i_a$.

We compute $n(w_1)^2 = \xi_s(w_1, w_1) = 2n\varepsilon_1^* \in \check{Q}$ and $n(w_2)^2 = 1/2((2n-1)\varepsilon_1^* + .. + (2n-1)\varepsilon_{2n}^*) \in \check{P}$ shows both lifts have order two in $\widehat{T}_{\mathrm{ad}}$.

Now, consider the lift the section into the centralizer as described in (3.4.1). Then $s(w_1) = t_{w_1} n(w_1), s(w_2) = t_{w_2} n(w_2)$, and since we are using the Langlands-Shelstad lift, the commutator $c(w_2, w_0)$ is represented by the half sum of the coroots for all roots occurring in Λ_{w_2, w_0}, where

$$\Lambda_{u,v} := \{\alpha > 0, (uv)^{-1}\alpha > 0\} \bigcap \big(\{u^{-1}\alpha < 0, v^{-1}\alpha > 0\} \cup \{u^{-1}\alpha > 0, v^{-1}\alpha < 0\}\big).$$

TABLE 4.4.2. w_0^{-1} action on D_{2n} roots

α	$w_0^{-1}(\varepsilon_i - \varepsilon_j), i < j$	$w_0^{-1}(\varepsilon_i + \varepsilon_j, i < j)$
$i, j \notin B$	$+$	$+$
$i = i_a, j < i_{a+1}$	$-$	$+$
$i > i'_{a+1}, j = i'_a$	$+$	$-$
$i = i_a, j \geq i_{a+1}$	$-$	$-$
$i = i'_a$	$-$	$-$
$i_a < i < i_{a+1}, j \geq i_{a+1}$	$+$	$+$
$i'_{a+1} < i < i'_a, j > i'_a$	$+$	$+$

Using tables 4.4.1, we find $\Lambda_{w_1,w_0} = \emptyset$, hence $n(w_1)$ is Ad-ϕ fixed and $t_{w_1} = 1$. Next we compute Λ_{w_2,w_0}:

$$\begin{aligned}\Lambda_{w_2,w_0} =& \{\varepsilon_i \pm \varepsilon_j : i = i_a, j < i_{a+1}\} \\ &\cup \{\varepsilon_i - \varepsilon_j : i = i_a, j \geq i_{a+1}, j \notin B\} \\ &\cup \{\varepsilon_i - \varepsilon_j : i = i'_a, j \notin B\} \\ &\cup \{\varepsilon_i + \varepsilon_j : j = i_a, i \notin B\} \\ &\cup \{\varepsilon_i + \varepsilon_j : j = i'_a, i \leq i'_{a+1}, i \notin B\}.\end{aligned}$$

Since we chose the section to be the Langlands-Shelstad section, the lift of

$$c(w_2, w_0)(-1) = \prod_{\alpha \in \Lambda_{w_2,w_0}} \check{\alpha}(-1) \in \widehat{T}$$

to V is represented by

$$\frac{1}{2}c(w_2, w_0) = \frac{1}{2} \sum_{\alpha \in \Lambda_{w_2,w_0}} \check{\alpha}.$$

To compute $c(w_2, w_0)$, we check the contributions for $\varepsilon^*_{i_b}$ and ε^*_i for $i_b < i < i_{b+1}$:

$$\begin{aligned}c(w_2, w_0)|_{i_b} = &\sum_{i_b < j < i_{b+1}} [(\varepsilon^*_{i_b} + \varepsilon^*_j) + (\varepsilon^*_{i_b} - \varepsilon^*_j)] + \\ &\sum_{\substack{j \geq i_{b+1} \\ j \notin B}} (\varepsilon^*_{i_b} - \varepsilon^*_j) + \sum_{\substack{i \notin B \\ i < i_b}} (\varepsilon^*_i + \varepsilon^*_{i_b}),\end{aligned}$$

where $c|_i$ denotes restrict summation to terms including ε^*_i. So the coefficient for $\varepsilon^*_{i_b}$ is $2(i_{b+1} - i_b - 1) + (2n - i_{b+1} + 1 - (|B| - b)) + (i_b - 1 - (b - 1)) = i_{b+1} - i_b + 2n - |B| - 1$

By doing the same thing for $\varepsilon^*_{i'_b}$, we see its coefficient is $i_b - i_{b+1} + 2n - |B| + 1$.

Next, we consider $i_b < i < i_{b+1}$:

$$\begin{aligned}c(w_2, w_0)|_i = &\sum_{a: i < i_{a+1}} (\varepsilon^*_{i_b} \pm \varepsilon_i) + \sum_{\substack{a: i \geq i_{a+1} \\ i \notin B}} (\varepsilon^*_{i_a} - \varepsilon^*_i) + \\ &\sum_{\substack{a: i < i_a \\ i \notin B}} (\varepsilon^*_i + \varepsilon^*_{i_a}) + \sum_{\substack{a: i \leq i'_{a+1} \\ i \notin B}} (\varepsilon^*_i + \varepsilon^*_{i'_a}).\end{aligned}$$

So the coefficient of ε^*_i is $(1 - 1) - (b - 1) + (k - b) + k = |B| - 2b + 1$.

In much a similar way, one finds coefficient for $\varepsilon_i^*, i'_{b+1} < i < i'_b$ is $2b - |B| - 1$. In conclusion,

$$\dot{C}(w_2, w_2)_i := 1/2(w_2.c(w_2, w_0) + c(w_2, w_0))_i = \begin{cases} i_{b+1} - i_b - 1 & \text{if } i = i_b \\ -i_{b+1} + i_b + 1 & \text{if } i = i'_b \\ 2k - 2b + 1 & \text{if } i_b < i < i_{b+1} \\ -2k + 2b - 1 & \text{if } i'_{b+1} < i < i'_b, \end{cases}$$

where $\dot{C}_i$ denotes the ith coordinate of $\dot{C}$ in the ε_i^* coordinates.

Finally, it is immediate to check $\dot{C}(w_2, w_2) = (\phi - 1)p$, where $p \in \check{P}$ is defined as $p = (p_i)$, with $p_{i_{b+1}-1} = (i_{b+1} - i_b - 1)(k - 1)$, and $p_{i_b-1-l} = p_{i_b-1} + l(2k - 2b + 1)$ for $1 < l < i_{b+1} - i_b$. Thus we have shown $s(w_2)^2 = (t_{w_2} n(w_2))^2 = 1$.

To summarize: $s(w_1) = n(w_1), s(w_2) = t_{w_2} n(w_2)$ and the computation for $\dot{C}(w_2, w_2)$ shows $s(w_2)^2 = (t_w n(w))^2 = 1$. Observe $[s(w_1), s(w_2)] = (t_{w_2}^{-1} \cdot {}^{w_1} t_{w_1})[n(w_1), n(w_2)]$ and $\Lambda_{w_1,w_2} = \{\varepsilon_1 \pm \varepsilon_j : 1 < j < 2n\}$ implies $1/2c(w_1, w_2) = 1/2(2n-2)(\varepsilon_1 + \varepsilon_{2n}) \in \check{Q}$, thus $[n(w_1), n(w_2)] = 1$. We compute directly $(\dot{t}_{w_2}^{-1} \cdot {}^{w_1} \dot{t}_{w_1}) = |B|/2\varepsilon_1 + |B|/2\varepsilon_{2n} \in \check{Q}$, thus $(t_{w_2}^{-1} \cdot {}^{w_1} t_{w_1}) = 1$ and $[s(w_1), s(w_2)] = 1$. Finally, $s(w_1)^2 = n(w_1)^2 = 2n\varepsilon_1 \in \check{Q}$. This all shows s is a splitting map, so $\text{Cent}(s, \varphi; \widehat{G}_{\text{ad}}) = \widehat{T}_{ad}^{\phi} \rtimes W_s^{\phi}$.

(5) First, we will show that

$$\widehat{T}_+ \cong (\mathbf{Z}/2)^{l-2} \times (\mathbf{Z}/4)^2,$$

where l is the number of cycles of w_0. First note that since $|\widehat{T}_{ad}^{\phi}| = 2^l$ and since $|Z(G_{sc})| = 4$, we know that $|\widehat{T}_+| = 2^{l+2}$. Now we give an explicit basis for $\widehat{T}_+$: $\{f_1, e_2, \dots e_k, e_{k'}, \dots, e_{2'}, f_2\}$, where

$$f_1 = \frac{1}{2}\varepsilon_1^*,$$
$$e_b = \begin{cases} \frac{1}{2}\varepsilon_{i_b}^* + \dots + \frac{1}{2}\varepsilon_{i_{b+1}-1}^* & i_{b+1} - i_b \text{ even} \\ \frac{1}{2}\varepsilon_{i_b}^* + \dots + \frac{1}{2}\varepsilon_{i_{b+1}-1}^* + \frac{1}{2}\varepsilon_{2n}^* & i_{b+1} - i_b \text{ odd}, \end{cases}$$
$$f_2 = \sum_{i_{b+1}-i_b \text{ even}} \left[\sum_{\substack{j=0 \\ j \text{ even}}}^{i_{b+1}-i_b-1} \frac{1}{2}\varepsilon_{i_b+j}^* + \frac{1}{2}\varepsilon_{i'_b-j}^* \right] + \sum_{i_{b+1}-i_b \text{ odd}} \left[\sum_{j=0}^{i_{b+1}-i_b-1} \frac{(-1)^j}{4}\varepsilon_{i_b+j}^* + \frac{(-1)^j}{4}\varepsilon_{i'_b-j}^* \right],$$

and $e_{b'} = m(e_b)$. These elements are all in $\widehat{T}_+$, and they are all linearly independent. Furthermore, f_1 and f_2 have order 4, while e_b has order 2 for all b. As a result, these elements generate $\widehat{T}_+$, so we know $\widehat{T}_+ \cong (\mathbf{Z}/2)^{l-2} \times (\mathbf{Z}/4)^2$.

We find

$$(w_0 - 1)\xi(w_2, w_2)_i = \begin{cases} -(2n-1) \text{ if } i \in B \\ 0 \text{ else,} \end{cases}$$

so take $p'_i = \begin{cases} \frac{2n-1}{2} \text{ if } i \in B \\ \frac{2n-1}{2} \text{ if } i \notin B, i \le n \\ -\frac{2n-1}{2} \text{ if } i \notin B, i > n. \end{cases}$

The p coming from $\dot{C} = (w_0 - 1)p$ is actually in $\check{Q}$, so $p = q$. Then $\dot{A}(w_2, w_2) = \dot{C}(w_2, w_2) + (w_0 - 1)\xi(w_2, w_2) = (w_0 - 1)q + (w_2 - 1)p'$ for some $q \in \check{Q}, p' \in \check{P}$, which shows $\text{Cent}(s, \varphi; \widehat{G}_{\text{ad}})_+$ is virtually split.

Case D_{2n+1}

(1) Notes: $\Omega = \mathbf{Z}/4\mathbf{Z} = \langle w \rangle$, w acts on extended Dynkin Diagram of type D_{2n+1}, by

$$\alpha_0 \to \alpha_{2n+1} \qquad \alpha_{2n+1} \to \alpha_1$$
$$\alpha_1 \to \alpha_{2n} \qquad \alpha_{2n} \to \alpha_0$$
$$\alpha_j \leftrightarrow \alpha_{2n+1-j} \text{ for } 2 \leq j \leq 2n-1.$$

(Note: there is a typo in [Bou02, Plate IV]) We assume a regular element is fixed by all of $\Omega = \langle w \rangle$. Then one finds it is of the form:

$$x = \sum_{i=0}^{2n} s_i v_i,$$

where $s_0 = s_1 = s_{2n} = s_{2n+1}$ and $s_i = s_{2n+1-i}$ for $2 \leq i \leq 2n-1$. Thus

$$\exp(x) = \mathrm{diag}(e^{2\pi i t_1}, \cdots e^{2\pi i t_{2n+1}}, e^{-2\pi i t_{2n+1}}, \cdots, e^{-2\pi i t_1}),$$

where $t_1 = 1/2 > t_2 > \cdots > t_{2n} > t_{2n+1} = 0$ and $t_i + t_{2n-i+2} = 1/2$ for all i. Then w acts on $\widehat{T}$ by

$$\varepsilon_i^* \to -\varepsilon_{2n+2-i}^* \text{ for } i < 2n+1, \varepsilon_{2n+1}^* \to \varepsilon_1^*.$$

(2) For the same reasons as in C_n, we see that

$$\widehat{T}_{ad}^{\phi} \cong (\mathbf{Z}/2)^{l-2} \times \mathbf{Z}/4.$$

The normal form for w_0 is similar to that for D_{2n}, with the exception of the action on the $\varepsilon_1, \varepsilon_{2n+1}$ coordinates. Let $i_1 = 1 < i_2 = 2 < \cdots < i_m < i_{m+1} := n+1 < i_{m+2} := n+2$. Define the action

$$(w_0')^{-1} : \varepsilon_i \to \varepsilon_{i-1}, \text{ if } i \neq \{i_2, \ldots, i_m\} \text{ and } \varepsilon_{i_b} \to -\varepsilon_{i_{n+1}-1},$$

$$e_{n+1} : \varepsilon_{n+1} \to -\varepsilon_{n+1},$$

$$(e_1')^{-1} : \varepsilon_1 \to \varepsilon_{2n+1}, \varepsilon_{2n+1} \to -\varepsilon_1.$$

Then $w_0 = e_1' e_{n+1} w_0' m w_0'^{-1}$ is the normal form that we will consider below.
(3) The restrictions that $\phi.s\phi^{-1} = s^p$ makes on (s, ϕ, p) for type D_n are very similar to that of type B_n and C_n. We omit it for that reason and invite the reader to attempt it. Let us know if you find a (nicer) characterization!
(4)

TABLE 4.4.3. W_s action of D_{2n+1} roots

α	$\varepsilon_1 - \varepsilon_j$	$\varepsilon_1 + \varepsilon_j$	$\varepsilon_i - \varepsilon_j, 1 < i$	$\varepsilon_i + \varepsilon_j, 1 < i$
$w^{-1}\alpha$	$\varepsilon_{2n+1} + \varepsilon_{2n+2-j}$	$\varepsilon_{2n+1} - \varepsilon_{2n+2-j}$	$\varepsilon_{2n+2-j} - \varepsilon_{2n+2-i}$	$-(\varepsilon_{2n+2-j} + \varepsilon_{2n+2-i})$
$w^2\alpha$	$-$	$-$	$+$	$+$

We first show $s(w)^2 = s(w^2)$, or equivalently

$$1/2(w_0 - 1)^{-1}[c(w, w_0) + w.c(w, w_0) - c(w^2, w_0) + \xi(w, w)] \in \check{P},$$

and then we finish by showing $s(w^2)$ has order 2 in $\widehat{T}_{ad}$.
We find

$$n(w)^2 = \xi(w, w) = (2n-1)/2\varepsilon_2^* + (2n-1)/2\varepsilon_3^* .. + (2n-1)/2\varepsilon_{2n+1}^*.$$

Thus,

$$1/2(\phi-1)\dot{\xi}(w,w)_i = \begin{cases} \frac{2n-1}{2} & \text{if } i=1 \\ -(2n-1) & \text{if } i \in B'' \text{ or } i=n+1 \\ -\frac{2n-1}{2} & \text{if } i=2n+1 \\ 0 & \text{else.} \end{cases}$$

Let $B'' = \{i_2,..,i_k,i'_2,..,i'_k\}$, $B' = B'' \cup \{n+1\}$, $B = B' \cup \{2n+1\}$, $i_{k+1} := n+1, i_{k+2} := n+2, \varepsilon_0 := \varepsilon_{2n+1}$. Then the D_{2n+1} table is the union of the subset of the D_{2n} table for B'' in $V'' = \operatorname{span}\{\varepsilon_2, \cdots, \varepsilon_n, \varepsilon_{n+2}, \cdots \varepsilon_{2n}\}$ (so the w_0^{-1} for D_{2n+1} coincides with the w_0^{-1} from D_{2n} when restricted to V'') and the following exceptions:

TABLE 4.4.4. w_0^{-1} action on D_{2n+1} (exception) roots

α	$w_0^{-1}(\varepsilon_i-\varepsilon_j)$	$w_0^{-1}(\varepsilon_i+\varepsilon_j)$	$w^{-1}w_0^{-1}(\varepsilon_i-\varepsilon_j)$	$w^{-1}w_0^{-1}(\varepsilon_i+\varepsilon_j)$
$i=1, j\notin B$	$-$	$+$	$-$	$-$
$i=1, j\in B, j\neq 2n+1$	$+$	$-$	$-$	$-$
$i=1, j=2n+1$	$+$	$-$	$-$	$-$
$i\notin B, i\neq 1, j=2n+1$	$+$	$-$	$-$	$-$
$i\in B, j=2n+1$	$+$	$-$	$+$	$+$
$i=n+1, j\notin B$	$-$	$-$	$+$	$+$
$i=n+1, j\in B, j\neq 2n+1$	$-$	$-$	$-$	$+$
$i\notin B, i\neq 1, j=n+1$	$+$	$+$	$-$	$-$
$i\in B, j=n+1$	$-$	$-$	$+$	$+$

We conclude

$$\begin{aligned}\Lambda_{w^2,w_0} =&\{\varepsilon_1+\varepsilon_j, j\notin B\} \cup \{\varepsilon_1-\varepsilon_j, j\in B, j\neq 2n+1\} \\ &\cup \{\varepsilon_i+\varepsilon_{2n+1}, i\notin B, i\neq 1\} \cup \{\varepsilon_i+\varepsilon_{2n+1}, i\in B\}.\end{aligned}$$

Thus

$$1/2c(w^2,w_0)_i = [1/2 \sum_{\alpha\in\Lambda_{w^2,w_0}} \check{\alpha}]_i = \begin{cases} (2n-1)/2 \text{ if } i=1,2n+1 \\ 0 \text{ if } i\in B'' \text{ or } i=n+1 \\ 1 \text{ if } i\notin B, i\neq 1, \end{cases}$$

where as usual, c_i denotes ith coordinate of c in ε_i^* basis. Thus,

$$-1/2c(w^2,w_0)+1/2(\phi-1)\xi(w,w)_i = \begin{cases} (2n-1) & \text{if } i=1 \\ 0 & \text{if } i=2n+1 \\ -(2n-1) & \text{if } i\in B'' \text{ or } i=n+1 \\ 1 & \text{else.} \end{cases} \tag{4.4.2}$$

We further find

$$\begin{aligned}\Lambda_{w,w_0} =&\{\varepsilon_i\pm\varepsilon_j : i=i_a\in B', j<i_{a+1}\} \\ &\cup \{\varepsilon_i-\varepsilon_j : i=i_a, j\geq i_{a+1}, j\notin B'\} \\ &\cup \{\varepsilon_i-\varepsilon_j : i=i'_a, j\notin B'\} \\ &\cup \{\varepsilon_i+\varepsilon_j : j=i_a, i\notin B'\} \\ &\cup \{\varepsilon_i+\varepsilon_j : j=i'_a, i\leq i'_{a+1}, i\notin B'\}.\end{aligned}$$

This implies,

$$\dot{C}(w,w) := 1/2(w.c(w,w_0)+c(w,w_0))_i = \begin{cases} 0 & \text{if } i = 1, n+1, 2n+1 \\ i_{b+1}-i_b-1 & \text{if } i = i_b \in B'' \\ -i_{b+1}+i_b+1 & \text{if } i = i'_b \in B'' \\ 2k-2b+1 & \text{if } i_b < i < i_{b+1} \\ -2k+2b-1 & \text{if } i'_{b+1} < i < i'_b. \end{cases} \tag{4.4.3}$$

We see $\dot{A}$ is the sum of equations $(4.4.2), (4.4.3)$ and it is not equal to $(w_0-1).p$ for any $p \in \check{P}$, which shows s is not a section. However, we still do have the decomposition $\dot{A}(w,w) = (w_0-1)q + (w_2-1)p$ by the following. Equation 4.4.2 is of the form $(w-1)p$ for $p = (p_i)$ and

$$p_i = \begin{cases} -(2n-1)/2 & \text{if } i = 1 \\ (2n-1)/2 & \text{if } i = n+1, 2n+1, \text{ or } i \in B'' \\ -1/2 & \text{else.} \end{cases} \tag{4.4.4}$$

Equation 4.4.3 is of the form $(w_0-1)q$ for $q \in \check{Q}$ for the same reason $\dot{C}$ occurring in D_{2n} was of this form. This shows that $\mathrm{Cent}(s,\varphi;\widehat{G}_{\mathrm{ad}})$ is virtually split.

(5) We will show that

$$\widehat{T}_+ \cong (\mathbf{Z}/2)^{l-1} \times \mathbf{Z}/8,$$

where l is the number of cycles of w_0. First note that since $|Z(G_{sc}| = 4$ and $|\widehat{T}^{\phi}_{ad}| = 2^l$, we must have $|\widehat{T}_+| = 2^{l+2}$. Now we give an explicit basis for $\widehat{T}_+$: $\{f_1, e_2, \ldots, e_k, e_{k'}, \ldots, e_{2'}, f_2\}$, where

$$f_1 = \frac{1}{2}\varepsilon^*_1 + \frac{1}{2}\varepsilon^*_{2n+1},$$

$$e_b = \begin{cases} \frac{1}{2}\varepsilon^*_{i_b} + \ldots + \frac{1}{2}\varepsilon^*_{i_{b+1}-1} & i_{b+1}-i_b \text{ even} \\ \frac{1}{2}\varepsilon^*_{i_b} + \ldots + \frac{1}{2}\varepsilon^*_{i_{b+1}-1} + \frac{1}{2}\varepsilon^*_{n+1} & i_{b+1}-i_b \text{ odd,} \end{cases}$$

$$f_2 = \sum_{i_{b+1}-i_b \text{ even}} \left[\sum_{\substack{j=0 \\ j \text{ even}}}^{i_{b+1}-i_b-1} \frac{1}{2}\varepsilon^*_{i_b+j} + \frac{1}{2}\varepsilon^*_{i'_b-j} \right] + \sum_{i_{b+1}-i_b \text{ odd}} \left[\sum_{j=0}^{i_{b+1}-i_b-1} \frac{(-1)^j}{4}\varepsilon^*_{i_b+j} + \frac{(-1)^j}{4}\varepsilon^*_{i'_b-j} \right].$$

and $e_{b'} = m(e_b)$. These are all elements of $\widehat{T}_+$, and they are all linearly independent. Furthermore, f_2 has order 8, while f_1 and e_b have order 2. As a result, these elements generate $\widehat{T}_+$, and so $\widehat{T}_+ \cong (\mathbf{Z}/2)^{l+1} \times \mathbf{Z}/8$.

$\mathrm{Cent}(s,\varphi;\widehat{G}_{\mathrm{ad}})_+$ is also virtually split for the same reason it is virtually split in the adjoint case.

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